

## $z_R$ -Ideals and $z_R^\circ$ -Ideals in Subrings of $\mathbb{R}^X$

Ali Rezaei Aliabad and Mehdi Parsinia\*

Department of Mathematics, Shahid Chamran University of Ahvaz,  
Ahvaz, Iran.

E-mail: aliabady\_r@scu.ac.ir

E-mail: parsiniamehdi@gmail.com

**ABSTRACT.** Let  $X$  be a topological space and  $R$  be a subring of  $\mathbb{R}^X$ . By determining some special topologies on  $X$  associated with the subring  $R$ , characterizations of maximal fixed and maximal  $g$ -ideals in  $R$  of the form  $M_x(R)$  are given. Moreover, the classes of  $z_R$ -ideals and  $z_R^\circ$ -ideals are introduced in  $R$  which are topological generalizations of  $z$ -ideals and  $z^\circ$ -ideals of  $C(X)$ , respectively. Various characterizations of these ideals are established. Also, coincidence of  $z_R$ -ideals with  $z$ -ideals and  $z_R^\circ$ -ideals with  $z^\circ$ -ideals in  $R$  are investigated. It turns out that some fundamental statements in the context of  $C(X)$  are extended to the subrings of  $\mathbb{R}^X$ .

**Keywords:**  $Z(R)$ -topology,  $Coz(R)$ -topology,  $g$ -ideal,  $z_R$ -ideal,  $z_R^\circ$ -ideal, invertible subring.

**2010 Mathematics subject classification:** 54C30, 46E25.

### 1. INTRODUCTION

For a topological space  $X$ ,  $\mathbb{R}^X$  denotes the algebra of all real-valued functions and  $C(X)$  (resp.,  $C^*(X)$ ) denotes the subalgebra of  $\mathbb{R}^X$  consisting of all continuous functions (resp., bounded continuous functions). Moreover, we use  $R$  to denote a unital subring of  $\mathbb{R}^X$ . Note that topological spaces which are considered in this paper are not necessarily Tychonoff. For each  $f \in \mathbb{R}^X$ ,

---

\*Corresponding Author

Received 12 May 2016; Accepted 14 January 2017

©2019 Academic Center for Education, Culture and Research TMU

$Z(f) = \{x \in X : f(x) = 0\}$  denotes the zero-set of  $f$  and  $Coz(f)$  denotes the complement of  $Z(f)$  with respect to  $X$ . We denote by  $Z(R)$  the collection of all the zero-sets of elements of  $R$ , we use  $Z(X)$  instead of  $Z(C(X))$ . We denote by  $M_x(R)$  the set  $\{f \in R : x \in Z(f)\}$ ,  $M_x(C(X))$  is denoted by  $M_x$ . The subring  $R$  is called invertible, if  $f \in R$  and  $Z(f) = \emptyset$  implies that  $f$  is invertible in  $R$ . Moreover,  $R$  is called a lattice-ordered subring if it is a sublattice of  $\mathbb{R}^X$  (i.e.,  $f \wedge g$  and  $f \vee g$  are in  $R$  for each  $f, g \in R$ ). It is clear that  $C(X)$  is an invertible lattice-ordered subring of  $\mathbb{R}^X$ . However, the same statement does not hold for  $C^*(X)$ . A proper ideal  $I$  of  $R$  is called a growing ideal, briefly, a  $g$ -ideal, if contains no invertible element of  $\mathbb{R}^X$ , i.e.,  $Z(f) \neq \emptyset$  for each  $f \in I$ . It is evident that a subring  $R$  is invertible if and only if every ideal every ideal of  $R$  is a  $g$ -ideal. Clearly,  $M^{*p}$ , for each  $p \in \beta X \setminus vX$ , is not a  $g$ -ideal of  $C^*(X)$ . An ideal  $I$  of  $R$  is called fixed if  $\bigcap_{f \in I} Z(f) \neq \emptyset$ , otherwise, it is called free. By a maximal fixed ideal of  $R$ , we mean a fixed ideal which is maximal in the set of all fixed ideals of  $R$ . An ideal  $I$  in a commutative ring  $S$  is called a  $z$ -ideal (resp.,  $z^\circ$ -ideal) if  $M_a(S) \subseteq I$  (resp.,  $P_a(S) \subseteq I$ ), for each  $a \in I$ , where  $M_a(S)$  (resp.,  $P_a(S)$ ) denotes the intersection of all the maximal (resp., minimal prime) ideals of  $S$  containing  $a$ . It is well-known that in  $C(X)$  an ideal  $I$  is a  $z$ -ideal (resp.,  $z^\circ$ -ideal) if and only if whenever  $Z(f) \subseteq Z(g)$  (resp.,  $\text{int}_X Z(f) \subseteq \text{int}_X Z(g)$ ),  $f \in I$  and  $g \in C(X)$ , then  $g \in I$ .

This paper consists of 4 sections. Section 1, as we have already noticed, is the introduction, in which we determine two special topologies on  $X$  which the subring  $R$  generate, namely,  $Z(R)$ -topology and  $Coz(R)$ -topology. Comparison and coincidence of these topologies are studied. Section 2 deals with maximal ideals in  $R$ , specially, maximal fixed and maximal  $g$ -ideals. Using the  $Z(R)$ -topology, characterizations of maximal fixed ideals of  $R$ , which are of the form  $M_x(R)$ , are given. Moreover, relations between mapping “ $x \rightarrow M_x(R)$ ” and the separation properties of the topological space  $(X, \tau_{Z(R)})$  will be found. In section 3, we introduce the notion of  $z_R$ -ideal in a subring  $R$  as a natural topological generalization of the notion of  $z$ -ideal in  $C(X)$ . Various characterizations of these ideals via  $Z(R)$ -topology are given and relations between  $z_R$ -ideals and  $z$ -ideals in  $R$  (by their algebraic descriptions) are discussed. Section 4 deals with  $z_R^\circ$ -ideals of  $R$  which are natural topological generalizations of  $z^\circ$ -ideals of  $C(X)$ . Using  $Coz(R)$ -topology, coincidence of  $z_R^\circ$ -ideals with  $z^\circ$ -ideals of  $R$  (by their algebraic descriptions) are established.

**Definition 1.1.** For each subring  $R$  of  $\mathbb{R}^X$ , clearly,  $Z(R)$  and  $Coz(R)$  constitute bases for some topologies on  $X$ . The induced topologies are called  $Z(R)$ -topology and  $Coz(R)$ -topology, respectively, and are denoted by  $\tau_{Z(R)}$  and  $\tau_{Coz(R)}$ , respectively.

In the next three statements we compare these topologies. Note that two subsets  $S_1, S_2$  of  $\mathbb{R}^X$  are called zero-set equivalent, if  $Z(S_1) = Z(S_2)$ .

**Proposition 1.2.** *Let  $R$  be a subring of  $\mathbb{R}^X$ , if  $S$  and  $C(\mathbb{R})$  are zero-set equivalent subsets of  $\mathbb{R}^{\mathbb{R}}$  and  $gof \in R$  for each  $f \in R$  and each  $g \in S$ , then  $\tau_{Coz(R)} \subseteq \tau_{Z(R)}$  and the equality does not hold, in general.*

*Proof.* We are to show that  $Coz(R) \subseteq \tau_{Z(R)}$ . If  $x \notin Z(f)$  where  $f \in R$ , then there is a  $g$  in  $S$  such that  $f(x) \in Z(g)$  and  $f^{-1}(Z(g)) \cap Z(f) = \emptyset$ . Therefore,  $gof \in R$ ,  $x \in Z(gof)$  and  $Z(gof) \cap Z(f) = \emptyset$  which proves the inclusion. Now, we show that the inclusion may be proper. Let  $(X, \tau_X)$  be a Tychonoff space which has at least one non-open zero-set  $Z$ . Set  $R = C(X)$ , then  $\tau_{Coz(R)} = \tau_X$ , whereas  $Z \notin \tau_X$  and hence,  $\tau_{Coz(R)} \subsetneq \tau_{Z(R)}$ .  $\square$

Proof of the following proposition is standard.

**Proposition 1.3.** *The following statements are equivalent.*

- (a)  $\tau_{Coz(R)} \subseteq \tau_{Z(R)}$ .
- (b) Every  $Z \in Z(R)$  is clopen under  $Z(R)$ -topology.

The annihilator of  $f \in R$  in  $R$  is defined to be the set  $\{g \in R : fg = 0\}$  and is denoted by  $Ann_R(f)$ . A simple reasoning shows that if  $X$  is equipped with the  $Coz(R)$ -topology, then  $Ann_R(f) = \{g \in R : Coz(g) \subseteq \text{int}_X Z(f)\} = \{g \in R : \text{cl}_X(Coz(g)) \subseteq Z(f)\}$ .

**Proposition 1.4.** *The following statements are equivalent.*

- (a)  $\tau_{Z(R)} \subseteq \tau_{Coz(R)}$ .
- (b)  $Z(f)$  is clopen in  $(X, \tau_{Coz(R)})$  for every  $f \in R$ .
- (c) For each  $f \in R$ ,  $Z(f) = \bigcup_{g \in Ann_R(f)} Coz(g)$ .
- (d) For each  $f \in R$ ,  $(Ann_R(f), f)$  is a free ideal.

*Proof.* The implications (a) $\Rightarrow$ (b) $\Rightarrow$ (c) are clear.

(c) $\Rightarrow$ (d). This clear by the hypothesis and the fact that whenever  $f \in R$  and  $I$  is an ideal of  $R$ , then  $\bigcap_{h \in (I, f)} Z(h) = \bigcap_{g \in I} (Z(f) \cap Z(g))$ .

(d) $\Rightarrow$ (a). Let  $f \in R$  and  $x \in Z(f)$ . By (d), there exists  $g \in Ann_R(f)$  such that  $x \notin Z(f) \cap Z(g)$ . Hence,  $x \notin Z(g)$  and  $x \in Coz(g) \subseteq Z(f)$  and so  $Z(f) \in \tau_{Coz(R)}$ .  $\square$

An immediate consequence of Propositions 1.3 and 1.4 is that  $\tau_{Coz(R)} = \tau_{Z(R)}$  if and only if  $Z(f)$  is clopen under both  $Z(R)$ -topology and  $Coz(R)$ -topology, for each  $f \in R$ .

## 2. CHARACTERIZATION OF MAXIMAL FIXED IDEALS IN SUBRINGS

We remind that maximal fixed ideals of  $C(X)$  coincide with its fixed maximal ideals and are of the form  $M_x = \{f \in C(X) : f(x) = 0\}$ , where  $x \in X$ . This fact is generalized for some special subalgebras of  $C(X)$ , such as intermediate subalgebras (subalgebras of  $C(X)$  containing  $C^*(X)$ , see [7]),  $C_c(X)$  (the subalgebra of  $C(X)$  consisting of all functions with countable image, see [9]) and the subalgebras of the form  $\mathbb{R} + I$  where  $I$  is an ideal of  $C(X)$ , see [13].

We will show that the same statement does not hold for arbitrary subrings of  $\mathbb{R}^X$ , in general.

*Remark 2.1.* (a) Every maximal fixed ideal and fixed maximal ideal of  $R$  is of the form  $M_x(R) = \{f \in R : f(x) = 0\}$  for some  $x \in X$ . However, parts (1) and (2) of Example 2.2 show that the ideals  $M_x(R)$  are not necessarily maximal ideals or even maximal fixed ideals in  $R$ .

(b) Every fixed maximal ideal is both a maximal fixed ideal and a maximal  $g$ -ideal. But the converse is not necessarily true, in general, see part (1) of Example 2.2 and Example 2.3.

(c) A maximal fixed ideal need not be a maximal  $g$ -ideal, see Example 2.3.

(d) Every fixed maximal  $g$ -ideal is a maximal fixed ideal.

EXAMPLE 2.2. (1) Let  $X$  be a Tychonoff space,  $x \in X$  and  $R = \mathbb{Z} + M_x$ . Then  $M_x(R) = M_x$  is not a maximal ideal in  $R$ , since  $2\mathbb{Z} + M_x$  is a proper ideal of  $R$  and  $M_x \subsetneq 2\mathbb{Z} + M_x$ . Therefore,  $M_x(R)$  is a maximal fixed ideal and a maximal  $g$ -ideal which is not a maximal ideal.

(2) Let  $X$  be a topological space with more than one point and  $a \in X$ . Also, let  $t \in \mathbb{R}$  be a transcendental number and define  $f : X \rightarrow \mathbb{R}$  by  $f(a) = 0$  and  $f(x) = t$ , for every  $x \neq a$ . Set  $R = \{\sum_{i=0}^n m_i f^i : n \in \mathbb{N} \cup \{0\}, m_i \in \mathbb{Z}\}$ . Evidently,  $M_a(R) = (f)$  and  $M_x(R) = \{0\}$ , for every  $x \neq a$ . Therefore,  $M_x(R)$  is not a maximal fixed ideal for any  $x \neq a$ .

In the next example we construct a subring  $R$  such that, for some  $x \in X$ ,  $M_x(R)$  is a maximal fixed ideal which is not a maximal  $g$ -ideal.

EXAMPLE 2.3. Let  $X = \mathbb{R}$ ,  $a \in \mathbb{R} \setminus \mathbb{Q}$ ,  $b \in \mathbb{R} \setminus \{0\}$  and  $t$  be a transcendental number. For every  $\epsilon > 0$ , define  $f_\epsilon : X \rightarrow \mathbb{R}$  by  $f_\epsilon(x) = 0$ , if  $|x - a| < \epsilon$  and  $f_\epsilon(x) = b$ , if  $|x - a| \geq \epsilon$ . Also, define  $f : X \rightarrow \mathbb{R}$  by  $f(x) = 0$ , if  $x \in \mathbb{Q}$  and  $f(x) = t$ , if  $x \in \mathbb{R} \setminus \mathbb{Q}$ . Let  $R$  be the algebra over  $\mathbb{Q}$  generated by  $\{f_\epsilon : \epsilon > 0\} \cup \{f, 1\}$ . Evidently,  $R$  is a subring of  $\mathbb{R}^X$ , and  $M_a(R)$  equals to  $(f_a)$  which is not a maximal ideal. It is easy to see that  $M_a(R)$  is a maximal fixed ideal and  $M_a(R) = I$ , where  $I$  is the ideal generated by  $\{f_\epsilon : \epsilon > 0\}$ . Clearly,  $Z(f) \cap Z(g) \neq \emptyset$ , for all  $g \in I$ . Hence  $J = (I, f)$  is a  $g$ -ideal which strictly contains  $I$ . Therefore,  $I$  is not a maximal  $g$ -ideal.

**Proposition 2.4.** *The following statements hold for a subring  $R$  of  $\mathbb{R}^X$ .*

(a)  $M_x(R)$  is a maximal  $g$ -ideal if and only if whenever  $Z \in Z(R)$  and  $x \notin Z$ , then  $x \notin \text{cl}_{\tau_{Z(R)}} Z$ .

(b) For each  $x \in X$ ,  $M_x(R)$  is a maximal  $g$ -ideal if and only if every  $Z \in Z(R)$  is clopen under  $Z(R)$ -topology.

*Proof.* (a  $\Rightarrow$ ). Let  $f \in R$  and  $x \notin Z(f)$ , thus, the ideal  $(M_x(R), f)$  contains an invertible element of  $\mathbb{R}^X$ . Hence, there are  $g \in M_x(R)$  and  $h \in R$  such that  $Z(g + fh) = \emptyset$ . Consequently,  $x \in Z(g)$  and  $Z(f) \cap Z(g) = \emptyset$ .

(a  $\Leftarrow$ ). Assume that  $f \notin M_x(R)$ . Then there is some  $g \in R$  such that  $x \in Z(g)$  and  $Z(f) \cap Z(g) = Z(f^2 + g^2) = \emptyset$ . Hence,  $(M_x(R), f)$  contains an invertible element of  $\mathbb{R}^X$ . Also, clearly,  $M_x(R)$  is a  $g$ -ideal. Thus,  $M_x(R)$  is a maximal  $g$ -ideal.

(b). An easy consequence of (a). □

**Corollary 2.5.** *If  $M_x(R)$  is a maximal ideal for each  $x \in X$ , then every  $Z \in Z(R)$  is clopen under  $Z(R)$ -topology.*

**Corollary 2.6.** *Let  $R$  be an invertible subring. Then every  $Z \in Z(R)$  is clopen under  $Z(R)$ -topology if and only if  $M_x(R)$  is a maximal ideal for each  $x \in X$ .*

*Proof.* By our hypothesis and Proposition 2.4, this is clear. □

The following lemma is a restatement of the fact that the transcendental degree of  $\mathbb{R}$  over  $\mathbb{Q}$  is uncountable, see [14].

**Lemma 2.7.** *Let  $S = \mathbb{Q}[y_1, \dots, y_n]$  be the ring of  $n$ -variable polynomials with rational coefficients. Then there exists an uncountable set  $X$  of transcendental numbers for which  $F(a_1, \dots, a_n) \neq 0$ , for every distinct elements  $a_1, \dots, a_n$  of  $X$  and every  $F \in S$ .*

The following example shows that the converse of Corollary 2.5 does not hold, in general.

**EXAMPLE 2.8.** Let  $S$  be the polynomial ring  $\mathbb{Q}[y_1, \dots, y_n]$ , where  $n \in \mathbb{N}$  and  $n > 1$ . By Lemma 2.7, there exists an infinite set of transcendental numbers  $X$  for which  $F(a_1, \dots, a_n) \neq 0$ , for every  $a_1, \dots, a_n \in X$  and every  $F \in S$ . For each  $a \in X$ , define the function  $f_a : X \rightarrow \mathbb{R}$  by  $f_a(a) = 0$  and  $f_a(x) = x$  for each  $x \neq a$ . Now, set

$$R = \{F(f_{a_1}, \dots, f_{a_n}) : F \in S, n \in \mathbb{N}, a_1, \dots, a_n \in X\}.$$

Hence,  $M_a(R) = (f_a)$ , for each  $a \in X$ , which is not a maximal ideal. However, every  $Z \in Z(R)$  is clopen under  $Z(R)$ -topology.

**Proposition 2.9.** *If  $R$  is a subalgebra of  $\mathbb{R}^X$ , then  $M_x(R)$  is a maximal  $g$ -ideal and a maximal fixed ideal for every  $x \in X$ .*

*Proof.* It suffices to prove that every element of  $Z(R)$  is closed under  $Z(R)$ -topology. To this aim, suppose that  $a \in X$  and  $a \notin Z(f)$ , for some  $f \in R$ . Put  $g = f - f(a)$ . Clearly,  $Z(g) \in Z(R)$ ,  $a \in Z(g)$  and  $Z(g) \cap Z(f) = \emptyset$ . □

**Corollary 2.10.** *If  $R$  is an invertible subalgebra of  $\mathbb{R}^X$ , then  $M_x(R)$  is a maximal ideal for each  $x \in X$ .*

The converse of Corollary 2.10 does not hold, in general. For example, let  $R$  denote the collection of all single variable polynomials over  $\mathbb{R}$ . Then,  $M_r(R)$  is the maximal ideal  $(x - r)$  for each  $r \in \mathbb{R}$ . However,  $f = x^2 + 1$  is invertible in

$\mathbb{R}^{\mathbb{R}}$  which is not invertible in  $R$ . Note that the subalgebras  $C_c(X)$  and  $\mathbb{R} + I$ , for each ideal  $I$  in  $C(X)$ , satisfy Corollary 2.10 and so  $M_x(C_c(X))$  and  $M_x(\mathbb{R} + I)$  are maximal ideals of  $C_c(X)$  and  $\mathbb{R} + I$ , respectively, for each  $x \in X$ . Remark that in parts (b) and (e) of the following proposition we assume that “=” is a partial order on  $X$ .

**Proposition 2.11.** *For a subring  $R$  of  $\mathbb{R}^X$ , the following statements hold.*

(a) *The mapping  $x \rightarrow M_x(R)$  is a one-one correspondence if and only if  $(X, \tau_{Z(R)})$  is a  $T_0$ -space.*

(b) *The mapping  $x \rightarrow M_x(R)$  is an order isomorphism between  $X$  and the set of all maximal fixed ideals of  $R$  if and only if  $(X, \tau_{Z(R)})$  is a  $T_1$ -space.*

(c) *For every two distinct elements  $x, y \in X$ ,  $M_x(R) + M_y(R)$  is not a  $g$ -ideal if and only if  $(X, \tau_{Z(R)})$  is a  $T_2$ -space.*

(d) *The mapping  $x \rightarrow M_x(R)$  is an order embedding between  $X$  and the set of all maximal  $g$ -ideals of  $R$  if and only if  $(X, \tau_{Z(R)})$  is a  $T_0$ -space and every element of  $Z(R)$  is clopen under  $Z(R)$ -topology.*

*Proof.* (a). Let  $x, y$  be distinct points of  $X$ , so  $M_x(R) \neq M_y(R)$ , say  $M_x(R) \not\subseteq M_y(R)$ . Hence, there exists  $f \in M_x(R) \setminus M_y(R)$ . Thus  $x \in Z(f)$  and  $y \notin Z(f)$ . It is clear that the above reasoning is reversible and hence we are done.

(b  $\Rightarrow$ ). Suppose that  $x$  and  $y$  are two distinct points of  $X$ . Since  $M_x(R) \not\subseteq M_y(R)$ , there exists  $f \in M_x(R) \setminus M_y(R)$ . Consequently,  $x \in Z(f)$  and  $y \notin Z(f)$ .

(b  $\Leftarrow$ ). Suppose that  $x \in X$  and  $I$  is a fixed ideal in  $R$  containing  $M_x(R)$ . Take  $y \in \bigcap_{f \in I} Z(f)$ . Clearly,  $M_x(R) \subseteq I \subseteq M_y(R)$ . It suffices to show  $x = y$ . Suppose that  $x \neq y$  and seek a contradiction. By our hypothesis, there exists  $f \in R$  such that  $x \in Z(f)$  and  $y \notin Z(f)$ . Therefore,  $M_x(R) \not\subseteq M_y(R)$  and this is a contradiction. Now, by part (a), the proof is complete.

(c). For any two distinct points  $x, y \in X$ , clearly,  $M_x(R) + M_y(R)$  is not a  $g$ -ideal if and only if there exist  $f \in M_x(R)$  and  $g \in M_y(R)$  such that  $Z(f) \cap Z(g) = \emptyset$ .

(d  $\Rightarrow$ ). By part (a), clearly,  $(X, \tau_{Z(R)})$  is a  $T_0$ -space. Now, Suppose that  $f \in R$  and  $x \notin Z(f)$ . Since  $M_x(R)$  is a maximal  $g$ -ideal, it follows that  $(M_x(R), f)$  has an invertible element of  $\mathbb{R}^X$  and so there exists  $g \in M_x(R)$ , such that  $Z(g) \cap Z(f) = \emptyset$ . Thus,  $Z(f)$  is closed and hence is clopen under  $Z(R)$ -topology.

(d  $\Leftarrow$ ). Suppose that  $x \in X$ , it suffices to show that  $M_x(R)$  is a maximal  $g$ -ideal. Assume that  $I$  is an ideal which properly contains  $M_x(R)$ . Hence, there exists  $f \in I$  such that  $x \notin Z(f)$ . By our hypothesis, there is  $g \in R$  such that  $x \in Z(g)$  and  $Z(g) \cap Z(f) = \emptyset$ . Therefore,  $Z(f^2 + g^2) = \emptyset$  and  $f^2 + g^2 \in I$ , hence,  $I$  is not a  $g$ -ideal.  $\square$

It is easy to see that  $M_x(R)$ , for each  $x \in X$ , is a prime ideal of  $R$  and thus the hull-kernel topology may be defined on the family  $\{M_x(R) : x \in X\}$ .

By considering this space, the next statement gives a relation between  $Z(R)$ -topology on  $X$  and points of  $X$ .

**Proposition 2.12.** *Let  $R$  be a subring of  $\mathbb{R}^X$  and  $X$  equipped with the  $\text{Coz}(R)$ -topology. Then the mapping  $\Phi : X \rightarrow \{M_x(R) : x \in X\}$  defined by  $x \mapsto M_x(R)$  is a homeomorphism if and only if  $(X, \tau_{Z(R)})$  is a  $T_0$ -space.*

*Proof.* By part (a) of Theorem 2.12,  $\Phi$  is a one-one correspondence if and only if  $(X, \tau_{Z(R)})$  is a  $T_0$ -space. Also, if  $f \in R$  and  $x \in Z(f)$ , then  $f \in M_x(R)$  which means that basic closed sets of  $X$  equipped with the  $\text{Coz}(R)$ -topology are mapped to the basic closed sets in  $\{M_x(R) : x \in X\}$  equipped with the hull-kernel topology by the mapping  $\Phi$  and therefore, it is a homeomorphism.  $\square$

### 3. $z_R$ -IDEALS AND $z$ -IDEALS IN SUBRINGS

In this section we introduce  $z_R$ -ideals in a subring  $R$  and via the  $Z(R)$ -topology and maximal  $g$ -ideals of  $R$ , various characterizations of these ideals are given.

**Definition 3.1.** A subset  $\mathcal{F}$  of  $Z(R)$  is called  $z_R$ -filter on  $X$ , if

- (a)  $\emptyset \notin \mathcal{F}$ .
- (b) If  $Z_1, Z_2 \in \mathcal{F}$ , then  $Z_1 \cap Z_2 \in \mathcal{F}$ .
- (c) If  $Z_1 \in \mathcal{F}$ ,  $Z_2 \in Z(R)$  and  $Z_1 \subseteq Z_2$ , then  $Z_2 \in \mathcal{F}$ .

Moreover,  $\mathcal{F}$  is called a prime  $z_R$ -filter, if whenever  $Z_1 \cup Z_2 \in \mathcal{F}$ , then  $Z_1 \in \mathcal{F}$  or  $Z_2 \in \mathcal{F}$  for each  $Z_1, Z_2 \in Z(R)$ . Also,  $\mathcal{F}$  is called a  $z_R$ -ultrafilter, if  $\mathcal{F}$  is maximal among  $z_R$ -filters on  $X$ .

The following proposition immediately follows from Definition 3.1.

**Proposition 3.2.** *For any subring  $R$ , the following statements hold.*

- (a)  $I \subseteq R$  is a  $g$ -ideal in  $R$  if and only if  $Z_R(I) = \{Z(f) : f \in I\}$  is a  $z_R$ -filter on  $X$ .
- (b)  $\mathcal{F}$  is a  $z_R$ -filter on  $X$  if and only if  $Z_R^{-1}(\mathcal{F}) = \{f \in R : Z(f) \in \mathcal{F}\}$  is a  $g$ -ideal.
- (c)  $\mathcal{F}$  is a prime  $z_R$ -filter on  $X$  if and only if  $Z_R^{-1}(\mathcal{F})$  is a prime  $g$ -ideal.
- (d)  $\mathcal{A}$  is a  $z_R$ -ultrafilter on  $X$  if and only if  $Z_R^{-1}(\mathcal{A})$  is a maximal  $g$ -ideal.
- (e) If  $M$  is a maximal  $g$ -ideal in  $R$ , then  $Z_R(M)$  is a  $z_R$ -ultrafilter on  $X$ .

It is easy to see that for an ideal  $I$  of  $R$  we always have  $I \subseteq Z_R^{-1}Z_R(I)$  and the inclusion may be proper. We call an ideal  $I$  in  $R$  a  $z_R$ -ideal, if  $I = Z_R^{-1}Z_R(I)$ . It follows that every  $z_R$ -ideal is semiprime and arbitrary intersections of  $z_R$ -ideals is a  $z_R$ -ideal. Also, the zero ideal, the ideals of the form  $M_x(R)$ , maximal  $g$ -ideals and  $Z^{-1}(\mathcal{F})$ , for each  $z_R$ -filter  $\mathcal{F}$ , are all  $z_R$ -ideals of  $R$ . For each  $f \in R$ , the intersection of all the maximal ideals, maximal  $g$ -ideals and maximal fixed ideals of  $R$  containing  $f$  are denoted by  $M_f(R)$ ,  $MG_f(R)$  and  $MF_f(R)$ , respectively. It is easy to observe that  $MG_f(R)$  is a  $z_R$ -ideal for each  $f \in R$ .

Obviously,  $MG_f \cap MG_g = MG_{fg}$ ,  $MF_f \cap MF_g = MF_{fg}$ ,  $MG_{f^2+g^2} = MG_{(f,g)}$  and  $MF_{f^2+g^2} = MF_{(f,g)}$  for all  $f, g \in R$ .

**Proposition 3.3.** *Let  $(X, \tau_{Z(R)})$  be a  $T_1$ -space. Then the following statements hold.*

(a) *The following statements are equivalent.*

(1)  $g \in MF_f(R)$ .

(2)  $MF_g(R) \subseteq MF_f(R)$ .

(3)  $Z(f) \subseteq Z(g)$ .

(b)  $MF_f(R) = \{g \in R : Z(f) \subseteq Z(g)\}$ .

(c) *An ideal  $I$  of  $R$  is a  $z_R$ -ideal if and only if  $MF_f(R) \subseteq I$  for every  $f \in I$ .*

*Proof.* (a: 1  $\Rightarrow$  2). Evident.

(a: 2  $\Rightarrow$  3). Let  $x \in Z(f)$ . Then  $f \in M_x(R)$  and thus  $MF_g(R) \subseteq MF_f(R) \subseteq M_x(R)$ . This implies  $g \in M_x(R)$  and hence  $x \in Z(g)$ .

(a: 3  $\Rightarrow$  1). If  $g \notin MF_f(R)$ , then there exists  $x \in X$  such that  $f \in M_x(R)$  and  $g \notin M_x(R)$ . Therefore,  $x \in Z(f) \setminus Z(g)$  and so  $Z(f) \not\subseteq Z(g)$ .

(b) and (c) obviously follow from part (a).  $\square$

**Lemma 3.4.** *Assume that every  $Z \in Z(R)$  is clopen under  $Z(R)$ -topology. Then  $MG_f(R) = MF_f(R)$ , for every  $f \in R$ .*

*Proof.* Suppose that  $f \in R$ . By part (b) of Proposition 2.4,  $M_x(R)$  is a maximal  $g$ -ideal for each  $x \in X$ . Consequently,  $MG_f(R) \subseteq MF_f(R)$ . Now, assume that  $g \notin MG_f(R)$ . Hence, there exists a maximal  $g$ -ideal  $M$  in  $R$  such that  $f \in M$  and  $g \notin M$ . Thus, there exists  $h \in M$  such that  $Z(g) \cap Z(h) = \emptyset$ . Since  $f^2 + h^2 \in M$  and  $M$  is a  $g$ -ideal, there is a point  $x \in Z(f^2 + h^2) = Z(f) \cap Z(h)$ . Clearly,  $g \notin M_x(R)$  and  $f \in M_x(R)$ . Therefore,  $g \notin MF_f(R)$ .  $\square$

Proposition 3.3 and Lemma 3.4 imply the next statement.

**Proposition 3.5.** *Let  $(X, \tau_{Z(R)})$  be a  $T_1$ -space and every  $Z \in Z(R)$  be a clopen set under  $Z(R)$ -topology. Then the following statements hold.*

(a) *The following statements are equivalent.*

(1)  $g \in MG_f(R)$ .

(2)  $MG_g(R) \subseteq MG_f(R)$ .

(3)  $Z(f) \subseteq Z(g)$ .

(b)  $MG_f(R) = \{g \in R : Z(f) \subseteq Z(g)\}$ .

(c) *An ideal  $I$  of  $R$  is  $z_R$ -ideal if and only if  $MG_f(R) \subseteq I$  for every  $f \in I$ .*

The following corollary follows from Corollary 2.6 and Proposition 3.5.

**Corollary 3.6.** *Let  $R$  be an invertible subalgebra of  $\mathbb{R}^X$ . Then the following statements hold.*

- (a) The following conditions are equivalent;
- (1)  $g \in M_f(R)$ .
  - (2)  $M_g(R) \subseteq M_f(R)$ .
  - (3)  $Z(f) \subseteq Z(g)$ .
- (b)  $M_f(R) = \{g \in R : Z(f) \subseteq Z(g)\}$ .
- (c) An ideal  $I$  of  $R$  is  $z_R$ -ideal if and only if  $M_f(R) \subseteq I$  for every  $f \in I$ .

It follows from Corollary 3.6 that for an invertible subalgebra  $R$ , the notion of  $z_R$ -ideal coincides with the notion of  $z$ -ideal. The next statement extend this fact and shows that this coincidence is equivalent to invertibility of  $R$ .

**Theorem 3.7.** *Let  $R$  be a subring of  $\mathbb{R}^X$ . The following statements are equivalent.*

- (a) Every maximal ideal in  $R$  is a  $g$ -ideal.
- (b) Every maximal  $g$ -ideal of  $R$  is a maximal ideal and if  $J$  is a maximal ideal of  $R$ , then every maximal element in the set of  $g$ -ideals contained in  $J$  is a prime ideal.
- (c) Every maximal ideal in  $R$  is a  $g$ -ideal.
- (d)  $R$  is an invertible subring.
- (e) Every  $z$ -ideal of  $R$  is a  $z_R$ -ideal.

Moreover, if  $R$  is a subalgebra and one of (a)-(c) holds, then every  $z_R$ -ideal is a  $z$ -ideal.

*Proof.* (a)  $\Rightarrow$  (b). This is clear.

(b) $\Rightarrow$ (c). Suppose that  $M$  is a maximal ideal and  $P$  is a maximal element of  $G_M$ , where  $G_M$  is the set of all  $g$ -ideals contained in  $M$ . Assume that  $J$  is a maximal ideal of  $R$  containing  $P$ . Then  $M \cap J = P$ . As  $M \cap J$  is prime and both  $M$  and  $J$  are maximal ideal, we have  $M = J$ . Hence,  $M$  is a maximal  $g$ -ideal.

(c) $\Rightarrow$ (d). Suppose that  $Z(f) = \emptyset$  for  $f \in R$  and, on the contrary,  $f$  is a non-unit element of  $R$ . Clearly, there exists a maximal ideal  $M$  of  $R$  containing  $f$ . By our hypothesis,  $M$  is a  $g$ -ideal which contradicts with  $Z(f) = \emptyset$ .

(d) $\Rightarrow$ (e). Suppose that  $I$  is a  $z$ -ideal and  $Z(f) \subseteq Z(g)$  where  $f \in I$  and  $g \in R$ . Since  $I$  is a  $z$ -ideal, it follows that  $M_f \subseteq I$ . It suffices to prove that  $g \in M_f$ . To see this, suppose that  $M$  is a maximal ideal containing  $f$ . As  $R$  is invertible,  $M$  is a  $g$ -ideal and so it is a maximal  $g$ -ideal. Obviously,  $M$  is a  $z_R$ -ideal and so  $g \in M$ .

(e) $\Rightarrow$ (a). Suppose that  $M$  is a maximal ideal and, on the contrary,  $M$  is not a  $g$ -ideal. Thus, there exists  $f \in M$  such that  $Z(f) = \emptyset$ . By (e),  $M$  is a  $z_R$ -ideal and since  $f \in M$ , it follows that  $M = R$ , which is a contradiction.

Now, suppose that one of (a)-(c) holds,  $R$  is a subalgebra and  $I$  is a  $z_R$ -ideal of  $R$ . By our hypothesis,  $MF_f(R) = M_f(R)$  for every  $f \in R$ , and thus we are done.  $\square$

It is well-known that every minimal prime ideals over a  $z$ -ideal is also a  $z$ -ideal, see [10, Theorem 14.7]. The same statement holds for  $z_R$ -ideals as the following proposition shows.

**Proposition 3.8.** *Let  $I$  be a  $z_R$ -ideal of  $R$  and  $P$  a prime ideal in  $R$  minimal over  $I$ . Then  $P$  is a  $z_R$ -ideal.*

*Proof.* Assume that  $Z(f) = Z(g)$  and  $f \in P$ . Thus, there exists  $h \notin P$ , such that  $fh \in I$ . Since  $Z(fh) = Z(gh)$  and  $I$  is a  $z_R$ -ideal, it follows that  $gh \in I \subseteq P$ . As  $h \notin P$ , clearly, this implies that  $g \in P$ .  $\square$

An immediate consequence of Proposition 3.8 is that every minimal prime ideal in a subring  $R$  is a  $z_R$ -ideal. By the following statement, we extend some fundamental statements about  $z$ -ideals in the literature of  $C(X)$  to the subrings of  $\mathbb{R}^X$ , namely, [10, 2.9, 5.3 and 5.5]. The proofs are left to the reader.

**Proposition 3.9.** *Let  $R$  be a lattice-ordered subring of  $\mathbb{R}^X$  and  $I$  be a  $z_R$ -ideal in  $R$ . Then the following statements hold.*

(a) *The following statements are equivalent*

- (1)  *$I$  is a prime ideal;*
- (2)  *$I$  contains a prime ideal;*
- (3) *if  $fg = 0$ , then  $f \in I$  or  $g \in I$ ;*
- (4) *for each  $f \in R$ , there is a  $Z \in Z_R(I)$  on which  $f$  does not change sign.*

(b) *Every prime  $g$ -ideal of  $R$  is contained in a unique maximal  $g$ -ideal.*

(c) *If  $P$  is a prime ideal of  $R$ , then  $Z_R(P)$  is a prime  $z_R$ -filter on  $X$ .*

(d) *If  $\mathcal{P}$  is a prime  $z_R$ -filter on  $X$ , then  $Z_R^{-1}(\mathcal{P})$  is a prime ideal in  $R$ .*

(e) *Every  $z_R$ -ideal of  $R$  is absolutely convex.*

*Thus, if  $I$  is an absolutely convex ideal of  $R$ , then  $R/I$  is a lattice ring.*

(f)  *$I(f) \geq 0$  if and only if  $f \geq 0$  on some  $Z \in Z_R(I)$ .*

(g) *Suppose that there exists  $Z \in Z_R(I)$  such that  $f(x) > 0$ , for every  $x \in Z$ , then  $I(f) > 0$ . The converse is true whenever  $I$  is a maximal  $g$ -ideal.*

#### 4. $z_R^\circ$ -IDEALS AND $z^\circ$ -IDEALS IN SUBRINGS

In this section we generalize the concept of  $z^\circ$ -ideals of  $C(X)$  to the subrings of  $\mathbb{R}^X$  and introduce  $z_R^\circ$ -ideal. Coincidence of  $z_R^\circ$ -ideals with  $z^\circ$ -ideals of  $R$  is discussed. Note that, for each element  $f$  of a commutative rings  $S$ , we use  $P_f(S)$  to denote the intersection of all the minimal prime ideals in  $S$  containing  $f$ .

**Definition 4.1.** An ideal  $I$  of a subring  $R$  of  $\mathbb{R}^X$  is called a  $z_R^\circ$ -ideal, if  $\text{int}_X Z(f) \subseteq \text{int}_X Z(g)$ , where  $f \in I$  and  $g \in R$ , implies  $g \in I$ .

The following statement investigates some characterizations of  $z_R^\circ$ -ideals in subrings.

**Theorem 4.2.** *Let  $R$  be a subring of  $\mathbb{R}^X$  and  $I$  be an ideal in  $R$ . The following statements are equivalent.*

- (a)  $I$  is a  $z_R^\circ$ -ideal.
- (b) Whenever  $\text{Ann}_C(f) \subseteq \text{Ann}_C(g)$  where  $f \in I$  and  $g \in R$ , then  $g \in I$ .
- (c)  $R \cap P_f(C) \subseteq I$  for each  $f \in I$ .
- (d) Whenever  $P_g(C) \cap R \subseteq P_f(C) \cap R$ , where  $f \in I$  and  $g \in R$ , then  $g \in I$ .

*Proof.* (a $\Rightarrow$ b). First note that by [3, Lemma 2.1] we have  $\text{Ann}_C(f) \subseteq \text{Ann}_C(g)$  if and only if  $\text{int}_X Z(f) \subseteq \text{int}_X Z(g)$  for each  $f, g \in C(X)$ . Now, let  $I$  be a  $z_R^\circ$ -ideal in  $R$  and  $\text{Ann}_C(f) \subseteq \text{Ann}_C(g)$  where  $f \in I$  and  $g \in R$ . Thus, by our hypothesis, we have  $\text{int}_X Z(f) \subseteq \text{int}_X Z(g)$  which implies that  $g \in I$ .

(b $\Rightarrow$ c). By [3, Proposition 2.3], we have  $P_f(C) = \{g \in C(X) : \text{Ann}_C(f) \subseteq \text{Ann}_C(g)\}$ . Thus the proof is evident.

(c $\Rightarrow$ d). Let  $P_g(C) \cap R \subseteq P_f(C) \cap R$ , where  $f \in I$  and  $g \in R$ . As  $f \in I$ , by our hypothesis,  $P_f(C) \cap R \subseteq I$  and thus  $P_g(C) \cap R \subseteq I$  which implies that  $g \in I$ .

(d $\Rightarrow$ a). Let  $\text{int}_X Z(f) \subseteq \text{int}_X Z(g)$  where  $f \in I$  and  $g \in R$ . Therefore, by [3, Lemma 2.1], we have  $P_f(C) \subseteq P_g(C)$  and hence  $P_f(C) \cap R \subseteq P_g(C) \cap R$ . Thus we are done by our hypothesis.  $\square$

**Lemma 4.3.** *Let  $R$  be a subring of  $\mathbb{R}^X$ , then for each  $f \in R$  we have  $P_f(C) \subseteq P_f(R)$ .*

*Proof.* Let  $g \in P_f(C)$ . By [3, Proposition 2.3.], we have  $\text{Ann}_C(f) \subseteq \text{Ann}_C(g)$ . Therefore,  $\text{Ann}_R(f) = \text{Ann}_C(f) \cap R \subseteq \text{Ann}_C(g) \cap R = \text{Ann}_R(g)$ . Thus, by [2, Proposition 1.5] we are done.  $\square$

**Theorem 4.4.** *Let  $R$  be a subring of  $\mathbb{R}^X$ . Then every  $z_R^\circ$ -ideal in  $R$  is a  $z^\circ$ -ideal if and only if  $P_f(R) = P_f(C)$  for each  $f \in R$ .*

*Proof.* ( $\Rightarrow$ ). Assume on the contrary that there exists some  $f \in R$  such that  $P_f(R) \neq P_f(C)$ . Thus, using Theorem 4.2 we have  $P_f(C) \subseteq P_f(R)$ . Again by Theorem 4.2,  $P_f(C) \cap R$  is a  $z_R^\circ$ -ideal in  $R$ . Also, it is clear that this ideal is not a  $z^\circ$ -ideal, since,  $P_f(R) \not\subseteq P_f(C) \cap R$ .

( $\Leftarrow$ ). Let  $I$  be a  $z_R^\circ$ -ideal in  $R$  and  $f \in I$ . By Theorem 4.2,  $P_f(C) \cap R \subseteq I$ . Thus, by our hypothesis,  $P_f(R) \subseteq I$  which means that  $I$  is a  $z^\circ$ -ideal in  $R$ .  $\square$

From Theorem 4.2 it follows that every  $z^\circ$ -ideal in a subring  $R$  is a  $z_R^\circ$ -ideal. However, the converse of this fact does not hold, in general. The following example gives an example of a subring  $R$  which has a  $z_R^\circ$ -ideal that is not a  $z^\circ$ -ideal.

**EXAMPLE 4.5.** Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be defined by  $f(x) = \begin{cases} x & x > 0 \\ 0 & x \leq 0 \end{cases}$ . It is clear

that  $f \in C(\mathbb{R})$ . Now, let  $R = \{\sum_{i=0}^n r_i f^i : r_i \in \mathbb{R}, n = 0, 1, \dots\}$ . It is easy to see that  $P_f(R) = R$ , however,  $P_f(C) \cap R \neq R$ . Also, by Theorem 4.2,  $P_f(C) \cap R$  is  $z_R^\circ$ -ideal and it is clear that this ideal is not a  $z^\circ$ -ideal.

The next theorem gives a sufficient conditions on  $X$  in order that  $z_R^\circ$ -ideals in a subring  $R$  coincide with  $z^\circ$ -ideals of  $R$ .

**Theorem 4.6.** *Let  $R$  be a subring of  $\mathbb{R}^X$  and  $X$  be equipped with the  $\text{Coz}(R)$ -topology. Then an ideal  $I$  in  $R$  is a  $z^\circ$ -ideal if and only if it is a  $z_R^\circ$ -ideal.*

*Proof.* Let  $I$  be a  $z_R^\circ$ -ideal in  $R$  and  $f \in I$ . As  $X$  is equipped with the  $\text{Coz}(R)$ -topology, we have  $g \in \text{Ann}_R(f)$  if and only if  $\text{Coz}(g) \subseteq \text{int}_X Z(f)$  for each  $f, g \in R$ . Therefore,  $P_f(R) = \text{Ann}_R \text{Ann}_R(f) = \{g \in R : \text{Coz}(g) \cap \text{int}_X Z(f) = \emptyset\} = \{g \in R : \text{Ann}_R(f) \subseteq \text{Ann}_R(g)\}$ . Hence,  $P_f(R) \subseteq I$  which means that  $I$  is a  $z^\circ$ -ideal in  $R$ . This completes the proof, since, as former stated, every  $z^\circ$ -ideal in  $R$  is a  $z_R^\circ$ -ideal.  $\square$

Note that the condition that  $X$  is equipped with the  $\text{Coz}(R)$ -topology is a sufficient condition for coincidence of  $z_R^\circ$ -ideals with  $z^\circ$ -ideals in a given subring  $R$ . The next example shows that this condition is not necessary.

EXAMPLE 4.7. Let  $X = \mathbb{R} \setminus \{0\}$  with the topology inherits from the usual topology on  $\mathbb{R}$ . Also, let  $f : X \rightarrow \mathbb{R}$  be defined by  $f(x) = \begin{cases} 1 & x > 0 \\ 0 & x < 0 \end{cases}$ . It is clear that  $f \in C(X)$  and  $f^2 = f$ . Now, set  $R = \{r + sf : r, s \in \mathbb{R}\}$ . It is clear that  $R$  is a subring of  $C(X)$ . Also, by a routine reasoning, one can prove that the only ideals of  $R$  are the ideals  $(0)$ ,  $(f)$ ,  $(1 - f)$  and  $R$ . Moreover, the minimal prime ideals of  $R$  are only the ideals  $(f)$  and  $(1 - f)$ . These imply that every  $z_R^\circ$ -ideal is a  $z^\circ$ -ideal in  $R$ . However, clearly,  $X$  is not equipped with the  $\text{Coz}(R)$ -topology.

It follows from Theorem 4.6 that for an intermediate subalgebra  $A(X)$  of  $C(X)$ ,  $z_A^\circ$ -ideals coincide with  $z^\circ$ -ideals of  $A(X)$ . However, the same statement does not true for  $z_A$ -ideals and  $z$ -ideals in  $A(X)$ , in general, see [6, Theorem 2.2]. Moreover, Theorem 3.7 together with Theorem 4.6 imply that in the subalgebras of  $C(X)$  which are of the form  $\mathbb{R} + I$ , where  $I$  is a free ideal in  $C(X)$ ,  $z_{\mathbb{R}+I}$ -ideals coincide with  $z$ -ideals of  $\mathbb{R} + I$  and  $z_{\mathbb{R}+I}^\circ$ -ideals coincide with  $z^\circ$ -ideals, too. Note that whenever  $I$  is a free ideal in  $C(X)$ , then  $\mathbb{R} + I$  determines the topology of  $X$ .

#### ACKNOWLEDGMENTS

The authors would like to thank to the referee for careful reading the paper.

#### REFERENCES

1. A.R. Aliabad, M. Parsinia, Remarks on Subrings of  $C(X)$  of the Form  $I + C^*(X)$ , *Quaest. Math.*, **40**(1), (2017), 63-73.
2. F. Azarpanah, O.A.S. Karamzadeh, A. Rezaei Aliabad, On Ideals Consisting Entirely of Zerodivisor, *Commun. Algebra*, **28**(2), (2000), 1061-1073.

3. F. Azarpanah, O.A.S. Karamzadeh, A. Rezaei Aliabad, On  $z^\circ$ -Ideals in  $C(X)$ , *Fund. Math.*, **160**, (1999), 15-25.
4. F. Azarpanah, M. Karavan, On Nonregular Ideals and  $z^\circ$ -Ideals in  $C(X)$ , *Cech. Math.*, **55**, (130), (2005), 397-407.
5. F. Azarpanah, R. Mohamadian,  $\sqrt{z}$ -Ideals and  $\sqrt{z^\circ}$ -Ideals in  $C(X)$ , *Acta. Math. Sinica*, English Series, **23** (2007), 989-1006.
6. F. Azarpanah, M. Parsinia, On the Sum of  $z$ -Ideals in Subring of  $C(X)$ , *J. Commut. Algebra*, to appear.
7. H.L. Byun, S. Watson, Prime and Maximal Ideals in Subrings of  $C(X)$ , *Topology. Appl.*, **40**, (1991), 45-62.
8. J.M. Dominguez, J. Gomez Perez, M.A. Mulero, Intermediate Algebras between  $C^*(X)$  and  $C(X)$  as Rings of Fractions of  $C^*(X)$ , *Topology Appl.*, **77**, (1997), 115-130.
9. M. Ghadermazi, O.A.S. Karamzadeh, M. Namdari, On the Functionally Countable Subalgebra of  $C(X)$ , *Rend. Semin. Math. Univ. Padova*, **129**, (2013), 47-69.
10. L. Gillman, M. Jerison, *Rings of Continuous Functions*, Springer-Verlag, New York., 1978.
11. M. Parsinia, Remarks on  $LBI$ -Subalgebras of  $C(X)$ , *Comment. Math. Univ. Carolin.*, **57**, (2016), 261-270.
12. D. Plank, On a Class of Subalgebras of  $C(X)$  with Application to  $\beta X \setminus X$ , *Fund. Math.*, **64**, (1969), 41-54.
13. D. Rudd, On Structure Spaces of Ideals in Rings of Continuous Functions, *Trans. Amer. Math. Soc.*, **190**, (1974), 393-403.
14. A.B. Sidlovskij, N. Koblitz., *Transcendental Numbers*, Walter de Gruyter. Berlin. New York, 1989.