

Some Generalizations of Locally Closed Sets

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ABSTRACT. Arenas et al. [1] introduced the notion of λ -closed sets as a generalization of locally closed sets. In this paper, we introduce the notions of λ -locally closed sets, Λ_λ -closed sets and λg -closed sets and obtain some decompositions of closed sets and continuity in topological spaces.

Keywords: λ -Open set, λ -Locally closed set, Λ_λ -Closed set, λg -Closed set, Decompositions of continuity.

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1. INTRODUCTION AND PRELIMINARIES

The study of locally closed sets was introduced by Bourbaki [3] in 1966 then the authors Ganster and Reilly [6] have studied it extensively. A subset A of a topological space X is called locally closed if $A = U \cap F$, where U is open and F is closed. It is interesting that a locally closed set is a generalization of both open sets and closed sets. The generalization has also been discussed in completely regular Hausdorff spaces [5] and has also been done on algebra with topology in [12] and [2].

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In this paper we consider a new type of sets in the topological space which is called λ -open sets. A set is said to be λ -open if it contains a nonempty open set. This idea is not a new idea. In literature, semi-open sets [7] and α -sets [11] are examples of that type of sets although preopen sets [10] is not an example of it. Because: let \mathbf{R} be the usual real line and Q the rational numbers. Then $\text{Cl}(Q) = \mathbf{R}$ and $Q \subseteq \text{Int}(\text{Cl}(Q)) = \mathbf{R}$ (where 'Cl' and 'Int' denote the closure and interior operators, respectively). But Q does not contain nonempty open set. However Dontchev [4] has introduced an S -space: A topological space X is called an S -space if every subset which contains a non-void open subset is open. But the concept of λ -open sets is different from Dontchev's S -spaces.

Definition 1.1. A subset A of a topological space X is said to be λ -open if A contains a nonempty open set. The complement of a λ -open set is said to be λ -closed.

For a subset A of a topological space X , $\text{Int}_\lambda(A)$ and $\text{Cl}_\lambda(A)$ are defined as follows:

Definition 1.2. Let X be a topological space and A be a subset of X .

$$\text{Int}_\lambda(A) = \bigcup\{U : U \subseteq A, U \text{ is } \lambda\text{-open in } X\};$$

$$\text{Cl}_\lambda(A) = \bigcap\{F : A \subseteq F, F \text{ is } \lambda\text{-closed in } X\}.$$

Lemma 1.3. Let X be a topological space and A, B subsets of X .

$$(1) \text{ if } A \subseteq B, \text{ then } \text{Int}_\lambda(A) \subseteq \text{Int}_\lambda(B) \text{ and } \text{Cl}_\lambda(A) \subseteq \text{Cl}_\lambda(B),$$

$$(2) X \setminus \text{Int}_\lambda(A) = \text{Cl}_\lambda(X \setminus A),$$

(3) For any index set Δ , if A_α is λ -open (resp. λ -closed), then $\bigcup\{A_\alpha : \alpha \in \Delta\}$ is λ -open (resp. $\bigcap\{A_\alpha : \alpha \in \Delta\}$ is λ -closed),

$$(4) \text{Int}_\lambda(A) \text{ is } \lambda\text{-open and } \text{Cl}_\lambda(A) \text{ is } \lambda\text{-closed.}$$

Remark 1.4. The finite intersection of λ -open sets need not be λ -open. Let \mathbf{R} be the usual real line, $A = (-1, 0]$ and $B = [0, 1)$. The A and B are λ -open but $A \cap B = \{0\}$ is not λ -open.

We generalize the locally closed set by using λ -open sets.

2. λ -LOCALLY CLOSED SETS

Definition 2.1. A subset A of a topological space X is said to be λ -locally closed if $A = U \cap F$, where U is λ -open and F is closed.

Corollary 2.2. Let $f : X \rightarrow Y$ be a continuous function. If L is a λ -locally closed subset of Y , then $f^{-1}(L)$ is λ -locally closed in X .

From Definition 1.1 it is obvious that every locally closed set is λ -locally closed. But the converse need not hold in general.

EXAMPLE 2.3. Let $X = \{a, b, c, d\}$, $\tau = \{\emptyset, X, \{a\}\}$. Then $C(X)$ (all closed sets in X) = $\{\emptyset, X, \{b, c, d\}\}$. And λ -open sets are: \emptyset , X , $\{a\}$, $\{a, b\}$, $\{a, b, c\}$, $\{a, c\}$,

$\{a, d\}$, $\{a, b, d\}$, $\{a, c, d\}$. Therefore, $\{d\} = \{a, d\} \cap \{b, c, d\}$ is a λ -locally closed set but it is not a locally closed set in X .

Remark 2.4. A subset A of a topological space X is λ -locally closed if and only if $X \setminus A$ is the union of a λ -closed set and an open set.

Remark 2.5. For a subset of a topological space, the following hold:

- (1) Every λ -open set is λ -locally closed,
- (2) Every closed set is λ -locally closed.

Theorem 2.6. *For a subset A of a topological space X , the following are equivalent:*

- (1) A is λ -locally closed;
- (2) $A = U \cap \text{Cl}(A)$ for some λ -open set U ;
- (3) $A \cup (X \setminus \text{Cl}(A))$ is λ -open;
- (4) $A \subseteq \text{Int}_\lambda[A \cup (X \setminus \text{Cl}(A))]$;
- (5) $\text{Cl}(A) \setminus A$ is λ -closed.

Proof. (1) \Rightarrow (2): Suppose A is λ -locally closed. Then $A = U \cap F$ where U is λ -open and F is closed. Then $\text{Cl}(A) = \text{Cl}(U \cap F) \subseteq \text{Cl}(F) = F$. Then $A \subseteq U \cap \text{Cl}(A) \subseteq U \cap F = A$ and hence $A = U \cap \text{Cl}(A)$.

(2) \Rightarrow (3): $X \setminus [A \cup (X \setminus \text{Cl}(A))] = (X \setminus A) \cap \text{Cl}(A) = \text{Cl}(A) \setminus A = \text{Cl}(A) \setminus (U \cap \text{Cl}(A)) = \text{Cl}(A) \setminus U = \text{Cl}(A) \cap (X \setminus U)$. Since U is λ -open, $\text{Cl}(A) \cap (X \setminus U)$ is λ -closed and hence $A \cup (X \setminus \text{Cl}(A))$ is λ -open.

(3) \Rightarrow (4): Since $A \cup (X \setminus \text{Cl}(A))$ is a λ -open set containing A , it is obvious that $A \subseteq \text{Int}_\lambda[A \cup (X \setminus \text{Cl}(A))]$.

(4) \Rightarrow (1): $A = A \cap \text{Cl}(A) \subseteq \text{Int}_\lambda[A \cup (X \setminus \text{Cl}(A))] \cap \text{Cl}(A) \subseteq [A \cup (X \setminus \text{Cl}(A))] \cap \text{Cl}(A) = A \cap \text{Cl}(A) = A$. Therefore, $A = \text{Int}_\lambda[A \cup (X \setminus \text{Cl}(A))] \cap \text{Cl}(A)$ and A is λ -locally closed.

(3) \Leftrightarrow (5): It is obvious. □

The union of two λ -locally closed sets need not be λ -locally closed.

EXAMPLE 2.7. Let $X = \{a, b, c, d\}$, $\tau = \{\emptyset, X, \{a, b\}, \{c, d\}\}$. Then $C(X) = \{\emptyset, X, \{c, d\}, \{a, b\}\}$ and λ -open sets are: \emptyset , X , $\{a, b\}$, $\{c, d\}$, $\{a, b, c\}$, $\{a, b, d\}$, $\{a, c, d\}$, $\{b, c, d\}$. λ -locally closed sets are: \emptyset , X , $\{a, b\}$, $\{c, d\}$, $\{a, b, c\}$, $\{a, b, d\}$, $\{a, c, d\}$, $\{b, c, d\}$, $\{c\}$, $\{d\}$, $\{a\}$, $\{b\}$. Therefore, $\{a\}$ and $\{c\}$ are λ -locally closed sets but their union $\{a, c\}$ is not a λ -locally closed set.

3. Λ_λ -CLOSED SETS

Locally closed sets in a topological space are introduced and investigated in [3] and [6]. As a generalization of locally closed sets, Arenas et al. [1] introduced the notion of λ -closed sets in a topological space. In this section, we introduce the notion of Λ_λ -closed sets which is a generalization of λ -closed sets. We obtain some characterizations of Λ_λ -closed sets and obtain decompositions of closed sets.

Definition 3.1. Let X be a topological space and A a subset of X . The subset $\Lambda_\lambda(A)$ is defined as follows: $\Lambda_\lambda(A) = \cap\{U : A \subseteq U, U \text{ is } \lambda\text{-open}\}$.

A subset A is called a Λ_λ -set if $A = \Lambda_\lambda(A)$. If U is open in Definition 3.1, then a Λ_λ -set A is called a Λ -set [9].

Lemma 3.2. For any subsets A and B of a topological space X , the following hold:

- (1) $A \subseteq \Lambda_\lambda(A)$,
- (2) If $A \subseteq B$, then $\Lambda_\lambda(A) \subseteq \Lambda_\lambda(B)$,
- (3) $\Lambda_\lambda(\Lambda_\lambda(A)) = \Lambda_\lambda(A)$,
- (4) $\Lambda_\lambda(\cap_{\alpha \in \Delta} A_\alpha) \subseteq \cap_{\alpha \in \Delta} \Lambda_\lambda(A_\alpha)$ for any index set Δ .

Lemma 3.3. For any subset A of a topological space X , the following hold:

- (1) $\Lambda_\lambda(A)$ is a Λ_λ -set,
- (2) If A is λ -open, then A is a Λ_λ -set,
- (3) If A_α is a Λ_λ -set for each $\alpha \in \Delta$, then $\cap_{\alpha \in \Delta} A_\alpha$ is a Λ_λ -set.

Remark 3.4. The converse of Lemma 3.3 (2) need not hold as shown by the following example: Let \mathbf{R} be the usual real line and $A = \{0\}$. Then A is a Λ_λ -set but it is not λ -open. Because $\{0\} \subseteq \Lambda_\lambda(\{0\}) \subseteq (-1, 0] \cap [0, 1) = \{0\}$ and hence $\Lambda_\lambda(\{0\}) = \{0\}$. Therefore, $A = \{0\}$ is a Λ_λ -set but it is not λ -open.

Definition 3.5. A subset A of a topological space X is said to be Λ_λ -closed (resp. λ -closed [1]) if $A = L \cap F$, where L is a Λ_λ -set (resp. Λ -set) and F is a closed set.

Lemma 3.6. For a subset of a topological space X , the following properties hold:

- (1) Every λ -locally closed set is Λ_λ -closed,
- (2) Every λ -closed set is Λ_λ -closed.

Proof. (1) By Lemma 3.3, every λ -open set is a Λ_λ -set and (1) holds.

(2) Let U be a Λ -set. Then,

$$U = \cap\{V : U \subseteq V, V \text{ is open}\} \supseteq \cap\{V : U \subset V, V \text{ is } \lambda\text{-open}\} \supseteq U$$

and hence U is a Λ_λ -set. Therefore, (2) holds. \square

Remark 3.7. By Lemma 3.6, we obtain the following diagram.

DIAGRAM I

$$\begin{array}{ccc} \text{locally closed} & \Rightarrow & \lambda\text{-locally closed} \\ \Downarrow & & \Downarrow \\ \lambda\text{-closed} & \Rightarrow & \Lambda_\lambda\text{-closed} \end{array}$$

Theorem 3.8. For a subset A of a topological space X , the following are equivalent:

- (1) A is Λ_λ -closed;
- (2) $A = U \cap \text{Cl}(A)$ for some Λ_λ -set U ;
- (3) $A = \Lambda_\lambda(A) \cap \text{Cl}(A)$.

Proof. (1) \Rightarrow (2): Let A be a Λ_λ -closed set. Then $A = U \cap F$, where U is a Λ_λ -set and F is a closed set. Thus, we have $A \subseteq U \cap \text{Cl}(A) \subseteq U \cap \text{Cl}(F) = U \cap F = A$. Therefore, $A = U \cap \text{Cl}(A)$.

(2) \Rightarrow (3): Let $A = U \cap \text{Cl}(A)$ for some Λ_λ -set U . Since $A \subseteq U$, by Lemma 3.2 $\Lambda_\lambda(A) \subseteq \Lambda_\lambda(U) = U$ and hence $A \subseteq \Lambda_\lambda(A) \cap \text{Cl}(A) \subseteq U \cap \text{Cl}(A) = A$. Therefore, we obtain $A = \Lambda_\lambda(A) \cap \text{Cl}(A)$.

(3) \Rightarrow (1): Let $A = \Lambda_\lambda(A) \cap \text{Cl}(A)$. By Lemma 3.3, $\Lambda_\lambda(A)$ is a Λ_λ -set and $\text{Cl}(A)$ is closed. Therefore, A is Λ_λ -closed. \square

Definition 3.9. Let X be a topological space. A subset A of X is said to be λg -closed (resp. g -closed [8]) if $\text{Cl}(A) \subseteq U$ whenever $A \subseteq U$ and U is a λ -open (resp. open) set.

Theorem 3.10. For a subset A of a topological space X , the following are equivalent:

- (1) A is closed;
- (2) A is λ -locally closed and λg -closed;
- (3) A is Λ_λ -closed and λg -closed.

Proof. (1) \Rightarrow (2): Let A be closed in X . Since $A = X \cap A$ and X is a Λ_λ -set, A is λ -locally closed. Let U be any λ -open set containing A . Then $\text{Cl}(A) = A \subseteq U$ and hence A is λg -closed.

(2) \Rightarrow (3): By Lemma 3.6, every λ -locally closed set is Λ_λ -closed.

(3) \Rightarrow (1): Let A be Λ_λ -closed and λg -closed. Since A is Λ_λ -closed, $A = P \cap L$, where P is a Λ_λ -set and L is closed in X . Let V be any λ -open set containing A . Since A is λg -closed, $\text{Cl}(A) \subseteq V$ and hence $\text{Cl}(A) \subseteq \cap\{V : A \subseteq V, V \text{ is } \lambda\text{-open}\} = \Lambda_\lambda(A)$. Therefore, $\text{Cl}(A) \subseteq \Lambda_\lambda(A) \subseteq \Lambda_\lambda(P) = P$. On the other hand, $A \subseteq L$ and $\text{Cl}(A) \subseteq \text{Cl}(L) = L$. Therefore, we obtain $\text{Cl}(A) \subseteq P \cap L = A$. Thus A is closed. \square

Theorem 3.11. Let X be a topological space. If A_α is a Λ_λ -closed set for each $\alpha \in \Delta$, then $\cap_{\alpha \in \Delta} A_\alpha$ is Λ_λ -closed.

Proof. Let A_α be a Λ_λ -closed set for each $\alpha \in \Delta$. Then $A_\alpha = U_\alpha \cap F_\alpha$, where U_α is a Λ_λ -set and F_α is a closed set for each $\alpha \in \Delta$. By Lemma 3.3, $\cap_{\alpha \in \Delta} U_\alpha$ is a Λ_λ -set, $\cap_{\alpha \in \Delta} F_\alpha$ is closed and $\cap_{\alpha \in \Delta} A_\alpha = (\cap_{\alpha \in \Delta} U_\alpha) \cap (\cap_{\alpha \in \Delta} F_\alpha)$. Therefore, $\cap_{\alpha \in \Delta} A_\alpha$ is Λ_λ -closed. \square

4. DECOMPOSITIONS OF CONTINUITY

In this section, we obtain the decompositions of continuity.

Definition 4.1. A function $f : X \rightarrow Y$ is said to be

- (1) λ -LC-continuous if $f^{-1}(V)$ is λ -locally closed in X for any closed set V of Y ,
- (2) Λ_λ -continuous if $f^{-1}(V)$ is Λ_λ -closed in X for any closed set V of Y ,
- (3) λg -continuous if $f^{-1}(V)$ is λg -closed in X for any closed set V of Y .

Theorem 4.2. For a function $f : X \rightarrow Y$, the following are equivalent:

- (1) f is continuous;
- (2) f is λ -LC-continuous and λg -continuous;
- (3) f is Λ_λ -continuous and λg -continuous.

Proof. This is an immediate consequence of Theorem 3.10 \square

Remark 4.3. The following facts are shown by Examples 4.4 and 4.5 and Remark 4.6:

- (1) λ -LC-continuity and λg -continuity are independent of each other,
- (2) Λ_λ -continuity and λg -continuity are independent of each other.

EXAMPLE 4.4. Let $X = Y = \{a, b, c, d\}$, $\tau = \sigma = \{\emptyset, X, \{a\}\}$. Then $C(X) = C(Y) = \{\emptyset, \{b, c, d\}\}$ and λ -open sets in X (resp. Y) are: \emptyset , X , $\{a\}$, $\{a, b\}$, $\{a, c\}$, $\{a, d\}$, $\{a, b, c\}$, $\{a, c, d\}$, $\{a, b, d\}$. λ -locally closed sets in X (resp. Y) are: \emptyset , X , $\{a\}$, $\{a, b\}$, $\{a, c\}$, $\{a, d\}$, $\{a, b, c\}$, $\{a, c, d\}$, $\{a, b, d\}$, $\{b, c, d\}$, $\{b, c\}$, $\{c, d\}$, $\{b, d\}$, $\{b\}$, $\{c\}$, $\{d\}$. Define a function $f : X \rightarrow Y$ by $f(a) = c$, $f(b) = b$, $f(c) = d$, $f(d) = a$. Then we have the following:

- (1) Since $f^{-1}(\{b, c, d\}) = \{a, b, c\}$, then f is not continuous.
- (2) Since $f^{-1}(\{b, c, d\}) = \{a, b, c\}$, then f is λ -LC-continuous.
- (3) Since $Cl(\{a, b, c\}) = X$ (i.e. $\{a, b, c\}$ is not λg -closed), then f is not λg -continuous.
- (4) Since $\{a, b, c\} \subseteq \cap\{U : \{a, b, c\} \subseteq U, U \text{ is } \lambda\text{-open}\} = \{a, b, c\}$ and $\{a, b, c\} = \{a, b, c\} \cap X = \{a, b, c\}$, then $\{a, b, c\}$ is Λ_λ -closed. Thus f is Λ_λ -continuous.

EXAMPLE 4.5. Let $X = Y = \{a, b, c, d\}$, $\tau = \sigma = \{\emptyset, X, \{a, b\}, \{c, d\}\}$. Then $C(X) = C(Y) = \{\emptyset, X, \{a, b\}, \{c, d\}\}$ and λ -open sets in X (resp. Y) are: \emptyset , X , $\{a, b\}$, $\{c, d\}$, $\{a, b, c\}$, $\{a, b, d\}$, $\{a, c, d\}$, $\{b, c, d\}$. And λ -locally closed sets in X (resp. Y) are: \emptyset , X , $\{a, b\}$, $\{c, d\}$, $\{a, b, c\}$, $\{a, b, d\}$, $\{a, c, d\}$, $\{b, c, d\}$, $\{a\}$, $\{b\}$, $\{c\}$, $\{d\}$. Define $g : X \rightarrow Y$ by $g(a) = c$, $g(b) = b$, $g(c) = a$, $g(d) = d$. Then we have the following:

- (1) Since $g^{-1}(\{c, d\}) = \{a, d\}$, then g is not a continuous function.
- (2) Since $g^{-1}(\{c, d\}) = \{a, d\}$, it is not a λ -locally closed set in X . Then g is not a λ -LC-continuous function.
- (3) Since $g^{-1}(\{a, b\}) = \{b, c\} \subseteq \cap\{U : \{b, c\} \subseteq U, U \text{ is } \lambda\text{-open in } X\} =$

$\{b, c\} \cap X = \{b, c\}$ and $g^{-1}(\{c, d\}) = \{a, d\} = \cap\{U : \{a, d\} \subseteq U, U \text{ is } \lambda\text{-open in } X\}$
 $= \{a, d\} \cap X = \{a, d\}$ are Λ_λ -closed, then Λ_λ -continuous.

Remark 4.6. (1) If every λg -continuous function is λ -LC-continuous, then it is continuous from Theorem 4.2. This is not true from Example 4.4(1).

(2) If every λg -continuous function is Λ_λ -continuous, then it is continuous from Theorem 4.2. This is not true from Example 4.5(1).

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