

## On $I$ -Statistical Convergence

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ABSTRACT. In this paper we investigate the notion of  $I$ -statistical convergence and introduce  $I$ -st limit points and  $I$ -st cluster points of real number sequence and also studied some of its basic properties.

**Keywords:**  $I$ -limit point,  $I$ -cluster point,  $I$ -statistically Convergent.

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### 1. INTRODUCTION

In 1951 Fast [6] and Steinhaus [18] introduced the concept of statistical convergence independently and established a relation with summability. Later on it was further investigated from sequence space point of view by Fridy [8], Salat [19] and many others. Some applications of statistical convergence in number theory and mathematical analysis can be found in [1, 2, 13, 14, 21].

The notion of  $I$ -convergence is a generalization of the statistical convergence which was introduced by Kostyrko et al. [12]. They used the notion of an ideal  $I$  of subsets of the set  $N$  to define such a concept. For an extensive view of this article we refer [4, 11, 20].

The idea of  $I$ -convergence was further extended to  $I$ -statistical convergence by Savas and Das [16]. Later on more investigation in this direction was done

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by Savas and Das [17], Debnath and Debnath [3], Mursaleen et.al [15], Et et al. [5] and many others [9, 10, 22, 23]. In [16], Savas and Das introduced the  $I$ -statistical convergence and  $I$ - $\lambda$ -statistical convergence and the relation between them. Also they studied these concept in the notion of  $[V, \lambda]$ -summability method.

In the present paper we return to the view of  $I$ -statistical convergence as a sequential limit concept and we extend this concept in a natural way to define a  $I$ -statistical analogue of the set of limit points and cluster points of a real number sequence.

## 2. DEFINITIONS AND PRELIMINARIES

**Definition 2.1.** [8] If  $K$  is a subset of the positive integers  $N$ , then  $K_n$  denotes the set  $\{k \in K : k \leq n\}$ . The natural density of  $K$  is given by  $D(K) = \lim_{n \rightarrow \infty} \frac{|K_n|}{n}$ .

**Definition 2.2.** [8] A sequence  $(x_n)$  is said to be statistically convergent to  $x_0$  if for each  $\varepsilon > 0$ , the set  $A(\varepsilon) = \{k \in N : d(x_k, x_0) \geq \varepsilon\}$  has natural density zero.  $x_0$  is called the statistical limit of the sequence  $(x_n)$  and we write  $\text{st-}\lim_{n \rightarrow \infty} x_n = x_0$ .

**Definition 2.3.** [7] If  $(x_{k(j)})$  be a subsequence of a sequence  $x = (x_n)$  and density of  $K = \{k(j) : j \in N\}$  is zero then  $(x_{k(j)})$  is called a thin subsequence. Otherwise  $(x_{k(j)})$  is called a non-thin subsequence of  $x$ .

$x_0$  is said to be a statistical limit point of a sequence  $(x_n)$ , if there exist a non-thin subsequence of  $(x_n)$  which converges to  $x_0$ .

Let  $A_x$  denotes the set of all statistical limit points of the sequence  $(x_n)$ .

**Definition 2.4.** [7]  $x_0$  is said to be a statistical cluster point of a sequence  $x = (x_n)$ , provided that for each  $\varepsilon > 0$  the density of the set  $\{k \in N : d(x_k, x_0) < \varepsilon\}$  is not equal to 0.

Let  $\Gamma_x$  denotes the set of all statistical cluster points of the sequence  $(x_n)$ .

**Definition 2.5.** [12] Let  $X$  is a non-empty set. A family of subsets  $I \subset P(X)$  is called an ideal on  $X$  if and only if

- (i)  $\emptyset \in I$ ;
- (ii) for each  $A, B \in I$  implies  $A \cup B \in I$ ;
- (iii) for each  $A \in I$  and  $B \subset A$  implies  $B \in I$ .

**Definition 2.6.** [12] Let  $X$  is a non-empty set. A family of subsets  $\mathcal{F} \subset P(X)$  is called a filter on  $X$  if and only if

- (i)  $\emptyset \notin \mathcal{F}$ ;
- (ii) for each  $A, B \in \mathcal{F}$  implies  $A \cap B \in \mathcal{F}$ ;
- (iii) for each  $A \in \mathcal{F}$  and  $B \supset A$  implies  $B \in \mathcal{F}$ .

An ideal  $I$  is called non-trivial if  $I \neq \emptyset$  and  $X \notin I$ . The filter  $\mathcal{F} = \mathcal{F}(I) = \{X - A : A \in I\}$  is called the filter associated with the ideal  $I$ . A non-trivial ideal  $I \subset P(X)$  is called an admissible ideal in  $X$  if and only if  $I \supset \{\{x\} : x \in X\}$

**Definition 2.7.** [12] Let  $I \subset P(N)$  be a non-trivial ideal on  $N$ . A sequence  $(x_n)$  is said to be  $I$ -convergent to  $x_0$  if for each  $\varepsilon > 0$ , the set  $A(\varepsilon) = \{k \in N : d(x_k, x_0) \geq \varepsilon\}$  belongs to  $I$ .  $x_0$  is called the  $I$ -limit of the sequence  $(x_n)$  and we write  $I\text{-}\lim_{n \rightarrow \infty} x_n = x_0$ .

**Definition 2.8.** [12]  $x_0$  is said to be  $I$ -limit point of a sequence  $x = (x_n)$  provided that there is a subset  $K = \{k_1 < k_2 < \dots\} \subset N$  such that  $K \notin I$  and  $\lim x_{k_i} = x_0$ .

Let  $I(A_x)$  denotes the set of all  $I$ -limit points of the sequence  $x$ .

**Definition 2.9.** [12]  $x_0$  is said to be  $I$ -cluster point of a sequence  $x = (x_n)$  provided that for each  $\varepsilon > 0$  the set  $\{k \in N : d(x_k, x_0) < \varepsilon\} \notin I$ .

Let  $I(I_x)$  denotes the set of all  $I$ -cluster points of the sequence  $x$ .

**Definition 2.10.** [16] A sequence  $x = (x_n)$  is said to be  $I$ -statistically convergent to  $x_0$  if for every  $\varepsilon > 0$  and every  $\delta > 0$ ,

$$\left\{n \in N : \frac{1}{n} |\{k \leq n : d(x_k, x_0) \geq \varepsilon\}| \geq \delta\right\} \in I.$$

$x_0$  is called  $I$ -statistical limit of the sequence  $(x_n)$  and we write,  $I\text{-st}\lim x_n = x_0$ .

Throughout the paper we consider  $I$  as an admissible ideal.

### 3. MAIN RESULTS

**Theorem 3.1.** If  $(x_n)$  be a sequence such that  $I\text{-st}\lim x_n = x_0$ , then  $x_0$  determined uniquely.

*Proof.* If possible let the sequence  $(x_n)$  be  $I$ -statistically convergent to two different numbers  $x_0$  and  $y_0$

i.e, for any  $\varepsilon > 0, \delta > 0$  we have,

$$A_1 = \left\{n \in N : \frac{1}{n} |\{k \leq n : d(x_k, x_0) \geq \varepsilon\}| < \delta\right\} \in \mathcal{F}(I)$$

$$\text{and } A_2 = \left\{n \in N : \frac{1}{n} |\{k \leq n : d(x_k, y_0) \geq \varepsilon\}| < \delta\right\} \in \mathcal{F}(I)$$

Therefore,  $A_1 \cap A_2 \neq \emptyset$ , since  $A_1 \cap A_2 \in \mathcal{F}(I)$ .

Let  $m \in A_1 \cap A_2$  and take  $\varepsilon = \frac{d(x_0, y_0)}{3} > 0$

$$\text{so, } \frac{1}{m} |\{k \leq m : d(x_k, x_0) \geq \varepsilon\}| < \delta$$

$$\text{and } \frac{1}{m} |\{k \leq m : d(x_k, y_0) \geq \varepsilon\}| < \delta$$

i.e, for maximum  $k \leq m$  will satisfy  $d(x_k, x_0) < \varepsilon$  and  $d(x_k, y_0) < \varepsilon$  for a very small  $\delta > 0$ .

Thus, we must have

$\{k \leq m : d(x_k, x_0) < \varepsilon\} \cap \{k \leq m : d(x_k, y_0) < \varepsilon\} \neq \emptyset$  a contradiction, as the neighbourhood of  $x_0$  and  $y_0$  are disjoint.

Hence the theorem is proved.  $\square$

**Theorem 3.2.** For any sequence  $(x_n)$ ,  $st\text{-}lim x_n = x_0$  implies  $I\text{-}st\text{-}lim x_n = x_0$ .

*Proof.* Let  $st\text{-}lim x_n = x_0$ .

Then for each  $\varepsilon > 0$ , the set  $A(\varepsilon) = \{k \leq n : d(x_k, x_0) \geq \varepsilon\}$  has natural density zero.

i.e,  $\lim_{n \rightarrow \infty} \frac{1}{n} |\{k \leq n : d(x_k, x_0) \geq \varepsilon\}| = 0$

So for every  $\varepsilon > 0$  and  $\delta > 0$ ,

$\{n \in N : \frac{1}{n} |\{k \leq n : d(x_k, x_0) \geq \varepsilon\}| \geq \delta\}$  is a finite set and therefore belongs to  $I$ , as  $I$  is an admissible ideal.

Hence  $I\text{-}st\text{-}lim x_n = x_0$ .  $\square$

But the converse is not true.

EXAMPLE 3.3. Let  $I = \zeta$  be the class of  $A \subset N$  that intersect a finite number of  $\Delta_j$ 's where  $N = \cup_{j=1}^{\infty} \Delta_j$  and  $\Delta_i \cap \Delta_j = \emptyset$  for  $i \neq j$ .

Let  $x_n = \frac{1}{n}$  and so  $\lim_{n \rightarrow \infty} d(x_n, 0) = 0$ . Put  $\epsilon_n = d(x_n, 0)$  for  $n \in N$ .

Now define a sequence  $(y_n)$  by  $y_n = x_j$  if  $n \in \Delta_j$

Let  $\eta > 0$ . Choose  $\nu \in N$  such that  $\epsilon_\nu < \eta$ . Then

$A(\eta) = \{n : d(y_n, 0) \geq \eta\} \subset \Delta_1 \cup \dots \cup \Delta_\nu \in \zeta$ .

Now,  $\{k \leq n : d(y_k, 0) \geq \eta\} \subseteq \{n \in N : d(y_n, 0) \geq \eta\}$

i.e,  $\frac{1}{n} |\{k \leq n : d(y_k, 0) \geq \eta\}| \leq |\{n \in N : d(y_n, 0) \geq \eta\}|$

so for any  $\delta > 0$ ,

$\{n \in N : \frac{1}{n} |\{k \leq n : d(y_k, 0) \geq \eta\}| \geq \delta\} \subseteq \{n \in N : d(y_n, 0) \geq \eta\} \in \zeta$ .

Therefore  $(y_n)$  is  $\zeta$ -statistically convergent to 0.

But  $(y_n)$  is not a statistically convergent.

**Theorem 3.4.** For any sequence  $(x_n)$ ,  $I\text{-}lim x_n = x_0$  implies  $I\text{-}st\text{-}lim x_n = x_0$ .

*Proof.* The proof is obvious. But the converse is not true.  $\square$

EXAMPLE 3.5. If we take  $I = I_f$  the sequence  $(x_n)$ ,

$$\text{where } x_n = \begin{cases} 0, & n = k^2, k \in N \\ 1, & \text{otherwise} \end{cases}$$

is  $I$ -statistically convergent to 1. But  $(x_n)$  is not  $I$ -convergent.

**Theorem 3.6.** If each subsequence of  $(x_n)$  is  $I$ -statistically convergent to  $\xi$  then  $(x_n)$  is also  $I$ -statistically convergent to  $\xi$ .

*Proof.* Suppose  $(x_n)$  is not  $I$ -statistically convergent to  $\xi$ , then there exists  $\varepsilon > 0$  and  $\delta > 0$  such that

$A = \{n \in N : \frac{1}{n} |\{k \leq n : d(x_k, \xi) \geq \varepsilon\}| \geq \delta\} \notin I$ . Since  $I$  is admissible ideal so  $A$  must be an infinite set.

Let  $A = \{n_1 < n_2 < \dots < n_m < \dots\}$ . Let  $y_m = x_{n_m}$  for  $m \in N$ . Then  $(y_m)_{m \in N}$  is a subsequence of  $(x_n)$  which is not  $I$ -statistically convergent to  $\xi$ , a contradiction. Hence the theorem is proved.  $\square$

But the converse is not true. We can easily show this from example 3.5.

**Theorem 3.7.** Let  $(x_n)$  and  $(y_n)$  be two sequences then

(i)  $I\text{-st}\lim x_n = x_0$  and  $c \in R$  implies  $I\text{-st}\lim cx_n = cx_0$ .

(ii)  $I\text{-st}\lim x_n = x_0$  and  $I\text{-st}\lim y_n = y_0$  implies  $I\text{-st}\lim (x_n + y_n) = x_0 + y_0$ .

*Proof.* (i) If  $c = 0$ , we have nothing to prove.

So we assume that  $c \neq 0$ .

$$\begin{aligned} \text{Now, } \frac{1}{n} |\{k \leq n : d(cx_k, cx_0) \geq \varepsilon\}| &= \frac{1}{n} |\{k \leq n : |c|d(x_k, x_0) \geq \varepsilon\}| \\ &\leq \frac{1}{n} |\{k \leq n : d(x_k, x_0) \geq \frac{\varepsilon}{|c|}\}| < \delta \end{aligned}$$

Therefore,  $\{n \in N : \frac{1}{n} |\{k \leq n : d(cx_k, cx_0) \geq \varepsilon\}| < \delta\} \in \mathcal{F}(I)$ .

i.e,  $I\text{-st}\lim cx_n = cx_0$ .

(ii) We have  $A_1 = \{n \in N : \frac{1}{n} |\{k \leq n : d(x_k, x_0) \geq \frac{\varepsilon}{2}\}| < \frac{\delta}{2}\} \in \mathcal{F}(I)$

and  $A_2 = \{n \in N : \frac{1}{n} |\{k \leq n : d(y_k, y_0) \geq \frac{\varepsilon}{2}\}| < \frac{\delta}{2}\} \in \mathcal{F}(I)$ .

Since  $A_1 \cap A_2 \neq \emptyset$ , therefore for all  $n \in A_1 \cap A_2$  we have,

$$\begin{aligned} \frac{1}{n} |\{k \leq n : d(x_k + y_k, x_0 + y_0) \geq \varepsilon\}| \\ \leq \frac{1}{n} |\{k \leq n : d(x_k, x_0) \geq \frac{\varepsilon}{2}\}| + \frac{1}{n} |\{k \leq n : d(y_k, y_0) \geq \frac{\varepsilon}{2}\}| < \delta. \end{aligned}$$

i.e,  $\{n \in N : \frac{1}{n} |\{k \leq n : d(x_k + y_k, x_0 + y_0) \geq \varepsilon\}| < \delta\} \in \mathcal{F}(I)$ .

Hence  $I\text{-st}\lim (x_n + y_n) = (x_0 + y_0)$ .  $\square$

**Definition 3.8.** A sequence  $x = (x_n)_{n \in N}$  of elements of  $X$  is said to be  $I^*$ -statistical convergent to  $\xi \in X$  if and only if there exists a set  $M = \{m_1 < m_2 < \dots < m_k < \dots\} \in \mathcal{F}(I)$ , such that  $st\text{-}\lim d(x_{m_k}, \xi) = 0$ .

**Theorem 3.9.** If  $I^*\text{-st}\lim_{n \rightarrow \infty} x_n = \xi$  then  $I\text{-st}\lim_{n \rightarrow \infty} x_n = \xi$ .

*Proof.* Let  $I^*\text{-st}\lim_{n \rightarrow \infty} x_n = \xi$ . By assumption there exist a set  $H \in I$  such that for  $M = N \setminus H = \{m_1 < m_2 < \dots < m_k < \dots\}$  we have  $st\text{-}\lim x_{m_k} = \xi$

i.e,  $\lim_{n \rightarrow \infty} \frac{1}{n} |\{m_k \leq n : d(x_{m_k}, \xi) \geq \varepsilon\}| = 0$

so for any  $\delta > 0$ ,  $\{n \in N : \frac{1}{n} |\{m_k \leq n : d(x_{m_k}, \xi) \geq \varepsilon\}| \geq \delta\} \in I$  since  $I$  is an admissible ideal.

Now,  $A(\varepsilon, \delta) = \{n \in N : \frac{1}{n} |\{k \leq n : d(x_k, \xi) \geq \varepsilon\}| \geq \delta\}$

$$\subset H \cup \{n \in N : \frac{1}{n} |\{m_k \leq n : d(x_{m_k}, \xi) \geq \varepsilon\}| \geq \delta\} \in I$$

i.e,  $I\text{-st}\lim_{n \rightarrow \infty} x_n = \xi$ .  $\square$

But the converse may not be true.

From example 3.3. we have  $\zeta\text{-st}\lim_{n \rightarrow \infty} y_n = 0$ .

Suppose that  $\zeta^*\text{-st}\lim_{n \rightarrow \infty} y_n = 0$ . Then there exist a set  $H \in \zeta$  such that for  $M = N \setminus H = \{m_1 < m_2 < \dots < m_k < \dots\}$  we have  $st\text{-}\lim y_{m_k} = 0$ . By definition of  $\zeta$  there exist a  $p \in N$  such that  $H \subset \Delta_1 \cup \dots \cup \Delta_p$ . But then  $\Delta_{p+1} \subset M$ , so for infinitely many  $m_k \in \Delta_{p+1}$ ,

$$D\{m_k \in \Delta_{p+1} : d(y_{m_k}, 0) \geq \eta\} = 2^{-(p+1)} > 0 \text{ for } 0 < \eta < \frac{1}{p+1}$$

i.e,  $D\{m_k \in \Delta_{p+1} : d(y_{m_k}, 0) \geq \eta\} \neq 0$ , which is a contradicts  $st\text{-}\lim y_{m_k} = 0$ .

Hence  $\zeta^*\text{-st}\lim_{n \rightarrow \infty} y_n \neq 0$ .

**Definition 3.10.** An element  $x_0$  is said to be an  $I$ -statistical limit point of a sequence  $x = (x_n)$  provided that for each  $\varepsilon > 0$  there is a set  $M = \{m_1 < m_2 < \dots\} \subset N$  such that  $M \notin I$  and  $st\text{-}\lim x_{m_k} = x_0$ .

$I\text{-}S(A_x)$  denotes the set of all  $I$ -statistical limit points of the sequence  $(x_n)$ .

**Theorem 3.11.** If  $(x_n)$  be a sequence such that  $I\text{-}st\lim x_n = x_0$  then  $I\text{-}S(A_x) = \{x_0\}$ .

*Proof.* Since  $(x_n)$  is  $I$ -statistically convergent to  $x_0$ , so for each  $\varepsilon > 0$  and  $\delta > 0$  the set,

$A = \{n \in N : \frac{1}{n} |\{k \leq n : d(x_k, x_0) \geq \varepsilon\}| \geq \delta\} \in I$ , where  $I$  is an admissible ideal.

Suppose  $I\text{-}S(A_x)$  contains  $y_0$  different from  $x_0$ . i.e,  $y_0 \in I\text{-}S(A_x)$ .

So there exist a  $M \subset N$  such that  $M \notin I$  and  $st\text{-}\lim x_{m_k} = y_0$ .

Let  $B = \{n \in M : \frac{1}{n} |\{k \leq n : d(x_k, y_0) \geq \varepsilon\}| \geq \delta\}$ . So  $B$  is a finite set and therefore  $B \in I$  and so  $B^c = \{n \in M : \frac{1}{n} |\{k \leq n : d(x_k, y_0) \geq \varepsilon\}| < \delta\} \in \mathcal{F}(I)$ .

Again let  $A_1 = \{n \in M : \frac{1}{n} |\{k \leq n : d(x_k, x_0) \geq \varepsilon\}| \geq \delta\}$ . So  $A_1 \subset A \in I$ . i.e,  $A_1^c \in \mathcal{F}(I)$ . Therefore  $A_1^c \cap B^c \neq \emptyset$ , since  $A_1^c \cap B^c \in \mathcal{F}(I)$

Let  $p \in A_1^c \cap B^c$  and take  $\varepsilon = \frac{d(x_0, y_0)}{3} > 0$

so  $\frac{1}{p} |\{k \leq p : d(x_k, x_0) \geq \varepsilon\}| < \delta$

and  $\frac{1}{p} |\{k \leq p : d(x_k, y_0) \geq \varepsilon\}| < \delta$

i.e, for maximum  $k \leq p$  will satisfy  $d(x_k, x_0) < \varepsilon$  and  $d(x_k, y_0) < \varepsilon$  for a very small  $\delta > 0$ .

Thus we must have,

$\{k \leq p : d(x_k, x_0) < \varepsilon\} \cap \{k \leq p : d(x_k, y_0) < \varepsilon\} \neq \emptyset$  a contradiction, as the neighbourhood of  $x_0$  and  $y_0$  are disjoint.

Hence  $I\text{-}S(A_x) = \{x_0\}$ .  $\square$

**Definition 3.12.** [15] An element  $x_0$  is said to be an  $I$ -statistical cluster point of a sequence  $x = (x_n)$  if for each  $\varepsilon > 0$  and  $\delta > 0$

$\{n \in N : \frac{1}{n} |\{k \leq n : d(x_k, x_0) \geq \varepsilon\}| < \delta\} \notin I$ .

$I\text{-}S(\Gamma_x)$  denotes the set of all  $I$ -statistical cluster points of the sequence  $(x_n)$ .

**Theorem 3.13.** For any sequence  $x = (x_n)$ ,  $I\text{-}S(\Gamma_x)$  is closed.

*Proof.* Let  $y_0$  be a limit point of the set  $I\text{-}S(\Gamma_x)$  then for any  $\varepsilon > 0$ ,  $I\text{-}S(\Gamma_x) \cap B(y_0, \varepsilon) \neq \emptyset$ , where  $B(y_0, \varepsilon) = \{z \in R : d(z, y_0) < \varepsilon\}$

Let  $z_0 \in I\text{-}S(\Gamma_x) \cap B(y_0, \varepsilon)$  and choose  $\varepsilon_1 > 0$  such that  $B(z_0, \varepsilon_1) \subseteq B(y_0, \varepsilon)$ .

Then we have  $\{k \leq n : d(x_k, z_0) \geq \varepsilon_1\} \supseteq \{k \leq n : d(x_k, y_0) \geq \varepsilon\}$

$\Rightarrow \frac{1}{n} |\{k \leq n : d(x_k, z_0) \geq \varepsilon_1\}| \geq \frac{1}{n} |\{k \leq n : d(x_k, y_0) \geq \varepsilon\}|$

Now for any  $\delta > 0$ ,

$\{n \in N : \frac{1}{n} |\{k \leq n : d(x_k, z_0) \geq \varepsilon_1\}| < \delta\}$

$$\subseteq \left\{ n \in N : \frac{1}{n} |\{k \leq n : d(x_k, y_0) \geq \varepsilon\}| < \delta \right\}$$

Since  $z_0 \in I\text{-}S(\Gamma_x)$  therefore,  $\left\{ n \in N : \frac{1}{n} |\{k \leq n : d(x_k, y_0) \geq \varepsilon\}| < \delta \right\} \notin I$ .  
i.e,  $y_0 \in I\text{-}S(\Gamma_x)$ . Hence the theorem is proved.  $\square$

**Theorem 3.14.** For any sequence  $x = (x_n)$ ,  $I\text{-}S(\Lambda_x) \subseteq I\text{-}S(\Gamma_x)$ .

*Proof.* Let  $x_0 \in I\text{-}S(\Lambda_x)$ . Then there exist a set  $M = \{m_1 < m_2 < \dots\} \notin I$  such that,  $st\text{-}lim x_{m_k} = x_0 \Rightarrow \lim_{k \rightarrow \infty} \frac{1}{k} |\{m_i \leq k : d(x_{m_i}, x_0) \geq \varepsilon\}| = 0$ .

Take  $\delta > 0$ , so there exist  $k_0 \in N$  such that for  $n > k_0$  we have,

$$\frac{1}{n} |\{m_i \leq n : d(x_{m_i}, x_0) \geq \varepsilon\}| < \delta.$$

$$\text{Let } A = \left\{ n \in N : \frac{1}{n} |\{m_i \leq n : d(x_{m_i}, x_0) \geq \varepsilon\}| < \delta \right\}.$$

Also,  $A \supset M / \{m_1 < m_2 < \dots < m_{k_0}\}$ . Since  $I$  is an admissible ideal and  $M \notin I$ , therefore  $A \notin I$ . So by definition of  $I$ -statistical cluster point  $x_0 \in I\text{-}S(\Gamma_x)$ .

Hence the theorem is proved.  $\square$

**Theorem 3.15.** If  $x = (x_n)$  and  $y = (y_n)$  be two sequences such that

$\{n \in N : x_n \neq y_n\} \in I$ , then

(i)  $I\text{-}S(\Lambda_x) = I\text{-}S(\Lambda_y)$  and (ii)  $I\text{-}S(\Gamma_x) = I\text{-}S(\Gamma_y)$ .

*Proof.* (i) Let  $x_0 \in I\text{-}S(\Lambda_x)$ . So by definition there exist a set

$K = \{k_1 < k_2 < k_3 < \dots\}$  of  $N$  such that  $K \notin I$  and  $st\text{-}lim x_{k_n} = x_0$ .

Since  $\{n \in K : x_n \neq y_n\} \subset \{n \in N : x_n \neq y_n\} \in I$ ,

therefore  $K' = \{n \in K : x_n = y_n\} \notin I$  and  $K' \subseteq K$ .

So we have  $st\text{-}lim y_{k'_n} = x_0$ .

This shows that  $x_0 \in I\text{-}S(\Lambda_y)$  and therefore  $I\text{-}S(\Lambda_x) \subseteq I\text{-}S(\Lambda_y)$ .

By symmetry  $I\text{-}S(\Lambda_y) \subseteq I\text{-}S(\Lambda_x)$ .

Hence  $I\text{-}S(\Lambda_y) = I\text{-}S(\Lambda_x)$ .

(ii) Let  $x_0 \in I\text{-}S(\Gamma_x)$ . So by definition for each  $\varepsilon > 0$  the set,

$$A = \left\{ n \in N : \frac{1}{n} |\{k \leq n : d(x_k, x_0) \geq \varepsilon\}| < \delta \right\} \notin I.$$

Let  $B = \left\{ n \in N : \frac{1}{n} |\{k \leq n : d(y_k, x_0) \geq \varepsilon\}| < \delta \right\}$ . We have to prove that  $B \notin I$ .

Suppose  $B \in I$ . So,  $B^c = \left\{ n \in N : \frac{1}{n} |\{k \leq n : d(y_k, x_0) \geq \varepsilon\}| \geq \delta \right\} \in \mathcal{F}(I)$ .

By hypothesis the set  $C = \{n \in N : x_n = y_n\} \in \mathcal{F}(I)$ .

Therefore  $B^c \cap C \in \mathcal{F}(I)$ . Also it is clear that  $B^c \cap C \subset A^c \in \mathcal{F}(I)$ ,

i.e,  $A \in I$ , which is a contradiction.

Hence  $B \notin I$  and thus the result is proved.  $\square$

**Theorem 3.16.** If  $g$  is a continuous function on  $X$  then it preserves  $I$ -statistical convergence in  $X$ .

*Proof.* Let  $I\text{-}st\text{-}lim_{n \rightarrow \infty} x_n = \xi$ .

Since  $g$  is continuous, then for each  $\varepsilon_1 > 0$ , there exist  $\varepsilon_2 > 0$  such that if  $x \in B(\xi, \varepsilon_1)$  then  $g(x) \in B(g(\xi), \varepsilon_2)$ .

Also we have,

$$C(\varepsilon_1, \delta) = \left\{ n \in N : \frac{1}{n} |\{k \leq n : d(x_k, \xi) \geq \varepsilon_1\}| < \delta \right\} \in \mathcal{F}(I)$$

$$\text{Now, } \{k \leq n : d(x_k, \xi) \geq \varepsilon_1\} \supseteq \{k \leq n : d(g(x_k), g(\xi)) \geq \varepsilon_2\}$$

$$\text{so, } \frac{1}{n} |\{k \leq n : d(x_k, \xi) \geq \varepsilon_1\}| \geq \frac{1}{n} |\{k \leq n : d(g(x_k), g(\xi)) \geq \varepsilon_2\}|$$

$$\text{for } \delta > 0, \left\{ n \in N : \frac{1}{n} |\{k \leq n : d(x_k, \xi) \geq \varepsilon_1\}| < \delta \right\}$$

$$\subseteq \left\{ n \in N : \frac{1}{n} |\{k \leq n : d(g(x_k), g(\xi)) \geq \varepsilon_2\}| < \delta \right\} \in \mathcal{F}(I)$$

since  $C(\varepsilon_1, \delta) \in \mathcal{F}(I)$ .

Hence the theorem is proved.  $\square$

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