On the 2-Adjointable Operators and Superstability of them between 2-Pre Hilbert C^*-Module Spaces

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ABSTRACT. In this paper, first, we introduce the new concept of 2-inner product on Banach modules over a C^*-algebra. Next, we present the concept of 2-linear operators over a C^*-algebra. Our result improve the main result of the paper [Z. Lewandowska, *On 2-normed sets*, Glasnik Mat., **38(58)** (2003), 99-110]. In the end of this paper, we define the notions 2-adjointable mappings between 2-pre Hilbert C*-modules and prove superstability of them in the spirit of Hyers-Ulam-Rassias.

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1. INTRODUCTION

The concept of 2-inner product has been intensively studied by many authors in the last three decades. The basic definitions and elementary properties of 2-inner product spaces can be found in [1] and [2].

Recently, M.Frank and e.t. defined the notion ϕ-perturbation of an adjointable mapping and proved the superstability of an adjointable mapping on Hilbert C^*-modules(see [3]).

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121

In this paper, first, we introduce the definition 2-pre Hibert C^*-module spaces and give several important properties. Next, we present the concept of 2-linear operators over a C^*-algebra which coincides with Lewandowska's definition (see [4, 5]). Also, we define 2-adjoinable mappings between 2-pre Hilbert C^*-modules and prove an analogue of ϕ-perturbation of adjoitable mappings in paper([3]).

We refer the interested reader to monographs [6, 7, 8, 9] and references therein for more information.

2. 2-Pre Hilbert Modules

Let X be a left module over a C^*-algebra A. An action of $a \in A$ on X is denoted by $a.x \in X$, $x \in X$.

Definition 2.1. A 2-pre Hilbert A-module is a left A-module X equipped with A-valued function defined on $X \times X \times X$ satisfing the following conditions:

I_1) $(x,x|z)$ is a positive element in A for any $x, z \in X$ and $(x,x|z) = 0$ if and only if x and z are linearly dependent;

I_2) $(x,x|z) = (z,z|x)$ for any $x, z \in X$;

I_3) $(y,x|z) = (x,y|z)^*$ for any $x, y, z \in X$;

I_4) $(\alpha x + x', y|z) = \alpha(x,y|z) + (x',y|z)$ for any $\alpha \in \mathbb{C}$ and $x, x', y, z \in X$;

I_5) $(ax,y|z) = a(x,y|z)$ for any $x, y, z \in X$ and any $a \in A$.

The map $(.,.|.)$ is called A-valued 2-inner product and $(X, (.,.|.))$ is called 2-pre Hilbert C^*-module space.

EXAMPLE 2.2. Every 2-inner product space is a 2-pre Hilbert $\mathbb{C}$-module.

EXAMPLE 2.3. Let A be a C^*-algebra and $J \subseteq A$ be a left ideal. Then J can be equipped with the structure of 2-pre Hilbert A-module with A-valued inner product $(x,y|z) := xy^*zz^* - xz^*zy^*$ for any $x, y, z \in A$.

Definition 2.4. Let X be a 2-pre Hilbert A-module. we can define a function $||.|.||_X$ on $X \times X$ by $|||x|z||_X = ||(x,x|z)||^{\frac{1}{2}}$ for all $x, z \in X$.

Lemma 2.5. $||.|.||_X$ *satisfies the following conditions:*

$N1$) $||ax|z||_X \leq ||a|| \; |||x|z||_X$ *for any* $x, z \in X$ *and* $a \in A$;

$N2$) $(x,y|z) \; (y,x|z) \leq ||y|z||_X^2 \; (x,x|z)$ *for any* $x, y, z \in X$;

$N3$) $||(x,y|z)||^2 \leq ||(x,x|z)|| \; ||(y,y|z)||$

Proof. $N1$ is obvious; $N3$ follows from $N2$, so let us prove $N2$.
Let ϕ be a positive linear functional on A. Then $\phi((.,.|.))$ is usual 2-inner product on X. Applying the Schwartz inequality for 2-inner product (see [2],

page 3) we obtain for all $x, y, z \in X$,

$$\phi((x, y|z) \ (y, x|z)) = \phi((x, y|z)y, x \ |z))$$

$$\leq \phi((x, x|z))^{\frac{1}{2}} \ \phi(((x, y|z)y \ , (x, y|z)y \ |z))^{\frac{1}{2}}$$

$$\leq \phi((x, x|z))^{\frac{1}{2}} \ \phi((x, y|z) \ (y, y|z) \ (x, y|z)^*)^{\frac{1}{2}}$$

$$\leq \phi((x, x|z))^{\frac{1}{2}} \ ||(y, y|z)||^{\frac{1}{2}} \ \phi((x, y|z) \ (y, x|z))^{\frac{1}{2}}.$$

Thus, for any positive linear functional ϕ, we have

$$\phi((x, y|z) \ (y, x|z)) \leq ||y|z||_X^2 \ \phi((x, x|z))$$

hence

$$(x, y|z) \ (y, x|z) \leq ||y|z||_X^2 \ (x, x|z).$$

$\square$

Theorem 2.6. *The function* $||.|.||_X$ *is a 2-norm on* X.

Proof. Now, we verify that $||.|.||_X$ satisfies the following properties of 2-norms:

1) I_3 and I_4 show that $||\alpha x|y||_X = ||(\alpha x, \alpha x|y)||^{\frac{1}{2}} = |\alpha| \ ||x|y||_X$ for all $x, y \in X$ and $\alpha \in \mathbb{C}$.

2) I_1 follows that $||x|y||_X = 0$ if and only if x and y are linearly dependent for all $x, y \in X$.

3) it follows from I_2 that $||x|y||_X = ||(x, x|y)||^{\frac{1}{2}} = ||y|x||_X$ for all $x, y \in X$.

4) By proposition 2.5 $(N3)$, we have

$$||x + x'|y||_X^2 = ||(x + x', x + x'|y)|| = ||(x, x|y) + (x', x|y + (x, x'|y) + (x', x'|y))||$$

$$\leq ||(x, x|y)|| + 2||(x, x'|y)|| + ||(x', x'|y)||$$

$$\leq (||(x, x|y)||^{\frac{1}{2}} + ||(x', x'|y)||^{\frac{1}{2}})^2 = (||x|y||_X + ||x'|y||_X)^2$$

for all $x, x', y \in X$. This show that $(X, ||.|.||_X)$ is a 2-normed space. $\square$

3. 2-ADJOINTABLE MAPPINGS

In continue, we let A be a C^*-algebra. Now, we start with following definition.

Definition 3.1. Let X and Y be two 2-pre Hilbert A-modules. An operator $f : X \times X \to Y$ is said to be A-2 linear if it satisfies the following conditions:

1) $f(x + y, z + w) = f(x, z) + f(x, w) + f(y, z) + f(y, w)$ for all $x, y, z, w \in X$;

2) $f(\alpha x, \beta y) = \alpha \ \overline{\beta} \ f(x, y)$ for all $\alpha, \beta \in \mathbb{C}$ and $x, y \in X$;

3) $f(ax, by) = a. \ b^*.f(x, y)$ for all $x, y \in X$ and $a, b \in A$.

EXAMPLE 3.2. Let X be a 2-pre Hilbert A-module and $z \in X$. Define $f : X \times X \to A$ by $f(x, y) = (x, y|z)$. Then f is a A- 2 linear operator.

Definition 3.3. Let X and Y be two 2-pre Hilbert A-modules. A mapping $f : X \times X \to Y$ is called 2-adjointable if there exists a mapping $g : Y \times Y \to X$ such that

$$(f(x,y), s \mid t) = (x, y \mid g(s,t)) \tag{3.1}$$

for all $x, y \in X$ and $s, t \in Y$. The mapping g is denoted by f^* and is called the 2-adjointable of f.

Lemma 3.4. *Let X be a 2-pre Hilbert A-module and $\dim(X) > 1$. If $(x, y|z) = 0$ for all $y, z \in X$, then $x = 0$.*

Proof. Suppose $x \neq 0$. Let x and y be linearly independent. Then by hypothesis $(x, x, |y) = 0$ and this is contradiction. $\square$

Lemma 3.5. *Every 2-adjonable mapping is A- 2 linear.*

Proof. Let $f : X \times X \to Y$ be a 2-adjoinable mapping. Then there exists a mapping $g : Y \times Y \to X$ such that (3.1) holds. For every $x, y, z, w \in X$, every $s, t \in Y$, every $\alpha, \beta \in \mathbb{C}$, every $a, b \in A$, we have

$$(f(\alpha a x + y, \beta b z + w), s \mid t) = (\alpha a x + y, \beta b z + w \mid g(s,t))$$
$$= \alpha \overline{\beta} a b^* (x, z \mid g(s,t)) + \alpha a (x, w \mid g(s,t)) + \overline{\beta} b^* (y, z \mid g(s,t)) + (y, w \mid g(s,t))$$
$$= \alpha \overline{\beta} a b^* (f(x,z), s \mid t) + \alpha a (f(x,w), s \mid t) + \overline{\beta} b^* (f(y,z), s \mid t) + (f(y,w), s \mid t)$$
$$= (\alpha \overline{\beta} a b^* f(x,z) + \alpha a f(x,w) + \overline{\beta} b^* f(y,z) + f(y,w), s \mid t).$$

It follows from lemma 3.4 that f is A-2 linear. $\square$

4. Superstability of 2-adjoinable mappings

In this section, X and Y denote 2-pre Hilbert A-modules and $\dim(X) > 1$, $\dim(Y) > 1$ and $\phi : X^2 \times Y^2 \to [0, \infty)$ is a function. We start our work with following definition.

Definition 4.1. *A* (not necessarily A-2 linear) mapping $f : X \times X \to Y$ is called
ϕ-perturbation of an 2-adjointable mapping if there exists a mapping (not necessarily $A - 2$- linear) $g : Y \times Y \to X$ such that

$$\|(f(x,y), s \mid t) - (x, y \mid g(s,t))\| \leq \phi(x,y,s,t) \tag{4.1}$$

for all $x, y \in X$ and $s, t \in Y$.

Theorem 4.2. *Let $f : X \times X \to Y$ be a ϕ-perturbation of a 2-adjointable mapping with corresponding mapping $g : Y \times Y \to X$. Suppose for some sequence c_n of non-zero complex numbers the following conditions hold:*

$$\lim_{n \to \infty} |c_n|^{-1} \phi(c_n x, y, s, t) = 0 \quad (x, y \in X, s, t \in Y) \tag{4.2}$$

$$\lim_{n \to \infty} |c_n|^{-1} \phi(x, y, c_n s, t) = 0 \quad (x, y \in X, s, t \in Y) \tag{4.3}$$

Then f is 2-adjointable and hence f is A-2 linear.

Proof. Let $\lambda \in \mathbb{C}$ be an arbitrary number. Putting λx instead x in (4.1), we get

$$\|(f(\lambda x, y), s \mid t) - (\lambda x, y \mid g(s, t))\| \leq \phi(\lambda x, y, s, t)$$

multiplication of (4.1) by $|\lambda|$, we have

$$\|(\lambda f(x, y), s \mid t) - \lambda(x, y \mid g(s, t))\| \leq |\lambda| \phi(x, y, s, t)$$

Thus,

$$\|(f(\lambda x, y), s \mid t) - (\lambda f(x, y), s \mid t)\| \leq \phi(\lambda x, y, s, t) + |\lambda| \phi(x, y, s, t) \tag{4.4}$$

Replacing $c_n s$ by s in (4.4), we get

$$\|f(\lambda x, y), s \mid t) - (\lambda f(x, y), s \mid t)\| \leq |c_n|^{-1} \phi(\lambda x, y, c_n s, t) + |\lambda| \, |c_n|^{-1} \phi(x, y, c_n s, t)$$

hence, as $n \to \infty$, applying (4.3) we obtain

$$(f(\lambda x, y), s \mid t) - (\lambda f(x, y), s \mid t) = 0 \quad (\lambda \in \mathbb{C}, x, y \in X, s, t \in Y).$$

It follows from proposition 3.4 that

$$f(\lambda x, y) = \lambda f(x, y) \quad (\lambda \in \mathbb{C}, x, y \in X) \tag{4.5}$$

Now, we take $c_n x$ instead x in (4.1) to get

$$\|(f(c_n x, y), s \mid t) - (c_n x, y \mid g(t, s))\| \leq \phi(c_n x, y, s, t).$$

It follows from (4.5) that

$$\|(f(x, y), s \mid t) - (x, y \mid g(s, t))\| \leq |c_n|^{-1} \phi(c_n x, y, s, t)$$

hence, as $n \to \infty$, applying (4.2) we get

$$(f(x, y), s \mid t) = (x, y \mid g(s, t) \quad (x, y \in X, s, t \in Y).$$

Therefore f is 2-adjointable and by Lemma 3.5, f is A-2 linear. $\quad\square$

In the following, we let $c_n = a^n$ that $a > 1$. we get the following results.

Corollary 4.3. *If $f : X \times X \to Y$ is a ϕ-perturbation of a 2-adjointable mapping, where*
$\phi(x, y, s, t) = \epsilon \, \|x|y\|_X^p \, \|s|t\|_Y^q \, (\epsilon \geq 0, \ 0 < p < 1, \ 0 < q < 1)$, *then f is 2-adjointable and hence f is A-2-linear.*

Corollary 4.4. *If $f : X \times X \to Y$ is a ϕ-perturbation of a 2-adjointable mapping, where*
$\phi(x, y, s, t) = \epsilon_1 \, \|x|y\|_X^p \, + \, \epsilon_2 \, \|s|t\|_Y^q \, (\epsilon_1 \geq 0, \ \epsilon_2 \geq 0, \ 0 < p < 1, \ 0 < q < 1)$.
Then f is 2-adjointable and hence f is A-2 linear.

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