

**On the 2-Adjointable Operators and Superstability of them  
between 2-Pre Hilbert  $C^*$ -Module Spaces**

Maryam Ramezani<sup>\*,a</sup>, Hamid Baghani<sup>b</sup>

<sup>a</sup>Department of Mathematics, University of Bojnord, Bojnord, Iran.

<sup>b</sup>Department of Mathematics, University of Sistan and Baluchestan,  
Zahedan, Iran.

E-mail: m.ramezani@ub.ac.ir

E-mail: h.baghani@gmail.com

ABSTRACT. In this paper, first, we introduce the new concept of 2-inner product on Banach modules over a  $C^*$ -algebra. Next, we present the concept of 2-linear operators over a  $C^*$ -algebra. Our result improve the main result of the paper [Z. Lewandowska, *On 2-normed sets*, *Glasnik Mat.*, **38(58)** (2003), 99-110]. In the end of this paper, we define the notions 2-adjointable mappings between 2-pre Hilbert  $C^*$ -modules and prove superstability of them in the spirit of Hyers-Ulam-Rassias.

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1. INTRODUCTION

The concept of 2-inner product has been intensively studied by many authors in the last three decades. The basic definitions and elementary properties of 2-inner product spaces can be found in [1] and [2].

Recently, M.Frank and e.t. defined the notion  $\phi$ -perturbation of an adjointable mapping and proved the superstability of an adjointable mapping on Hilbert  $C^*$ -modules(see [3]).

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\*Corresponding Author

In this paper, first, we introduce the definition 2-pre Hilbert  $C^*$ -module spaces and give several important properties. Next, we present the concept of 2-linear operators over a  $C^*$ -algebra which coincides with Lewandowska's definition (see [4, 5]). Also, we define 2-adjoinable mappings between 2-pre Hilbert  $C^*$ -modules and prove an analogue of  $\phi$ -perturbation of adjointable mappings in paper([3]).

We refer the interested reader to monographs [6, 7, 8, 9] and references therein for more information.

## 2. 2-PRE HILBERT MODULES

Let  $X$  be a left module over a  $C^*$ -algebra  $A$ . An action of  $a \in A$  on  $X$  is denoted by  $a.x \in X$ ,  $x \in X$ .

**Definition 2.1.** A 2-pre Hilbert  $A$ -module is a left  $A$ -module  $X$  equipped with  $A$ -valued function defined on  $X \times X \times X$  satisfying the following conditions:

$I_1$ )  $(x, x|z)$  is a positive element in  $A$  for any  $x, z \in X$  and  $(x, x|z) = 0$  if and only if  $x$  and  $z$  are linearly dependent;

$I_2$ )  $(x, x|z) = (z, z|x)$  for any  $x, z \in X$ ;

$I_3$ )  $(y, x|z) = (x, y|z)^*$  for any  $x, y, z \in X$ ;

$I_4$ )  $(\alpha x + x', y|z) = \alpha(x, y|z) + (x', y|z)$  for any  $\alpha \in \mathbb{C}$  and  $x, x', y, z \in X$ ;

$I_5$ )  $(ax, y|z) = a(x, y|z)$  for any  $x, y, z \in X$  and any  $a \in A$ .

The map  $(\cdot, \cdot|\cdot)$  is called  $A$ -valued 2-inner product and  $(X, (\cdot, \cdot|\cdot))$  is called 2-pre Hilbert  $C^*$ -module space.

EXAMPLE 2.2. Every 2-inner product space is a 2-pre Hilbert  $\mathbb{C}$ -module.

EXAMPLE 2.3. Let  $A$  be a  $C^*$ -algebra and  $J \subseteq A$  be a left ideal. Then  $J$  can be equipped with the structure of 2-pre Hilbert  $A$ -module with  $A$ -valued inner product  $(x, y|z) := xy^*zz^* - xz^*zy^*$  for any  $x, y, z \in A$ .

**Definition 2.4.** Let  $X$  be a 2-pre Hilbert  $A$ -module. we can define a function  $\|\cdot\|_X$  on  $X \times X$  by  $\|x|z\|_X = \|(x, x|z)\|_A^{\frac{1}{2}}$  for all  $x, z \in X$ .

**Lemma 2.5.**  $\|\cdot\|_X$  satisfies the following conditions:

N1)  $\|ax|z\|_X \leq \|a\| \|x|z\|_X$  for any  $x, z \in X$  and  $a \in A$ ;

N2)  $(x, y|z)(y, x|z) \leq \|y|z\|_X^2 (x, x|z)$  for any  $x, y, z \in X$ ;

N3)  $\|(x, y|z)\|^2 \leq \|(x, x|z)\| \|(y, y|z)\|$

*Proof.* N1 is obvious; N3 follows from N2, so let us prove N2.

Let  $\phi$  be a positive linear functional on  $A$ . Then  $\phi((\cdot, \cdot|\cdot))$  is usual 2-inner product on  $X$ . Applying the Schwartz inequality for 2-inner product (see [2],

page 3) we obtain for all  $x, y, z \in X$ ,

$$\begin{aligned} \phi((x, y|z) (y, x|z)) &= \phi((x, y|z)y, x |z)) \\ &\leq \phi((x, x|z))^{\frac{1}{2}} \phi(((x, y|z)y, (x, y|z)y |z))^{\frac{1}{2}} \\ &\leq \phi((x, x|z))^{\frac{1}{2}} \phi((x, y|z) (y, y|z) (x, y|z)^*)^{\frac{1}{2}} \\ &\leq \phi((x, x|z))^{\frac{1}{2}} \|(y, y|z)\|^{\frac{1}{2}} \phi((x, y|z) (y, x|z))^{\frac{1}{2}}. \end{aligned}$$

Thus, for any positive linear functional  $\phi$ , we have

$$\phi((x, y|z) (y, x|z)) \leq \|y|z\|_X^2 \phi((x, x|z))$$

hence

$$(x, y|z) (y, x|z) \leq \|y|z\|_X^2 (x, x|z).$$

□

**Theorem 2.6.** *The function  $\|\cdot\|_X$  is a 2-norm on  $X$ .*

*Proof.* Now, we verify that  $\|\cdot\|_X$  satisfies the following properties of 2-norms:

1)  $I_3$  and  $I_4$  show that  $\|\alpha x|y\|_X = \|(\alpha x, \alpha x|y)\|^{\frac{1}{2}} = |\alpha| \|x|y\|_X$  for all  $x, y \in X$  and  $\alpha \in \mathbb{C}$ .

2)  $I_1$  follows that  $\|x|y\|_X = 0$  if and only if  $x$  and  $y$  are linearly dependent for all  $x, y \in X$ .

3) it follows from  $I_2$  that  $\|x|y\|_X = \|(x, x|y)\|^{\frac{1}{2}} = \|y|x\|_X$  for all  $x, y \in X$ .

4) By proposition 2.5 ( $N3$ ), we have

$$\begin{aligned} \|x + x'|y\|_X^2 &= \|(x + x', x + x'|y)\| = \|(x, x|y) + (x', x'|y) + (x, x'|y) + (x', x'|y)\| \\ &\leq \|(x, x|y)\| + 2\|(x, x'|y)\| + \|(x', x'|y)\| \\ &\leq (\|(x, x|y)\|^{\frac{1}{2}} + \|(x', x'|y)\|^{\frac{1}{2}})^2 = (\|x|y\|_X + \|x'|y\|_X)^2 \end{aligned}$$

for all  $x, x', y \in X$ . This show that  $(X, \|\cdot\|_X)$  is a 2-normed space. □

### 3. 2-ADJOINTABLE MAPPINGS

In continue, we let  $A$  be a  $C^*$ -algebra. Now, we start with following definition.

**Definition 3.1.** Let  $X$  and  $Y$  be two 2-pre Hilbert  $A$ -modules. An operator  $f : X \times X \rightarrow Y$  is said to be  $A$ -2 linear if it satisfies the following conditions:

- 1)  $f(x + y, z + w) = f(x, z) + f(x, w) + f(y, z) + f(y, w)$  for all  $x, y, z, w \in X$ ;
- 2)  $f(\alpha x, \beta y) = \alpha \bar{\beta} f(x, y)$  for all  $\alpha, \beta \in \mathbb{C}$  and  $x, y \in X$ ;
- 3)  $f(ax, by) = a \cdot b^* \cdot f(x, y)$  for all  $x, y \in X$  and  $a, b \in A$ .

**EXAMPLE 3.2.** Let  $X$  be a 2-pre Hilbert  $A$ -module and  $z \in X$ . Define  $f : X \times X \rightarrow A$  by  $f(x, y) = (x, y|z)$ . Then  $f$  is a  $A$ - 2 linear operator.

**Definition 3.3.** Let  $X$  and  $Y$  be two 2-pre Hilbert  $A$ -modules. A mapping  $f : X \times X \rightarrow Y$  is called 2-adjointable if there exists a mapping  $g : Y \times Y \rightarrow X$  such that

$$(f(x, y), s | t) = (x, y | g(s, t)) \quad (3.1)$$

for all  $x, y \in X$  and  $s, t \in Y$ . The mapping  $g$  is denoted by  $f^*$  and is called the 2-adjointable of  $f$ .

**Lemma 3.4.** Let  $X$  be a 2-pre Hilbert  $A$ -module and  $\dim(X) > 1$ . If  $(x, y|z) = 0$  for all  $y, z \in X$ , then  $x = 0$ .

*Proof.* Suppose  $x \neq 0$ . Let  $x$  and  $y$  be linearly independent. Then by hypothesis  $(x, x|y) = 0$  and this is contradiction.  $\square$

**Lemma 3.5.** Every 2-adjonable mapping is  $A$ -2 linear.

*Proof.* Let  $f : X \times X \rightarrow Y$  be a 2-adjonable mapping. Then there exists a mapping  $g : Y \times Y \rightarrow X$  such that (3.1) holds. For every  $x, y, z, w \in X$ , every  $s, t \in Y$ , every  $\alpha, \beta \in \mathbb{C}$ , every  $a, b \in A$ , we have

$$\begin{aligned} (f(\alpha ax + y, \beta bz + w), s | t) &= (\alpha ax + y, \beta bz + w | g(s, t)) \\ &= \alpha \bar{\beta} ab^* (x, z | g(s, t)) + \alpha a (x, w | g(s, t)) + \bar{\beta} b^* (y, z | g(s, t)) + (y, w | g(s, t)) \\ &= \alpha \bar{\beta} ab^* (f(x, z), s | t) + \alpha a (f(x, w), s | t) + \bar{\beta} b^* (f(y, z), s | t) + (f(y, w), s | t) \\ &= (\alpha \bar{\beta} ab^* f(x, z) + \alpha a f(x, w) + \bar{\beta} b^* f(y, z) + f(y, w), s | t). \end{aligned}$$

It follows from lemma 3.4 that  $f$  is  $A$ -2 linear.  $\square$

#### 4. SUPERSTABILITY OF 2-ADJOINABLE MAPPINGS

In this section,  $X$  and  $Y$  denote 2-pre Hilbert  $A$ -modules and  $\dim(X) > 1$ ,  $\dim(Y) > 1$  and  $\phi : X^2 \times Y^2 \rightarrow [0, \infty)$  is a function. We start our work with following definition.

**Definition 4.1.** A (not necessarily  $A$ -2 linear) mapping  $f : X \times X \rightarrow Y$  is called

$\phi$ -perturbation of an 2-adjointable mapping if there exists a mapping (not necessarily  $A$ -2-linear)  $g : Y \times Y \rightarrow X$  such that

$$\|(f(x, y), s | t) - (x, y | g(s, t))\| \leq \phi(x, y, s, t) \quad (4.1)$$

for all  $x, y \in X$  and  $s, t \in Y$ .

**Theorem 4.2.** Let  $f : X \times X \rightarrow Y$  be a  $\phi$ -perturbation of a 2-adjointable mapping with corresponding mapping  $g : Y \times Y \rightarrow X$ . Suppose for some sequence  $c_n$  of non-zero complex numbers the following conditions hold:

$$\lim_{n \rightarrow \infty} |c_n|^{-1} \phi(c_n x, y, s, t) = 0 \quad (x, y \in X, s, t \in Y) \quad (4.2)$$

$$\lim_{n \rightarrow \infty} |c_n|^{-1} \phi(x, y, c_n s, t) = 0 \quad (x, y \in X, s, t \in Y) \quad (4.3)$$

Then  $f$  is 2-adjointable and hence  $f$  is A-2 linear.

*Proof.* Let  $\lambda \in \mathbb{C}$  be an arbitrary number. Putting  $\lambda x$  instead  $x$  in (4.1), we get

$$\|(f(\lambda x, y), s | t) - (\lambda x, y | g(s, t))\| \leq \phi(\lambda x, y, s, t)$$

multiplication of (4.1) by  $|\lambda|$ , we have

$$\|(\lambda f(x, y), s | t) - \lambda(x, y | g(s, t))\| \leq |\lambda| \phi(x, y, s, t)$$

Thus,

$$\|(f(\lambda x, y), s | t) - (\lambda f(x, y), s | t)\| \leq \phi(\lambda x, y, s, t) + |\lambda| \phi(x, y, s, t) \quad (4.4)$$

Replacing  $c_n s$  by  $s$  in (4.4), we get

$$\|f(\lambda x, y), s | t) - (\lambda f(x, y), s | t)\| \leq |c_n|^{-1} \phi(\lambda x, y, c_n s, t) + |\lambda| |c_n|^{-1} \phi(x, y, c_n s, t)$$

hence, as  $n \rightarrow \infty$ , applying (4.3) we obtain

$$(f(\lambda x, y), s | t) - (\lambda f(x, y), s | t) = 0 \quad (\lambda \in \mathbb{C}, x, y \in X, s, t \in Y).$$

It follows from proposition 3.4 that

$$f(\lambda x, y) = \lambda f(x, y) \quad (\lambda \in \mathbb{C}, x, y \in X) \quad (4.5)$$

Now, we take  $c_n x$  instead  $x$  in (4.1) to get

$$\|(f(c_n x, y), s | t) - (c_n x, y | g(s, t))\| \leq \phi(c_n x, y, s, t).$$

It follows from (4.5) that

$$\|(f(x, y), s | t) - (x, y | g(s, t))\| \leq |c_n|^{-1} \phi(c_n x, y, s, t)$$

hence, as  $n \rightarrow \infty$ , applying (4.2) we get

$$(f(x, y), s | t) = (x, y | g(s, t)) \quad (x, y \in X, s, t \in Y).$$

Therefore  $f$  is 2-adjointable and by Lemma 3.5,  $f$  is A-2 linear.  $\square$

In the following, we let  $c_n = a^n$  that  $a > 1$ . we get the following results.

**Corollary 4.3.** *If  $f : X \times X \rightarrow Y$  is a  $\phi$ -perturbation of a 2-adjointable mapping, where*

$\phi(x, y, s, t) = \epsilon \|x|y\|_X^p \|s|t\|_Y^q$  ( $\epsilon \geq 0$ ,  $0 < p < 1$ ,  $0 < q < 1$ ), then  $f$  is 2-adjointable and hence  $f$  is A-2-linear.

**Corollary 4.4.** *If  $f : X \times X \rightarrow Y$  is a  $\phi$ -perturbation of a 2-adjointable mapping, where*

$\phi(x, y, s, t) = \epsilon_1 \|x|y\|_X^p + \epsilon_2 \|s|t\|_Y^q$  ( $\epsilon_1 \geq 0$ ,  $\epsilon_2 \geq 0$ ,  $0 < p < 1$ ,  $0 < q < 1$ ). Then  $f$  is 2-adjointable and hence  $f$  is A-2 linear.

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## REFERENCES

1. Y.J. Cho, P.C.S. Lin, S.S. Kim, A. Misiak, *Theory of 2-Inner Product Spaces*, Nova Science Publishers, Inc., New York, 2001.
2. S.S. Dragomir, Y.J. Cho, S.S. Kim, A. Sofo, Some Boas-Bellman Type Inequalities in 2-Inner Product Space, *JIPAM*, **6**(2),(2005), article 55.
3. M. Frank, P. Găvruta, M.S. Moslehian, Superstability of Adjointable Mappings on Hilbert  $C^*$ - Modules, *Appl. Anal. Discrete Math.*, **No3**, (2009), 39-45.
4. Z. Lewandowska, On 2-Normed Sets, *Glasnik Mat.*, **38**(58), (2003), 99-110.
5. Z. Lewandowska, Bounded 2-Linear Operators on 2-Normed Sets, *Glas. Mat. Ser. III*, **39**(59)(2), (2004), 301-312.
6. A. Ashyani, H. Mohammadinejad, O. RabieiMotlagh, Stability Analysis of Mathematical Model of Virus Therapy for Cancer, *Iranian Journal of Mathematical Sciences and Informatics*, **11**( 2), (2016), 97-110.
7. H. Sadeghi, Generalized Approximate Amenability of Direct Sum of Banach Algebras, *Iranian Journal of Mathematical Sciences and Informatics*, **13**(1), (2018), 75-87.
8. M.E. Gordji, M. Ramezani, Approximate inner products on Hilbert  $C^*$ -modules; A Fixed Point Approach, *Operators and Matrices*, **6**(4), (2012), 757-766.
9. M.E. Gordji, M. Ramezani, Y.J. Cho, H. Baghani, Approximate Lie brackets: A Fixed Point Approach, *Journal of Inequalities and Applications*, (2012), doi:10.1186/1029-242X-2012-125.