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## Graded r-Ideals

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ABSTRACT. Let G be a group with identity e and R be a commutative Ggraded ring with nonzero unity 1. In this article, we introduce the concept of graded r-ideals. A proper graded ideal P of a graded ring R is said to be a graded r-ideal if whenever  $a, b \in h(R)$  such that  $ab \in P$  and Ann(a) = $\{0\}$ , then  $b \in P$ . We study and investigate the behavior of graded r-ideals to introduce several results. We introduced several characterizations for graded r-ideals; we proved that P is a graded r-ideal of R if and only if  $aP = aR \cap P$  for all  $a \in h(R)$  with  $Ann(a) = \{0\}$ . Also, P is a graded rideal of R if and only if P = (P : a) for all  $a \in h(R)$  with  $Ann(a) = \{0\}$ . Moreover, P is a graded r-ideal of R if and only if whenever A, B are graded ideals of R such that  $AB \subseteq P$  and  $A \cap r(h(R)) \neq \phi$ , then  $B \subseteq P$ . In this article, we introduce the concept of a huz-rings. A graded ring R is said to be a huz-ring if every homogeneous element of R is either a zero divisor or a unit. In fact, we proved that R is a huz-ring if and only if every graded ideal of R is a graded r-ideal. Moreover, assuming that Ris a graded domain, we proved that  $\{0\}$  is the only graded r-ideal of R.

**Keywords:** Graded ideals, Graded prime ideals, Graded r-ideals.

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### 1. Introduction

Let G be a group with identity e. A ring R is said to be a G-graded ring if there exist additive subgroups  $R_g$  of R such that  $R = \bigoplus_{g \in G} R_g$  and  $R_g R_h \subseteq R_{gh}$ 

for all  $g,h \in G$ . The elements of  $R_g$  are called homogeneous of degree g and  $R_e$  (the identity component of R) is a subring of R and  $1 \in R_e$ . For  $x \in R$ , x can be written uniquely as  $\sum_{g \in G} x_g$  where  $x_g$  is the component of x in  $R_g$ . Also

we write  $h(R) = \bigcup_{g \in G} R_g$  and  $supp(R, G) = \{g \in G : R_g \neq 0\}$ . For more details, see [3].

Let R be a G-graded ring and P be an ideal of R. Then P is called a G-graded ideal if  $P = \bigoplus_{g \in G} (P \cap R_g)$ , i.e., if  $x \in P$  and  $x = \sum_{g \in G} x_g$ , then  $x_g \in P$  for all  $g \in G$ . An ideal of a graded ring need not be graded; see the following example.

EXAMPLE 1.1. Consider  $R = \mathbf{Z}[i]$  and  $G = \mathbf{Z}_2$ . Then R is G-graded by  $R_0 = \mathbf{Z}$  and  $R_1 = i\mathbf{Z}$ . Now,  $P = \langle 1+i \rangle$  is an ideal of R with  $1+i \in P$ . If P is a graded ideal, then  $1 \in P$ , so 1 = a(1+i) for some  $a \in R$ , i.e., 1 = (x+iy)(1+i) for some  $x, y \in \mathbf{Z}$ . Thus 1 = x - y and 0 = x + y, i.e., 2x = 1 and hence  $x = \frac{1}{2}$  a contradiction. So, P is not graded ideal.

Throughout this article, R will be a commutative ring with nonzero unity 1. For  $a \in R$ , we define  $Ann(a) = \{r \in R : ra = 0\}$ . An element  $a \in R$  is said to be a regular element if  $Ann(a) = \{0\}$ , the set of all regular elements of R is denoted by r(R). If A is a subset of R and P is an ideal of R, then we define  $(P:A) = \{r \in R : rA \subseteq P\}$ .

The notion of r-ideals was introduced and studied by Rostam Mohamadian in [2]. A proper ideal P of R is said to be an r-ideal (resp. pr-ideal) if whenever  $a, b \in R$  such that  $ab \in P$  and  $Ann(a) = \{0\}$ , then  $b \in P$  (resp.  $b^n \in P$  for some  $n \in \mathbb{N}$ ).

In this article, we introduce the concept of graded r-ideals. A proper graded ideal P of a graded ring R is said to be a graded r-ideal (resp. graded pr-ideal) if whenever  $a, b \in h(R)$  such that  $ab \in P$  and  $Ann(a) = \{0\}$ , then  $b \in P$  (resp.  $b^n \in P$  for some  $n \in \mathbb{N}$ ). We study and investigate the behavior of graded r-ideals to introduce several results.

We introduce several characterizations for graded r-ideals; we prove that P is a graded r-ideal of R if and only if  $aP = aR \cap P$  for all  $a \in h(R)$  with  $Ann(a) = \{0\}$ . Also, P is a graded r-ideal of R if and only if P = (P : a) for all  $a \in h(R)$  with  $Ann(a) = \{0\}$ . Moreover, P is a graded r-ideal of R if and only if whenever A, B are graded ideals of R such that  $AB \subseteq P$  and  $A \cap r(h(R)) \neq \emptyset$ , then  $B \subseteq P$ .

A proper graded ideal of a graded ring R is said to be graded prime if whenever  $a, b \in h(R)$  such that  $ab \in P$ , then either  $a \in P$  or  $b \in P$  ([1]). We prove that the intersection of two graded r-ideals is a graded r-ideal. On the other hand, if the intersection of two non-comparable graded prime ideals is a graded r-ideal, then both ideals are graded r-ideals. Moreover, we prove that every graded maximal r-ideal is graded prime.

If P is a graded r-ideal of R, we prove that  $P_e$  is an r-ideal of  $R_e$  and (P:a)is a graded r-ideal of R for all  $a \in h(R) - P$ . Also, we prove that if R is  $\mathbb{Z}$ -graded, then P is a graded pr-ideal of R if and only if  $\sqrt{P}$  is a graded r-ideal

In this article, we introduce the concept of huz-rings. A graded ring R is said to be a huz-ring if every homogeneous element of R is either a zero divisor or a unit. In fact, we prove that R is a huz-ring if and only if every graded ideal of R is a graded r-ideal. Moreover, assuming that R is a graded domain, we prove that  $\{0\}$  is the only graded r-ideal of R.

#### 2. Graded r-Ideals

In this section, we introduce and study the concept of graded r-ideals.

**Definition 2.1.** Let R be a G-graded ring. A proper graded ideal P of R is said to be a graded r-ideal (resp. graded pr-ideal) if whenever  $a, b \in h(R)$  such that  $ab \in P$  and  $Ann(a) = \{0\}$ , then  $b \in P$  (resp.  $b^n \in P$  for some  $n \in \mathbb{N}$ ).

Note that for a graded ideal P of a G-graded ring R,  $P_q = P \cap R_q$  for  $g \in G$ .

**Theorem 2.2.** Let R be a G-graded ring and P be a graded ideal of R. Then P is a graded r-ideal if and only if  $aP = aR \cap P$  for every  $a \in h(R)$  with  $Ann(a) = \{0\}.$ 

*Proof.*  $(\Rightarrow)$  Let  $a \in h(R)$  such that  $Ann(a) = \{0\}$ . Then  $aP \subseteq P$  and  $aP \subseteq aR$ , i.e.,  $aP \subseteq aR \cap P$ . Let  $x \in aR \cap P$ . Then  $x = az \in P$  for some  $z \in R$ . Since R is G-graded,  $z = \sum_{g \in G} z_g$  and then  $x = \sum_{g \in G} az_g \in P$  and since P is a graded ideal,  $az_g \in P$  for all  $g \in G$ . Since P is a graded r-ideal,  $z_g \in P$  for all  $g \in G$  and then  $z = \sum_{g \in G} z_g \in P$  which implies that  $x = az \in aP$ . Hence,

 $aP = aR \cap P$ .  $(\Leftarrow)$  Let  $a, b \in h(R)$  such that  $ab \in P$  and  $Ann(a) = \{0\}$ . Then  $ab \in aR \cap P = aP$  and then ab = ax for some  $x \in P$  which implies that a(b-x) = 0. Since  $Ann(a) = \{0\}, b-x = 0$ , i.e.,  $b = x \in P$ . Hence, P is a graded r-ideal.

**Theorem 2.3.** Let R be a G-graded ring and P be a graded ideal of R. If  $aP_g = aR_h \cap P_g$  for all  $g, h \in G$  and for all  $a \in h(R)$  with  $Ann(a) = \{0\}$ , then P is a graded r-ideal of R.

Proof. Let  $a, b \in h(R)$  such that  $ab \in P$  and  $Ann(a) = \{0\}$ . Then there exist  $g, h \in G$  such that  $a \in R_g$  and  $b \in R_h$  and then  $ab \in R_g R_h \cap P \subseteq R_{gh} \cap P = P_{gh}$ . Now,  $ab \in aR_h \cap P_{gh} = aP_{gh}$ , i.e., ab = ay for some  $y \in P_{gh}$  and then a(b-y) = 0. Since  $Ann(a) = \{0\}$ ,  $b = y \in P_{gh} \subseteq P$ . Hence, P is a graded r-ideal of R.

**Theorem 2.4.** Let R be a G-graded ring and P be a graded ideal of R. Then P is a graded r-ideal if and only if P = (P : a) for all  $a \in h(R)$  with  $Ann(a) = \{0\}$ .

Proof. Suppose that P is a graded r-ideal of R. Let  $a \in h(R)$  with  $Ann(a) = \{0\}$ . Clearly,  $P \subseteq (P:a)$ . Let  $y \in (P:a)$ . Then  $ya \in P$ . Since R is G-graded,  $y = \sum_{g \in G} y_g$  and then  $ya = \sum_{g \in G} y_g a \in P$  and since P is graded,  $y_g a \in P$  for all  $g \in G$ . Since P is a graded r-ideal,  $y_g \in P$  for all  $g \in G$  and then  $y = \sum_{g \in G} y_g \in P$ . Hence, P = (P:a). Conversely, let  $a, b \in h(R)$  such that  $ab \in P$  and  $Ann(a) = \{0\}$ . Then  $b \in (P:a) = P$ . Hence, P is a graded r-ideal of R.

**Theorem 2.5.** Let R be a G-graded ring and P be a graded ideal of R. If  $P_g = (P_g :_{R_h} a)$  for all  $g, h \in G$  and for all  $a \in h(R)$  with  $Ann(a) = \{0\}$ , then P is a graded r-ideal of R.

*Proof.* Let  $a, b \in h(R)$  such that  $ab \in P$  and  $Ann(a) = \{0\}$ . Then  $a \in R_g$  and  $b \in R_h$  for some  $g, h \in G$  and then  $ab \in R_g R_h \cap P \subseteq R_{gh} \cap P = P_{gh}$ , i.e.,  $b \in (P_{gh}:_{R_h} a) = P_{gh} \subseteq P$ . Hence, P is a graded r-ideal of R.

**Theorem 2.6.** Let R be a G-graded ring and P be a graded ideal of R. Then P is a graded r-ideal if and only if whenever A, B are graded ideals of R such that  $AB \subseteq P$  and  $A \cap r(h(R)) \neq \phi$ , then  $B \subseteq P$ .

Proof. Suppose that P is a graded r-ideal of R. Let A,B be two graded ideals of R such that  $AB \subseteq P$  and  $A \cap r(h(R)) \neq \phi$ . Since  $A \cap r(h(R)) \neq \phi$ , there exists  $a \in A \cap r(h(R))$ . Let  $g \in G$  and  $b \in B_g$ . Then  $ab \in AB_g \subseteq AB \subseteq P$ . Since P is a graded r-ideal,  $b \in P$ . So,  $B_g \subseteq P$  for all  $g \in G$  which implies that  $B \subseteq P$ . Conversely, let  $a,b \in h(R)$  such that  $ab \in P$  and  $Ann(a) = \{0\}$ . Then  $A = \langle a \rangle$  and  $B = \langle b \rangle$  are graded ideals of R such that  $AB \subseteq P$  and  $a \in A \cap r(h(R))$ . By assumption,  $B \subseteq P$  and then  $b \in P$ . Hence, P is a graded r-ideal of R.

**Theorem 2.7.** If R is a G-graded domain, then  $\{0\}$  is a unique graded r-ideal of R.

*Proof.* Let P be a nonzero proper graded ideal of R. Then there exists  $0 \neq a = \sum_{g \in G} a_g \in P$  and then  $a_g \in P$  for all  $g \in G$  since P is graded. Since R is

5

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a domain,  $Ann(a_g) = \{0\}$  with  $1.a_g \in P$ . If P is a graded r-ideal, then  $1 \in P$  which is a contradiction. Hence,  $\{0\}$  is the only graded r-ideal of R.

**Lemma 2.8.** If R is a G-graded ring, then  $R_e$  contains all homogeneous idempotent elements of R.

*Proof.* Let  $0 \neq x \in h(R)$  be an idempotent. Then  $x \in R_g$  for some  $g \in G$  and then  $x = x^2 \in R_g \cap R_{g^2}$ . Since  $0 \neq x \in R_g \cap R_{g^2}$ ,  $g^2 = g(\in G)$  which implies that g = e. Hence,  $x \in R_e$ .

**Theorem 2.9.** Let R be a G-graded ring. Suppose that  $\{x_i : i \in \Gamma\}$  is a set of homogeneous idempotent elements in  $R_e$ . Then  $P = \sum_{i \in \Gamma} R_e x_i$  is an r-ideal of  $R_e$ .

*Proof.* Let  $a, b \in R_e$  such that  $ab \in P$  and  $Ann(a) = \{0\}$ . Let  $z = \prod_{k=1}^{n} (1 - x_{i_k})$ 

where  $ab = \sum_{j=1}^{n} r_j x_{i_j}$  for some  $r_1 \dots r_n \in R_e$ . Then abz = 0. Since Ann(a) =

 $\{0\}$ , bz = 0. On the other hand, there exists  $r \in P$  such that z = 1 - r and then b(1 - r) = 0 which implies that  $b = br \in P$ . Hence, P is an r-ideal of  $R_e$ .

The next lemma is well known and clear; so we omit the proof.

**Lemma 2.10.** If  $P_1$  and  $P_2$  are graded ideals of a graded ring R, then  $P_1 \cap P_2$  is a graded ideal of R.

**Theorem 2.11.** Let R be a G-graded ring. If  $P_1$  and  $P_2$  are graded r-ideals of R, then  $P_1 \cap P_2$  is a graded r-ideal of R.

*Proof.* By Lemma 2.10,  $P_1 \cap P_2$  is a graded ideal of R. Let  $a, b \in h(R)$  such that  $ab \in P_1 \cap P_2$  and  $Ann(a) = \{0\}$ . Then  $ab \in P_1$ . Since  $P_1$  is a graded r-ideal,  $b \in P_1$ . Similarly,  $b \in P_2$  and hence  $b \in P_1 \cap P_2$ . Therefore,  $P_1 \cap P_2$  is a graded r-ideal of R.

**Theorem 2.12.** Let R be a G-graded ring and  $P_1, P_2$  be graded prime ideals of R which are not comparable. If  $P_1 \cap P_2$  is a graded r-ideal of R, then  $P_1$  and  $P_2$  are graded r-ideals of R.

*Proof.* Let  $a, b \in h(R)$  such that  $ab \in P_1$  and  $Ann(a) = \{0\}$ . Suppose that  $y \in P_2 - P_1$ . Then  $aby \in P_1 \cap P_2$ . Since  $P_1 \cap P_2$  is graded r-ideal,  $by \in P_1 \cap P_2$  and then  $by \in P_1$ . Since  $P_1$  is graded prime and  $y \notin P_1$ ,  $b \in P_1$ . Hence,  $P_1$  is a graded r-ideal of R. Similarly,  $P_2$  is a graded r-ideal of R.

If P is a graded ideal of a G-graded ring R, then  $\sqrt{P}$  need not to be a graded ideal of R; see ([4], Exercises 17 and 13 on pp. 127-128). We introduce the following.

**Lemma 2.13.** If P is a graded ideal of a  $\mathbb{Z}$ -graded ring R, then  $\sqrt{P}$  is a graded ideal of R.

*Proof.* Clearly,  $\sqrt{P}$  is an ideal of R. Let  $x \in \sqrt{P}$  and write  $x = \sum_{i=1}^{t} x_i$  where

 $x_i \in R_{n_i}$  and  $n_1 < n_2 < \dots < n_t$ . Then  $x^k \in P$  for some positive integer k. Of course,  $x^k = x_1^k +$  (higher terms) and as P is graded, we should have that  $x_1^k \in P$ . Thus,  $x_1 \in \sqrt{P}$  which implies that  $x - x_1 \in \sqrt{P}$ . Now, induct on the number of homogeneous components to conclude that  $x_i \in \sqrt{P}$  for all  $1 \le i \le t$ . Hence,  $\sqrt{P}$  is a graded ideal of R.

**Theorem 2.14.** Let R be a  $\mathbb{Z}$ -graded ring and P be a graded ideal of R. Then P is a graded pr-ideal of R if and only if  $\sqrt{P}$  is a graded r-ideal of R.

Proof. Suppose that P is a graded pr-ideal of R. By Lemma 2.13,  $\sqrt{P}$  is a graded ideal of R. Let  $a,b\in h(R)$  such that  $ab\in \sqrt{P}$  and  $Ann(a)=\{0\}$ . Then  $a^nb^n=(ab)^n\in P$  for some  $n\in \mathbf{N}$ . Since  $a,b\in h(R)$ , there exist  $g,h\in G$  such that  $a\in R_g$  and  $b\in R_h$  and then  $a^n\in R_{g^n}$  and  $b^n\in R_{h^n}$  which implies that  $a^n,b^n\in h(R)$  such that  $a^nb^n\in P$ . Clearly,  $Ann(a^n)=\{0\}$  and since P is a graded pr-ideal,  $b^{nm}=(b^n)^m\in P$  for some  $m\in \mathbf{N}$  which implies that  $b\in \sqrt{P}$ . Hence,  $\sqrt{P}$  is a graded r-ideal of R. Conversely, let  $a,b\in h(R)$  such that  $ab\in P$  and  $Ann(a)=\{0\}$ . Then  $ab\in \sqrt{P}$  and since  $\sqrt{P}$  is a graded r-ideal,  $b\in \sqrt{P}$  which implies that  $b^n\in P$  for some  $n\in \mathbf{N}$ . Hence, P is a graded pr-ideal of R.

Using Theorem 2.14 and Theorem 2.2, we have the next corollary.

**Corollary 2.15.** Let R be a  $\mathbb{Z}$ -graded ring and P be a graded ideal of R. Then P is a graded pr-ideal if and only if  $a\sqrt{P} = aR \bigcap \sqrt{P}$  for every  $a \in h(R)$  with  $Ann(a) = \{0\}$ .

Using Theorem 2.14 and Theorem 2.4, we have the next corollary.

**Corollary 2.16.** Let R be a  $\mathbb{Z}$ -graded ring and P be a graded ideal of R. Then P is graded pr-ideal if and only if  $\sqrt{P} = (\sqrt{P} : a)$  for all  $a \in h(R)$  with  $Ann(a) = \{0\}$ .

**Theorem 2.17.** If P is a graded r-ideal of a G-graded ring R, then (P : a) is a graded r-ideal of R for all  $a \in h(R) - P$ .

Proof. Let  $a \in h(R)-P$ . Clearly, (P:a) is an ideal of R. Let  $x \in (P:a)$ . Then  $x \in R$  such that  $xa \in P$ . Since R is graded,  $x = \sum_{g \in G} x_g$  where  $x_g \in R_g$ . Since

 $a \in h(R), a \in R_h$  for some  $h \in G$  and then  $x_g a \in R_g R_h \subseteq R_{gh}$ , i.e.,  $x_g a \in h(R)$  for all  $g \in G$ . Now,  $xa = \sum_{g \in G} x_g a \in P$ . Since P is a graded,  $x_g a \in P$  for all

 $g \in G$ , i.e.,  $x_g \in (P:a)$  for all  $g \in G$ . Hence, (P:a) is a graded ideal of R.

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Let  $b, c \in h(R)$  such that  $bc \in (P:a)$  and  $Ann(b) = \{0\}$ . Then  $bca \in P$ . Since P is a graded r-ideal,  $ca \in P$  which implies that  $c \in (P:a)$ . Therefore, (P:a) is a graded r-ideal of R.

**Theorem 2.18.** Every graded maximal r-ideal of a graded ring R is graded prime.

*Proof.* Let P be a graded maximal r-ideal of R. Suppose that  $a, b \in h(R)$  such that  $ab \in P$  and  $a \notin P$ . Then by Theorem 2.17, (P:a) is a graded r-ideal of R. Clearly,  $P \subseteq (P:a)$  and  $b \in (P:a)$ . By maximality of P, P = (P:a) and then  $b \in P$ . Hence, P is a graded prime ideal of R.

**Definition 2.19.** A graded ring R is said to be an huz-ring if every homogeneous element of R is either a zero divisor or a unit.

The next theorem gives an example on huz-rings.

**Theorem 2.20.** Every graded finite ring is an huz-ring.

Proof. Let R be a G-graded finite ring. Assume that  $a \in h(R)$ . Then  $a \in R_g$  for some  $g \in G$ . Define  $\phi: R_{g^{-1}} \to R_e$  by  $\phi(x) = ax$ . If  $\phi$  is injective, then since R is finite,  $\phi$  is surjective and as  $1 \in R_e$ , 1 = ax for some  $x \in R_{g^{-1}}$  and then a is a unit. Suppose that  $\phi$  is not injective. Then there exist  $x, y \in R_{g^{-1}}$  with  $x \neq y$  such that ax = ay. But then a(x - y) = 0 and  $x - y \neq 0$ , so a is a zero divisor.

If we drop the finite condition in Theorem 2.20, then the result is not true in general. See the following example.

EXAMPLE 2.21. Let  $G = \mathbb{Z}$ . Then clearly, the semigroup ring  $R[X;\mathbb{Z}]$  is a  $\mathbb{Z}$ -graded ring. If R is a field, then  $R[X;\mathbb{Z}]$  is a huz-ring; and if  $R = \mathbb{Z}$ , then  $R[X;\mathbb{Z}]$  is not a huz-ring.

Finally, we prove that a graded ring R is an huz-ring if and only if every proper graded ideal of R is a graded r-ideal.

**Theorem 2.22.** A graded ring R is a huz-ring if and only if every proper graded ideal of R is a graded r-ideal.

Proof. Suppose that R is an huz-ring. Let P be a proper graded ideal of R. Assume that  $a,b \in h(R)$  such that  $ab \in P$  and  $Ann(a) = \{0\}$ . Since  $Ann(a) = \{0\}$ , a is not zero divisor and since R is huz, a is a unit and then  $b = a^{-1}(ab) \in P$ . Hence, P is a graded r-ideal of R. Conversely, let  $a \in h(R)$  such that a is not a zero divisor. Then  $Ann(a) = \{0\}$ . Suppose that  $P = \langle a \rangle$ . If P is proper, then P is a graded r-ideal of R by assumption. Let  $b \in h(R)$ . Then  $ab \in P$  and then  $b \in P$  since P is a graded r-ideal. So,  $h(R) \subseteq P$ . Since  $1 \in R_e \subseteq h(R)$ ,  $1 \in P$  which is a contradiction. So, P = R, then  $1 \in P$  and then 1 = xa for some  $x \in R$  which implies that a is a unit and hence R is an huz-ring.

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