On Generalizations of Hadamard Inequalities for Fractional Integrals

Ghulam Farid*, Atiq Ur Rehman, Moquddsa Zahra

Department of Mathematics
COMSATS University Islamabad
Attock Campus, Pakistan.

E-mail: faridphdsms@hotmail.com
E-mail: atiq@mathcity.org
E-mail: moquddsazahra@gmail.com

ABSTRACT. Fejér Hadamard inequality is generalization of Hadamard
inequality. In this paper we prove certain Fejér Hadamard inequalities
for k-fractional integrals. We deduce Fejér Hadamard-type inequalities for
Riemann-Liouville fractional integrals. Also as a special case Hadamard
inequalities for k-fractional as well as fractional integrals are given.

1. INTRODUCTION

Fractional calculus is a branch of mathematics that deals with the equations
concerning integrals and derivatives of fractional orders. The history of frac-
tional calculus is as old as the history of differential calculus. In fact, fractional
calculus is a natural extension of standard mathematics. In view of the fact
that the commencement of the hypothesis of differential and integral calculus,

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*Corresponding Author

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mathematicians such as Euler and Liouville developed their ideas on the calculation of derivatives and integral of non-integers. Possibly the topic would be more suitably called "integration and differentiation of arbitrary order" (see, [4]).

Fractional calculus has lots of applications in the fields of science counting rheology, fluid flow, diffusive transport, electrical networks, electromagnetic theory and probability (see, [3]).

In mathematical analysis inequalities have been fascinating for researchers of all ages (see, [1, 2, 5, 11, 15] and references there in). Now a days fractional integral inequalities especially diverse versions of Hadamard and Ostrowski fractional inequalities have been established (see, [6, 7, 8, 12] and references there in). We are interested to give generalizations of fractional Hadamard inequalities via k-fractional integrals.

In [9] k-fractional Riemann-Liouville integrals are defined.

Let $f \in L_1[a,b]$. Then k-fractional integrals of order $\alpha, k > 0$ with $a \geq 0$ are defined as:

$$I_{a+}^{\alpha,k} f(x) = \frac{1}{k\Gamma_k(\alpha)} \int_a^x (x-t)^{\frac{\alpha}{k}-1} f(t) dt, \quad x > a \tag{1.1}$$

and

$$I_{b-}^{\alpha,k} f(x) = \frac{1}{k\Gamma_k(\alpha)} \int_x^b (t-x)^{\frac{\alpha}{k}-1} f(t) dt, \quad x < b, \tag{1.2}$$

where $\Gamma_k(\alpha)$ is the k-Gamma function defined as:

$$\Gamma_k(\alpha) = \int_0^\infty t^{\alpha-1} e^{-\frac{t^k}{k}} dt,$$

also

$$\Gamma_k(\alpha+k) = \alpha\Gamma_k(\alpha)$$

and $I_{a+}^{0,1} f(x) = I_{b-}^{0,1} f(x) = f(x)$.

For $k = 1$, k-fractional integrals give Riemann-Liouville integrals.

Following results for k-fractional integrals hold [7].

Theorem 1.1. *Let $f : [a,b] \to \mathbb{R}$ be a positive function with $0 \leq a < b$. If f is a convex function on $[a,b]$, then the following inequalities for k-fractional integrals hold:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma_k(\alpha+k)}{2(b-a)^{\frac{\alpha}{k}}} \left[I_{a+}^{\alpha,k} f(b) + I_{b-}^{\alpha,k} f(a)\right] \leq \frac{f(a)+f(b)}{2} \tag{1.3}$$

with $\alpha, k > 0$.

Theorem 1.2. *Let $f : [a, b] \to \mathbb{R}$ be a differentiable mapping on (a, b) with $a < b$. If $|f'|$ is convex on $[a, b]$, then the following inequality for k-fractional integral holds:*

$$\left| \frac{f(a) + f(b)}{2} - \frac{\Gamma_k(\alpha + k)}{2(b-a)^{\frac{\alpha}{k}}} \left[I_{a+}^{\alpha,k} f(b) + I_{b-}^{\alpha,k} f(a) \right] \right| \tag{1.4}$$

$$\leq \frac{b-a}{2(\frac{\alpha}{k}+1)} \left(1 - \frac{1}{2^{\frac{\alpha}{k}}} \right) [|f'(a)| + |f'(b)|]$$

with $\alpha, k > 0$.

There in [7] we remark that for $k = 1$ in above theorems we get the results of [12], and for $\alpha = 1$ along with $k = 1$ we get the classical Hadamard inequality. In [8] the following results related to Fejér Hadamard-type inequalities for fractional integrals are given.

Theorem 1.3. *Let $f : [a, b] \to \mathbb{R}$ be a convex function with $a < b$. If $g : [a, b] \to \mathbb{R}$ is nonnegative, integrable and symmetric to $\frac{a+b}{2}$, then following inequalities for fractional integrals hold:*

$$f \left(\frac{a + b}{2} \right) \left[I_{a+}^{\alpha} g(b) + I_{b-}^{\alpha} g(a) \right]$$

$$\leq \left[I_{a+}^{\alpha} (fg)(b) + I_{b-}^{\alpha} (fg)(a) \right]$$

$$\leq \frac{f(a) + f(b)}{2} \left[I_{a+}^{\alpha} g(b) + I_{b-}^{\alpha} g(a) \right]$$

with $\alpha > 0$.

Theorem 1.4. *Let $f : I \to \mathbb{R}$ be a differentiable mapping on I^o and $f' \in L[a, b]$ with $a < b$. If $|f'|$ is convex on $[a, b]$ and $g : [a, b] \to \mathbb{R}$ is continuous and symmetric to $\frac{a+b}{2}$, then the following inequality for fractional integrals holds:*

$$\left| \left(\frac{f(a) + f(b)}{2} \right) [I_{a+}^{\alpha} g(b) + I_{b-}^{\alpha} g(a)] - [I_{a+}^{\alpha} (fg)(b) + I_{b-}^{\alpha} (fg)(a)] \right|$$

$$\leq \frac{(b-a)^{\alpha+1} \|g\|_{\infty}}{(\alpha+1)\Gamma(\alpha+1)} \left(1 - \frac{1}{2^{\alpha}} \right) [|f'(a)| + |f'(b)|]$$

with $\alpha > 0$.

Theorem 1.5. *Let $f : I \to \mathbb{R}$ be a differentiable mapping on I^o with $a < b$. If $|f'|^q$, $q > 1$ is convex on $[a, b]$ and $g : [a, b] \to \mathbb{R}$ is continuous and symmetric to $\frac{a+b}{2}$, then the following inequality for fractional integrals holds:*

$$\left| \left(\frac{f(a) + f(b)}{2} \right) [I_{a+}^{\alpha} g(b) + I_{b-}^{\alpha} g(a)] - [I_{a+}^{\alpha} (fg)(b) + I_{b-}^{\alpha} (fg)(a)] \right|$$

$$\leq \frac{2(b-a)^{\alpha+1} \|g\|_{\infty}}{(\alpha+1)\Gamma(\alpha+1)(b-a)^{\frac{1}{q}}} \left(1 - \frac{1}{2^{\alpha}} \right) \left(\frac{|f'(a)|^q + |f'(b)|^q}{2} \right)^{\frac{1}{q}}$$

with $\alpha > 0$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Theorem 1.6. *Let $f : I \to \mathbb{R}$ be a differentiable mapping on I^o with $a < b$. If $|f'|^q$, $q > 1$ is convex on $[a, b]$ and $g : [a, b] \to \mathbb{R}$ is continuous and symmetric to $\frac{a+b}{2}$, then the following inequalities for fractional integrals hold:*

$$(i) \left| \left(\frac{f(a) + f(b)}{2} \right) \left[I_{a+}^{\alpha} g(b) + I_{b-}^{\alpha} g(a) \right] - \left[I_{a+}^{\alpha}(fg)(b) + I_{b-}^{\alpha}(fg)(a) \right] \right|$$

$$\leq \frac{2^{\frac{1}{p}}(b-a)^{\alpha+1} \|g\|_{\infty}}{(\alpha p + 1)^{\frac{1}{p}} \Gamma(\alpha + 1)} \left(1 - \frac{1}{2^{\alpha p}} \right)^{\frac{1}{p}} \left(\frac{|f'(a)|^q + |f'(b)|^q}{2} \right)^{\frac{1}{q}}$$

with $\alpha > 0$.

$$(ii) \left| \left(\frac{f(a) + f(b)}{2} \right) \left[I_{a+}^{\alpha} g(b) + I_{b-}^{\alpha} g(a) \right] - \left[I_{a+}^{\alpha}(fg)(b) + I_{b-}^{\alpha}(fg)(a) \right] \right|$$

$$\leq \frac{(b-a)^{\alpha+1} \|g\|_{\infty}}{(\alpha p + 1)^{\frac{1}{p}} \Gamma(\alpha + 1)} \left(\frac{|f'(a)|^q + |f'(b)|^q}{2} \right)^{\frac{1}{q}}$$

with $0 < \alpha \leq 1$, where $\frac{1}{p} + \frac{1}{q} = 1$.

In this paper we give Fejér Hadamard inequality for k-fractional integrals and note that the results in [8] are special cases of these results. Also we present new results as generalizations of Hadamard inequalities for fractional integrals and deduce some results of [12, 7].

2. Main Results

The following lemma is given in [14].

Lemma 2.1. *For $0 < \lambda \leq 1$ and $0 \leq a < b$, we have*

$$|a^{\lambda} - b^{\lambda}| \leq (b - a)^{\lambda}.$$

Here first we prove the following result.

Lemma 2.2. *If $g : [a, b] \to \mathbb{R}$ is integrable and symmetric to $\frac{a+b}{2}$ with $a < b$, then*

$$I_{a+}^{\alpha,k} g(b) = I_{b-}^{\alpha,k} g(a) = \frac{1}{2} [I_{a+}^{\alpha,k} g(b) + I_{b-}^{\alpha,k} g(a)]$$

with $\alpha, k > 0$.

Proof. By symmetricity of g we have $g(a + b - x) = g(x)$, where $x \in [a, b]$. Setting $x = a + b - x$ in the following integral we have

$$\begin{aligned} I_{a+}^{\alpha,k} g(b) &= \frac{1}{k\Gamma_k(\alpha)} \int_a^b (b - x)^{\frac{\alpha}{k} - 1} g(x) dx \\ &= \frac{1}{k\Gamma_k(\alpha)} \int_a^b (x - a)^{\frac{\alpha}{k} - 1} g(a + b - x) dx \\ &= \frac{1}{k\Gamma_k(\alpha)} \int_a^b (x - a)^{\frac{\alpha}{k} - 1} g(x) dx \\ &= I_{b-}^{\alpha,k} g(a). \end{aligned}$$

$\square$

Using the above lemma we prove the following results.

Theorem 2.3. *Let $f : [a, b] \to \mathbb{R}$ be a convex function with $a < b$. If $g : [a, b] \to \mathbb{R}$ is nonnegative, integrable and symmetric to $\frac{a+b}{2}$, then following inequalities for k-fractional integrals hold:*

$$f\left(\frac{a+b}{2}\right)\left[I_{a+}^{\alpha,k}g(b) + I_{b-}^{\alpha,k}g(a)\right] \tag{2.1}$$

$$\leq [I_{a+}^{\alpha,k}(fg)(b) + I_{b-}^{\alpha,k}(fg)(a)]$$

$$\leq \frac{f(a)+f(b)}{2}\left[I_{a+}^{\alpha,k}g(b) + I_{b-}^{\alpha,k}g(a)\right]$$

with $\alpha, k > 0$.

Proof. By convexity of f we have

$$f\left(\frac{a+b}{2}\right) = f\left(\frac{ta + (1-t)b + tb + (1-t)a}{2}\right) \tag{2.2}$$

$$\leq \frac{f(ta + (1-t)b) + f(tb + (1-t)a)}{2},$$

where $t \in [0, 1]$. Multiplying both sides of the above inequality with $2t^{\frac{\alpha}{k}-1}g(tb + (1-t)a)$ and integrating the resulting inequality over $[0, 1]$ we have,

$$2f\left(\frac{a+b}{2}\right)\int_0^1 t^{\frac{\alpha}{k}-1}g(tb + (1-t)a)dt$$

$$\leq \int_0^1 t^{\frac{\alpha}{k}-1}f(ta + (1-t)b)g(tb + (1-t)a)dt$$

$$+ \int_0^1 t^{\frac{\alpha}{k}-1}f(tb + (1-t)a)g(tb + (1-t)a)dt.$$

Putting $x = tb + (1-t)a$ we get

$$\frac{2}{(b-a)^{\frac{\alpha}{k}}}f\left(\frac{a+b}{2}\right)\int_a^b (x-a)^{\frac{\alpha}{k}-1}g(x)dx$$

$$\leq \frac{1}{(b-a)^{\frac{\alpha}{k}}}\left[\int_a^b (x-a)^{\frac{\alpha}{k}-1}f(a+b-x)g(x)dx + \int_a^b (x-a)^{\frac{\alpha}{k}-1}f(x)g(x)dx\right]$$

$$= \frac{1}{(b-a)^{\frac{\alpha}{k}}}\left[\int_a^b (b-x)^{\frac{\alpha}{k}-1}f(x)g(a+b-x)dx + \int_a^b (x-a)^{\frac{\alpha}{k}-1}f(x)g(x)dx\right]$$

$$= \frac{1}{(b-a)^{\frac{\alpha}{k}}}\left[\int_a^b (b-x)^{\frac{\alpha}{k}-1}f(x)g(x)dx + \int_a^b (x-a)^{\frac{\alpha}{k}-1}f(x)g(x)dx\right].$$

By using Lemma 2.2, we get the first inequality of (2.1).

For the second inequality of (2.1) convexity of f gives

$$f(ta + (1-t)b) + f(tb + (1-t)a) \leq f(a) + f(b),$$

where $t \in [0, 1]$. Multiplying both sides of the above inequality with $t^{\frac{\alpha}{k}-1}g(tb + (1-t)a)$ and integrating the resulting inequality over $[0, 1]$ we get

$$\int_0^1 t^{\frac{\alpha}{k}-1} f(ta + (1-t)b)g(tb + (1-t)a)dt$$

$$+ \int_0^1 t^{\frac{\alpha}{k}-1} f(tb + (1-t)a)g(tb + (1-t)a)dt$$

$$\leq (f(a) + f(b)) \int_0^1 t^{\frac{\alpha}{k}-1} g(tb + (1-t)a)dt,$$

which after some computations the result follows. $\square$

Remark 2.4. If we take $k = 1$ in Theorem 2.3, then we get Theorem 1.3. If we take $\alpha = 1$ along with $k = 1$, then we get [8, Theorem 1]. If we take $g(x) = 1$ in Theorem 2.3, then we get inequality (1.3). If we take $g(x) = 1$ along with $k = 1$ in Theorem 2.3, then we get [12, Theorem 2].

Next we need the following lemma.

Lemma 2.5. *Let $f : [a, b] \to \mathbb{R}$ be a differentiable mapping on (a, b) with $a < b$ and $f' \in L[a, b]$. If $g : [a, b] \to \mathbb{R}$ is integrable and symmetric to $\frac{a+b}{2}$, then the following equality for k-fractional integrals hold:*

$$\left(\frac{f(a) + f(b)}{2} \right) [I_{a+}^{\alpha,k} g(b) + I_{b-}^{\alpha,k} g(a)] - [I_{a+}^{\alpha,k}(fg)(b) + I_{b-}^{\alpha,k}(fg)(a)]$$

$$= \frac{1}{k\Gamma_k(\alpha)} \int_a^b \left[\int_a^t (b-s)^{\frac{\alpha}{k}-1} g(s)ds - \int_t^b (s-a)^{\frac{\alpha}{k}-1} g(s)ds \right] f'(t)dt$$

with $\alpha, k > 0$.

Proof. Note that

$$\frac{1}{k\Gamma_k(\alpha)} \int_a^b \left[\int_a^t (b-s)^{\frac{\alpha}{k}-1} g(s)ds - \int_t^b (s-a)^{\frac{\alpha}{k}-1} g(s)ds \right] f'(t)dt$$

$$= \frac{1}{k\Gamma_k(\alpha)} \left[\int_a^b \left(\int_a^t (b-s)^{\frac{\alpha}{k}-1} g(s)ds \right) f'(t)dt \right. \qquad (2.3)$$

$$\left. + \int_a^b \left(- \int_t^b (s-a)^{\frac{\alpha}{k}-1} g(s)ds \right) f'(t)dt \right].$$

By simple calculations one can get

$$\int_a^b \left(\int_a^t (b-s)^{\frac{\alpha}{k}-1} g(s)ds \right) f'(t)dt$$

$$= \left[\left(\int_a^b (b-s)^{\frac{\alpha}{k}-1} g(s)ds \right) f(b) - \int_a^b (b-t)^{\frac{\alpha}{k}-1} fg(t)dt \right]$$

$$= k\Gamma_k(\alpha) \left[f(b) I_{a+}^{\alpha,k} g(b) - I_{a+}^{\alpha,k}(fg)(b) \right]$$

$$= k\Gamma_k(\alpha) \left[\frac{f(b)}{2} [I_{a+}^{\alpha,k} g(b) + I_{b-}^{\alpha,k} g(a)] - I_{a+}^{\alpha,k} fg(b) \right]$$

and

$$\int_a^b \left(- \int_t^b (s-a)^{\frac{\alpha}{k}-1} g(s)ds \right) f'(t)dt$$

$$= \left(\int_a^b (s-a)^{\frac{\alpha}{k}-1} g(s)ds \right) f(a) - \int_a^b (t-a)^{\frac{\alpha}{k}-1}(fg)(t)dt$$

$$= k\Gamma_k(\alpha) \left[\frac{f(a)}{2} [I_{a+}^{\alpha,k} g(b) + I_{b-}^{\alpha,k} g(a)] - I_{a+}^{\alpha,k} fg(a)) \right].$$

Hence using (2.3) implies the result. □

Theorem 2.6. *Let $f : I \to \mathbb{R}$ be a differentiable mapping on I^o the interior of I, and $f' \in L[a,b], a,b \in I^o$ with $a < b$. If $|f'|$ is convex on $[a,b]$ and $g : [a,b] \to \mathbb{R}$ is continuous and symmetric to $\frac{a+b}{2}$, then the following inequality for k-fractional integrals holds:*

$$\left| \left(\frac{f(a)+f(b)}{2} \right) [I_{a+}^{\alpha,k} g(b) + I_{b-}^{\alpha,k} g(a)] - [I_{a+}^{\alpha,k}(fg)(b) + I_{b-}^{\alpha,k}(fg)(a)] \right|$$

$$\leq \frac{(b-a)^{\frac{\alpha}{k}+1} \|g\|_\infty}{(\frac{\alpha}{k}+1)\Gamma_k(\alpha+k)} \left(1 - \frac{1}{2^{\frac{\alpha}{k}}} \right) [|f'(a)| + |f'(b)|]$$

with $\alpha, k > 0$.

Proof. Using Lemma 2.5 we have

$$\left| \left(\frac{f(a)+f(b)}{2} \right) [I_{a+}^{\alpha,k} g(b) + I_{b-}^{\alpha,k} g(a)] - [I_{a+}^{\alpha,k}(fg)(b) + I_{b-}^{\alpha,k}(fg)(a)] \right|$$

$$\leq \frac{1}{k\Gamma_k(\alpha)} \int_a^b \left| \int_a^t (b-s)^{\frac{\alpha}{k}-1} g(s)ds - \int_t^b (s-a)^{\frac{\alpha}{k}-1} g(s)ds \right| |f'(t)|dt. \quad (2.4)$$

Using convexity of $|f'|$ we have

$$|f'(t)| \leq \frac{b-t}{t-a} |f'(a)| + \frac{t-a}{b-a} |f'(b)|, \quad (2.5)$$

where $t \in [a,b]$.

From symmetricity of g we have

$$\int_t^b (s-a)^{\frac{\alpha}{k}-1} g(s) ds = \int_a^{a+b-t} (b-s)^{\frac{\alpha}{k}-1} g(a+b-s) ds$$

$$= \int_a^{a+b-t} (b-s)^{\frac{\alpha}{k}-1} g(s) ds.$$

This gives

$$\left| \int_a^t (b-s)^{\frac{\alpha}{k}-1} g(s) ds - \int_t^b (s-a)^{\frac{\alpha}{k}-1} g(s) ds \right| \tag{2.6}$$

$$= \left| \int_t^{a+b-t} (b-s)^{\frac{\alpha}{k}-1} g(s) ds \right| \leq \begin{cases} \int_t^{a+b-t} |(b-s)^{\frac{\alpha}{k}-1} g(s)| ds, & t \in [a, \frac{a+b}{2}] \\ \int_{a+b-t}^t |(b-s)^{\frac{\alpha}{k}-1} g(s)| ds, & t \in [\frac{a+b}{2}, b]. \end{cases}$$

By virtue of (2.4), (2.5), (2.6), we have

$$\left| \left(\frac{f(a)+f(b)}{2} \right) \left[I_{a+}^{\alpha,k} g(b) + I_{b-}^{\alpha,k} g(a) \right] - \left[I_{a+}^{\alpha,k}(fg)(b) + I_{b-}^{\alpha,k}(fg)(a) \right] \right|$$

$$\leq \frac{1}{k\Gamma_k(\alpha)} \left[\int_a^{\frac{a+b}{2}} \left(\int_t^{a+b-t} \left| (b-s)^{\frac{\alpha}{k}-1} g(s) \right| ds \right) \left(\frac{b-t}{t-a} |f'(a)| + \frac{t-a}{b-a} |f'(b)| \right) dt \right.$$

$$+ \int_{\frac{a+b}{2}}^b \left(\int_{a+b-t}^t |(b-s)^{\frac{\alpha}{k}-1} g(s) ds| \right) \left(\frac{b-t}{t-a} |f'(a)| + \frac{t-a}{b-a} |f'(b)| \right) dt \right]$$

$$\leq \frac{\|g\|_\infty}{\Gamma_k(\alpha+k)(b-a)} \left[\int_a^{\frac{a+b}{2}} \left((b-t)^{\frac{\alpha}{k}} - (t-a)^{\frac{\alpha}{k}} \right) \left((b-t)|f'(a)| + (t-a)|f'(b)| \right) dt \right.$$

$$+ \int_{\frac{a+b}{2}}^b \left((t-a)^{\frac{\alpha}{k}} - (b-t)^{\frac{\alpha}{k}} \right) \left((b-t)|f'(a)| + (t-a)|f'(b)| \right) dt \right]. \tag{2.7}$$

One can have

$$\int_a^{\frac{a+b}{2}} ((b-t)^{\frac{\alpha}{k}} - (t-a)^{\frac{\alpha}{k}})(b-t) dt = \int_{\frac{a+b}{2}}^b ((t-a)^{\frac{\alpha}{k}} - (b-t)^{\frac{\alpha}{k}})(t-a) dt$$

$$= \frac{(b-a)^{\frac{\alpha}{k}+2}}{\frac{\alpha}{k}+1} \left(\frac{\frac{\alpha}{k}+1}{\frac{\alpha}{k}+1} - \frac{1}{2^{\frac{\alpha}{k}+1}} \right) \tag{2.8}$$

and

$$\int_a^{\frac{a+b}{2}} ((b-t)^{\frac{\alpha}{k}} - (t-a)^{\frac{\alpha}{k}})(t-a) dt = \int_{\frac{a+b}{2}}^b ((t-a)^{\frac{\alpha}{k}} - (b-t)^{\frac{\alpha}{k}})(b-t) dt$$

$$= \frac{(b-a)^{\frac{\alpha}{k}+2}}{(\frac{\alpha}{k}+1)} \left(\frac{1}{\frac{\alpha}{k}+2} - \frac{1}{2^{\frac{\alpha}{k}+1}} \right). \tag{2.9}$$

Using (2.8), (2.9) in (2.7) we get required result.   □

Remark 2.7. If we take $k=1$, then we get Theorem 1.4. If we take $g(x)=1$ in the above theorem, then we get inequality (1.4). If we take $g(x)=1$ along with $k=1$ in above theorem, then we get [12, Theorem 3].

Theorem 2.8. *Let $f : I \to \mathbb{R}$ be a differentiable mapping on I^o the interior of I, and $f' \in L[a,b], a, b \in I^o$ with $a < b$. If $|f'|^q, q > 1$ is convex on $[a,b]$ and $g : [a,b] \to \mathbb{R}$ is continuous and symmetric to $\frac{a+b}{2}$, then the following inequality for k-fractional integrals holds:*

$$\left| \left(\frac{f(a) + f(b)}{2} \right) \left[I_{a+}^{\alpha,k} g(b) + I_{b-}^{\alpha,k} g(a) \right] - \left[I_{a+}^{\alpha,k}(fg)(b) + I_{b-}^{\alpha,k}(fg)(a) \right] \right|$$

$$\leq \frac{2(b-a)^{\frac{\alpha}{k}+1} \|g\|_\infty}{(\frac{\alpha}{k}+1)\Gamma_k(\alpha+k)(b-a)^{\frac{1}{q}}} \left(1 - \frac{1}{2^{\frac{\alpha}{k}}} \right) \left(\frac{|f'(a)|^q + |f'(b)|^q}{2} \right)^{\frac{1}{q}}$$

with $\alpha, k > 0$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. By Using Lemma 2.5, Hölder inequality, inequality (2.6) and convexity of $|f'|^q$ we have

$$\left| \left(\frac{f(a) + f(b)}{2} \right) \left[I_{a+}^{\alpha,k} g(b) + I_{b-}^{\alpha,k} g(a) \right] - \left[I_{a+}^{\alpha,k}(fg)(b) + I_{b-}^{\alpha,k}(fg)(a) \right] \right|$$

$$\leq \frac{1}{k\Gamma_k(\alpha)} \left[\int_a^b \left| \int_t^{a+b-t} (b-s)^{\frac{\alpha}{k}-1} g(s) ds \right| dt \right]^{1-\frac{1}{q}}$$

$$\left[\int_a^b \left| \int_t^{a+b-t} (b-s)^{\frac{\alpha}{k}-1} g(s) ds \right| |f'(t)|^q dt \right]^{\frac{1}{q}}$$

$$\leq \frac{1}{k\Gamma_k(\alpha)} \left[\int_a^{\frac{a+b}{2}} \left(\int_t^{a+b-t} |(b-s)^{\frac{\alpha}{k}-1} g(s)| ds \right) dt \right.$$

$$\left. + \int_{\frac{a+b}{2}}^b \left(\int_{a+b-t}^t |(b-s)^{\frac{\alpha}{k}-1} g(s)| ds \right) dt \right]^{1-\frac{1}{q}}$$

$$\left[\int_a^{\frac{a+b}{2}} \left(\int_t^{a+b-t} |(b-s)^{\frac{\alpha}{k}-1} g(s)| ds \right) |f'(t)|^q dt \right.$$

$$\left. + \int_{\frac{a+b}{2}}^b \left(\int_{a+b-t}^t |(b-s)^{\frac{\alpha}{k}-1} g(s)| ds \right) |f'(t)|^q dt \right]^{\frac{1}{q}}$$

$$\leq \frac{\|g\|_\infty}{k\Gamma_k(\alpha)} \left[\left(\frac{2k(b-a)^{\frac{\alpha}{k}+1}}{\alpha(\frac{\alpha}{k}+1)} \left(1 - \frac{1}{2^{\frac{\alpha}{k}}} \right) \right)^{1-\frac{1}{q}} \right.$$

$$\left. \left(\frac{k(|f'(a)|^q + |f'(b)|^q)(b-a)^{\frac{\alpha}{k}+1}}{\alpha(\frac{\alpha}{k}+1)(b-a)^{\frac{1}{q}}} \left(1 - \frac{1}{2^{\frac{\alpha}{k}}} \right) \right)^{\frac{1}{q}} \right].$$

From which after a little computation one can have required result.   □

Theorem 2.9. *Let $f : I \to \mathbb{R}$ be a differential mapping on I^o the interior of I, and $f' \in L[a,b], a, b \in I^o$ with $a < b$. If $|f'|^q, q > 1$ is convex on $[a,b]$ and*

$g : [a, b] \to \mathbb{R}$ *is continuous and symmetric to* $\frac{a+b}{2}$*, then following inequalities*
for k-fractional integrals hold:

$$(i) \left| \left(\frac{f(a) + f(b)}{2} \right) \left[I_{a+}^{\alpha,k} g(b) + I_{b-}^{\alpha,k} g(a) \right] - \left[I_{a+}^{\alpha,k}(fg)(b) + I_{b-}^{\alpha,k}(fg)(a) \right] \right|$$

$$\leq \frac{2^{\frac{1}{p}}(b-a)^{\frac{\alpha}{k}+1}\|g\|_\infty}{(\frac{\alpha p}{k}+1)^{\frac{1}{p}}\Gamma_k(\alpha+k)} \left(1 - \frac{1}{2^{\frac{\alpha p}{k}}} \right)^{\frac{1}{p}} \left(\frac{|f'(a)|^q + |f'(b)|^q}{2} \right)^{\frac{1}{q}}$$

with $\alpha, k > 0$.

$$(ii) \left| \left(\frac{f(a) + f(b)}{2} \right) \left[I_{a+}^{\alpha,k} g(b) + I_{b-}^{\alpha,k} g(a) \right] - \left[I_{a+}^{\alpha,k}(fg)(b) + I_{b-}^{\alpha,k}(fg)(a) \right] \right|$$

$$\leq \frac{(b-a)^{\frac{\alpha}{k}+1}\|g\|_\infty}{(\frac{\alpha p}{k}+1)^{\frac{1}{p}}\Gamma_k(\alpha+k)} \left(\frac{|f'(a)|^q + |f'(b)|^q}{2} \right)^{\frac{1}{q}} \tag{2.10}$$

with $0 < \alpha \leq 1$*, where* $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. By Using Lemma 2.5, Hölder inequality, inequality (2.6) and convexity
of $|f'|^q$ we have

$$\left| \left(\frac{f(a) + f(b)}{2} \right) \left[I_{a+}^{\alpha,k} g(b) + I_{b-}^{\alpha,k} g(a) \right] - \left[I_{a+}^{\alpha,k}(fg)(b) + I_{b-}^{\alpha,k}(fg)(a) \right] \right|$$

$$\leq \frac{1}{k\Gamma_k(\alpha)} \left(\int_a^b \left| \int_t^{a+b-t} (b-s)^{\frac{\alpha}{k}-1} g(s)ds \right|^p dt \right)^{\frac{1}{p}} \left(\int_a^b |f'(t)|^q dt \right)^{\frac{1}{q}}$$

$$\leq \frac{1}{k\Gamma_k(\alpha)} \left[\int_a^{\frac{a+b}{2}} \left(\int_t^{a+b-t} |(b-s)^{\frac{\alpha}{k}-1} g(s)|ds \right) dt \right.$$

$$+ \int_{\frac{a+b}{2}}^b \left(\int_{a+b-t}^t |(b-s)^{\frac{\alpha}{k}-1} g(s)|ds \right) dt \right]^{\frac{1}{p}} \left[\int_a^b \left(\frac{b-t}{b-a}|f'(a)|^q + \frac{t-a}{b-a}|f'(b)|^q \right) dt \right]^{\frac{1}{q}}$$

$$\leq \frac{\|g\|_\infty}{k\Gamma_k(\alpha+k)} \left[\int_a^{\frac{a+b}{2}} \left((b-t)^{\frac{\alpha}{k}} - (t-a)^{\frac{\alpha}{k}} \right)^p dt + \int_{\frac{a+b}{2}}^b \left((t-a)^{\frac{\alpha}{k}} - (b-t)^{\frac{\alpha}{k}} \right)^p dt \right]^{\frac{1}{p}}$$

$$\left[\int_a^b \left(\frac{b-t}{b-a}|f'(a)|^q + \frac{t-a}{b-a}|f'(b)|^q \right) dt \right]^{\frac{1}{q}}. \tag{2.11}$$

Now

$$(A - B)^q \leq A^q - B^q, \quad A \geq B \geq 0$$

gives

$$[(b-t)^{\frac{\alpha}{k}} - (t-a)^{\frac{\alpha}{k}}]^p \leq (b-t)^{\frac{\alpha p}{k}} - (t-a)^{\frac{\alpha p}{k}} \tag{2.12}$$

for $t \in [a, \frac{a+b}{2}]$, and

$$[(t-a)^{\frac{\alpha}{k}} - (b-t)^{\frac{\alpha}{k}}]^p \leq (t-a)^{\frac{\alpha p}{k}} - (b-t)^{\frac{\alpha p}{k}} \tag{2.13}$$

for $t \in [\frac{a+b}{2}, b]$.

Using (2.12) and (2.13) in inequality (2.11) and solving we get required result.

For (2.10) use (2.11) and Lemma 2.1.   □

Remark 2.10. If we take $k = 1$ in above theorem we get Theorem 1.6.

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