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A Shorter and Simple Approach to Study Fixed Point Results via b-Simulation Functions

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ABSTRACT. The purpose of this short note is to consider much shorter and nicer proofs about fixed point results on b-metric spaces via b-simulation function introduced very recently by Demma et al. [M. Demma, R. Saadati, P. Vetro, Fixed point results on b-metric space via Picard sequences and b-simulation functions, Iranian J. Math. Sci. Infor. 11 (1) (2016) 123-136].

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1. INTRODUCTION AND PRELIMINARIES

In 2015, Khojasteh et al. [4] gave a new approach to study fixed point results in the framework of metric spaces via simulation function as follows:

A mapping $\zeta : [0, +\infty)^2 \to \mathbb{R}$ is called a simulation function if it satisfies the following:

 $(\zeta_1) \zeta (0,0) = 0;$

 $(\zeta_2) \zeta(t,s) < s-t \text{ for all } t,s > 0;$

 (ζ_3) if $\{t_n\}, \{s_n\}$ are sequences in $(0, +\infty)$ such that $\lim_{n \to \infty} t_n = \lim_{n \to \infty} s_n > 0$, then $\overline{\lim \zeta(t_n, s_n)} < 0$.

Also, they denoted the set of all simulation functions by \mathcal{Z} .

It is worth noticing that Argoubi et al. [1] revised the above definition by withdrawing the condition (ζ_1) (also, see [7]). Also, Roldan et al. [8] revised (ζ_3) by taking $t_n < s_n$. Hence, we can say that a mapping $\zeta : [0, +\infty)^2 \to \mathbb{R}$ is called a simulation function if it satisfies:

 $(\zeta_2) \zeta(t,s) < s-t$ for all t,s > 0;

 (ζ_3) if $\{t_n\}, \{s_n\}$ are sequences in $(0, +\infty)$ such that $\lim_{n \to \infty} t_n = \lim_{n \to \infty} s_n > 0$ and $t_n < s_n$ for all $n \in \mathbb{N}$, then $\overline{\lim_{n \to \infty}} \zeta(t_n, s_n) < 0$.

For several examples of simulation functions, see [1, 2, 4, 6, 7, 8].

Definition 1.1. [4] Let (X, d) be a metric space and $\zeta \in \mathcal{Z}$. Then a mapping $T: X \to X$ is called a \mathcal{Z} -contraction with respect to ζ if the following condition is satisfied:

$$\zeta \left(d\left(Tx, Ty\right), d\left(x, y\right) \right) \ge 0 \qquad \forall x, y \in X.$$

$$(1.1)$$

Now, it is clear that $\zeta(t,t) < 0$ when t > 0; further (1.1) implies that d(Tx,Ty) < d(x,y) when $x \neq y$ for each $x,y \in X$. This means that each \mathcal{Z} -contraction with respect to ζ is continuous.

Theorem 1.2. [4] Let (X, d) be a complete metric space and $T : X \to X$ be a \mathbb{Z} -contraction with respect to ζ . Then T has a unique fixed point in X and for every $x_0 \in X$, the Picard sequence $\{x_n\}$, where $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$, converges to the fixed point of T.

One very important and significant kind of generalized (standard) metric spaces are so-called b-metric spaces (or metric type spaces). Namely, (X, d) is b-metric space if $X \neq \emptyset$ and $d : X \times X \rightarrow [0, +\infty)$ be a mapping such that for all $x, y, z \in X$ hold: $d(x, y) = 0 \Leftrightarrow x = y; d(x, y) = d(y, x)$ and $d(x, y) \leq b(d(x, y) + d(y, z))$ for $b \geq 1$. Then d is called b-metric. For more details on b-metric spaces, see [2, 3, 5] and the references contained therein.

Recently, Demma et al. [2] introduced the b-simulation function in the framework of b-metric spaces as follows.

Definition 1.3. Let (X, d) be a b-metric space. A b-simulation function is a function $\zeta : [0, +\infty)^2 \to \mathbb{R}$ satisfying the following:

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$$(\xi_1) \ \xi \ (t,s) < s-t \text{ for all } t,s > 0;$$

 $(\xi_2) \text{ if } \{t_n\}, \{s_n\} \text{ are sequences in } (0,+\infty) \text{ such that }$

$$0 < \lim_{n \to +\infty} t_n \le \underline{\lim}_{n \to +\infty} s_n \le \overline{\lim}_{n \to \infty} s_n \le b \lim_{n \to +\infty} t_n < +\infty,$$
(1.2)

then $\overline{\lim_{n \to \infty}} \xi(bt_n, s_n) < 0.$

It is clear if b = 1, then b-simulation function is in the fact the simulation function in the framework of (standard) metric spaces.

EXAMPLE 1.4. [2] Let $\xi : [0, +\infty)^2 \to \mathbb{R}$ be defined by

(i) $\xi(t,s) = \lambda s - t$ for all $t, s \in [0, +\infty)$, where $\lambda \in [0, 1)$.

(ii) $\xi(t,s) = \psi(s) - \varphi(t)$ for all $t,s \in [0,+\infty)$, where $\varphi, \psi: [0,+\infty) \rightarrow [0,+\infty)$ are two continuous functions such that $\psi(t) = \varphi(t) = 0$ if and only if t = 0 and $\psi(t) < t \leq \varphi(t)$ for all t > 0.

(iii) $\xi(t,s) = s - \frac{f(t,s)}{g(t,s)}t$ for all $t,s \in [0,+\infty)$, where $f,g : [0,+\infty)^2 \to (0,+\infty)$ are two continuous functions with respect to each variable such that f(t,s) > g(t,s) for all t,s > 0.

(iv) $\xi(t,s) = s - \varphi(s) - t$ for all $t, s \in [0, +\infty)$, where $\varphi: [0, +\infty) \to [0, +\infty)$ is a lower semi-continuous function such that $\varphi(t) = 0$ if and only if t = 0.

(v) $\xi(t,s) = s\varphi(s) - t$ for all $t, s \in [0, +\infty)$, where $\varphi : [0, +\infty) \to [0, 1)$ is such that $\lim_{t \to \infty} \varphi(t) < 1$ for all r > 0.

Each of the function considered in (i)-(v) is a b-simulation function.

The following important and very interesting results are proved in [2].

Lemma 1.5. Let (X,d) be a b-metric space and $f: X \to X$ be a mapping. Suppose that there exists a b-simulation function ξ such that following condition holds.

$$\xi \left(bd \left(fx, fy \right), d \left(x, y \right) \right) \ge 0 \qquad \forall x, y \in X.$$
(1.3)

Let $\{x_n\}$ be a sequence of Picard of initial at point $x_0 \in X$ and $x_{n-1} \neq x_n$ for all $n \in \mathbb{N}$. Then

$$\lim_{n \to \infty} d\left(x_{n-1}, x_n\right) = 0.$$

Lemma 1.6. Let (X, d) be a b-metric space and $f : X \to X$ be a mapping. Suppose that there exists a b-simulation function ξ such that (1.3) holds. Let $\{x_n\}$ be a sequence of Picard of initial at point $x_0 \in X$ and $x_{n-1} \neq x_n$ for all $n \in \mathbb{N}$. Then $\{x_n\}$ is a bounded sequence.

Lemma 1.7. Let (X,d) be a b-metric space and $f: X \to X$ be a mapping. Suppose that there exists a b-simulation function ξ such that (1.3) holds. Let $\{x_n\}$ be a sequence of Picard of initial at point $x_0 \in X$ and $x_{n-1} \neq x_n$ for all $n \in \mathbb{N}$. Then $\{x_n\}$ is a Cauchy sequence. **Theorem 1.8.** Let (X, d) be a complete b-metric space and let $f : X \to X$ be a mapping. Suppose that there exists a b-simulation function ξ such that (1.3) holds; that is,

$$\xi \left(bd \left(fx, fy \right), d \left(x, y \right) \right) \ge 0 \qquad \forall x, y \in X.$$

Then f has a unique fixed point.

For the proof of Theorem 1.8, Demma et al. [2] used Lemmas 1.5-1.7.

2. Main results

In this section we improve the main result from [2]; that is, we prove Theorem 1.8 without using all three lemmas 1.5-1.7. At the first, we quote some well known results from b-metric spaces. The following lemma was used (and proved) in the course of proofs of several fixed point results in the framework of b-metric spaces in [3].

Lemma 2.1. Let $\{y_n\}$ be a sequence in a b-metric space (X, d) such that

$$d(y_n, y_{n+1}) \le \lambda d(y_{n-1}, y_n) \tag{2.1}$$

for some λ , $0 \leq \lambda < \frac{1}{b}$ and each $n = 1, 2, \cdots$. Then $\{y_n\}$ is a Cauchy sequence in (X, d).

By utilizing Lemma 2.1, Jovanović et al. [3] proved following result.

Theorem 2.2. Let (X, d) be a complete b-metric space and $f : X \to X$ be a map such that

$$d\left(fx, fy\right) \le \lambda d\left(x, y\right) \tag{2.2}$$

holds for all $x, y \in X$, where $0 \le \lambda < \frac{1}{b}$. Then f has a unique fixed point z and for every $x_0 \in X$, the sequence $\{f^n x_0\}$ converges to z.

Now we formulate and prove Theorem 1.8 via a shorter and simple approach.

Theorem 2.3. Let (X, d) be a complete b-metric space and $f : X \to X$ be a mapping. Suppose that there exists a b-simulation function ξ such that (1.3) holds; that is,

$$\xi \left(bd \left(fx, fy \right), d \left(x, y \right) \right) \ge 0 \qquad \forall x, y \in X.$$

$$(2.3)$$

Then f has a unique fixed point.

Proof. It is enough clear that (2.3) implies

$$bd(fx, fy) \le d(x, y) \qquad \forall x, y \in X.$$

$$(2.4)$$

Indeed, (2.4) holds if x = y. In the case that $x \neq y$ there are two possibilities, either fx = fy or $fx \neq fy$. In the first case we have that $b \cdot d(fx, fy) = 0 < d(x, y)$, while in second case the result follows from (ξ_1) . This means that (2.3) implies (2.4) for all $x, y \in X$. Further, obviously, (2.4) implies that

$$d\left(f^{2}x, f^{2}y\right) \leq \frac{1}{b^{2}}d\left(x, y\right) = \lambda d\left(x, y\right).$$

$$(2.5)$$

Since $\lambda = \frac{1}{b^2} \in [0, \frac{1}{b})$, then according to Theorem 2.2, f^2 has a unique fixed point (say z) in X. This further means that f has a unique fixed point z in X. Now, the proof of this theorem is complete.

Obviously, our proof is much shorter than the corresponding ones from Demma et al.'s work [2]. It is very interesting that all four Corollaries 4.1-4.4 from [2] follows immediately according to our easy approach. Thus we have following corollary.

Corollary 2.4. Let (X,d) be a complete b-metric space and let $f: X \to X$ be a mapping. Suppose that

- (i) $\lambda \in [0,1)$ such that $bd(fx, fy) \leq \lambda d(x,y)$;
- (ii) a lower semi-continuous function $\varphi : [0, +\infty) \to [0, \infty)$ with $\varphi^{-1}(0) = \{0\}$ such that $bd(fx, fy) \leq d(x, y) \varphi(d(x, y));$
- (iii) φ : $[0, +\infty) \rightarrow [0, 1)$ with $\lim_{t \to r^+} \varphi(t) < 1$ for all r > 0 such that $bd(fx, fy) \leq \varphi(d(x, y)) d(x, y);$
- (iv) $\eta : [0, +\infty) \to [0, \infty)$ with $\eta(t) < t$ for all t > 0 and $\eta(0) = 0$ such that $bd(fx, fy) \le \eta(d(x, y))$

for all $x, y \in X$. Then f has a unique fixed point in each one of above condition.

Proof. Obviously, each one of mentioned conditions implies the condition (2.4) by selecting the appropriate b-simulation function in Example 1.4. Hence, we obtain that $bd(fx, fy) \leq d(x, y)$ for all $x, y \in X$. The result then follows according to Theorem 2.3.

EXAMPLE 2.5. Now, we consider Example 4.5 from [2]. Let X = [0, 1] and $d: X \times X \to \mathbb{R}$ be defined by $d(x, y) = |x - y|^2$. Then (X, d) is a complete b-metric space with b = 2. Consider a mapping $f: X \to X$ by

$$fx = \frac{ax}{1+x}$$

for all $x \in X$, where $a \in [0, \frac{1}{\sqrt{2}}]$. Now, we have

$$2d(fx, fy) = 2\left|\frac{ax}{1+x} - \frac{ay}{1+y}\right|^2 = 2a^2 \frac{|x-y|^2}{(1+x)^2(1+y)^2} \le |x-y|^2 = d(x,y)$$
(2.6)

for all $x, y \in X$. Further, (2.6) implies that

$$d\left(f^{2}x, f^{2}y\right) \leq \frac{1}{4}d\left(x, y\right)$$

that is, f^2 has a unique fixed point according to Theorem 2.2. This means that f has a unique fixed point. Here it is z = 0.

The next result is probably known, but our proof is very condensed.

Theorem 2.6. Let (X, d) be a complete b-metric space and let $f : X \to X$ be a mapping such that

$$d\left(fx, fy\right) \le \lambda d\left(x, y\right) \tag{2.7}$$

for all $x, y \in X$, where $\lambda \in [0, 1)$. Then f has a unique fixed point (say z) in X and for $x_0 \in X$ the sequence $\{f^n x_0\}_{n \in \mathbb{N}}$ converges to z.

Proof. The condition (2.7) implies that

$$d(f^n x, f^n y) \le \lambda d(f^{n-1} x, f^{n-1} y) \le \dots \le \lambda^n d(x, y)$$

for all $x, y \in X$ and $n \in \mathbb{N}$. Since $\lambda^n \to 0$ as $n \to \infty$, there is $k \in \mathbb{N}$ such that $\lambda^k < \frac{1}{k}$. Therefore, we have

$$d(f^{k+1}x, f^{k+1}y) \le \frac{1}{b^2}d(x, y).$$

The result now follows by Theorem 2.2.

Question 1. Does Theorem 2.3 holds if $\xi(d(fx, fy), d(x, y)) \ge 0$ for all $x, y \in X$, where (X, d) is a given complete b-metric space and $f: X \to X$ be a mapping and ξ a given b-simulation function?

Question 2. Can you obtain this results by considering ordered b-metric spaces or cone b-metric spaces instead of b-metric spaces?

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