Iranian Journal of Mathematical Sciences and Informatics
Vol. 13, No. 2 (2018), pp 29-43
DOI: 10.7508/ijmsi.2018.13.003

Existence Results for Generalized ε-Vector Equilibrium Problems

Malek Abbasi, Mahboubeh Rezaei*

Department of Mathematics, University of Isfahan, P.O. Box 81745-163, Isfahan, Iran.

E-mail: malek.abbasi@sci.ui.ac.ir
E-mail: mrezaie@sci.ui.ac.ir

Abstract. This paper studies some existence results for generalized ε-vector equilibrium problems and generalized ε-vector variational inequalities. The existence results for solutions are derived by using the celebrated KKM theorem. The results achieved in this paper generalize and improve the works of many authors in references.

Keywords: Generalized ε-vector equilibrium problems, Generalized ε-vector variational inequalities, KKM theorem, Existence results, Painlevé-Kuratowski set-convergence.

2000 Mathematics subject classification: 49J40, 47J20, 47J25, 65K15, 65K10.

1. Introduction

Equilibrium problems and variational inequalities in a variety of disciplines play vital roles. Market equilibrium problems, economic equilibrium problems, traffic network equilibrium problems and so on, are some instances each of which has a long history in economic or industry or other branches of applied sciences. In recent years, variational inequality theory has proved a very useful tool in computation of various equilibrium problems. With any such close

* Corresponding Author

Received 23 October 2015; Accepted 27 October 2016

relationship between equilibrium problems and variational inequalities and the role each concept plays in applied sciences, it certainly will be of high importance to generalize the results achieved by authors who have worked in these fields. In references [1, 2, 3, 4, 7, 8, 10, 18] the reader could find a lot of materials for the history of the work. The following paper considers some kinds of generalized equilibrium problems and generalized variational inequalities and furnishes some new results. Using the celebrated KKM theorem (or Fan's theorem), we follow some existence results consisting of some sufficient conditions guaranteeing the solvability of the mentioned problems.

The paper is organized as follows. In Section 2 we present definitions and notations needed in addressing our study. In Section 3 some existence theorems for generalized ε-vector equilibrium problems ($(GVEP)_\varepsilon$, in short) and generalized ε-vector variational inequalities ($(GVVI)_\varepsilon$, in short) are verified. We hope the reader will find something of interest in this article.

2. Preliminaries

Throughout this paper, unless otherwise specified, let X be a Banach space with its dual $X^\star$ and let $X_m^\star = L(X, \mathbb{R}^m)$ denote the set of all linear continuous operators from X to $\mathbb{R}^m$. Let $K \subset \mathbb{R}^m$ be a closed convex pointed cone with $\text{int}K \neq \emptyset$, where $\text{int}K$ denotes the topological interior of K. We denote

$$\mathbb{R}^m_+ = \{x = (x_1, x_2, \ldots, x_m) \in \mathbb{R}^m : x_i \geq 0, i = 1, 2, \ldots, m\}.$$

Given $x, y \in \mathbb{R}^m$, we consider the following ordering relations [10]:

$$y <_K x \Leftrightarrow y - x \in -\text{int}K, \qquad y \not<_K x \Leftrightarrow y - x \notin -\text{int}K$$

and

$$y \leq_K x \Leftrightarrow y - x \in -K, \qquad y \not\leq_K x \Leftrightarrow y - x \notin -K.$$

Let $h : X \times X \to \mathbb{R}^m$ and $f : X \to \mathbb{R}^m$ be two vector-valued mappings such that $h(x, x) = 0$ for all $x \in X$. Consider the following generalized ε-vector equilibrium problem $(\text{GVEP})_\varepsilon$:
Find $x_0 \in X$ such that

$$h(x_0, x) + f(x) - f(x_0) + \varepsilon\|x - x_0\|1_{\mathbb{R}^m} \not<_K 0, \quad \forall x \in X. \tag{2.1}$$

An element $x_0 \in X$ satisfying (2.1) is called an ε-solution of $(\text{GVEP})_\varepsilon$.

If $\varepsilon = 0$, then $(\text{GVEP})_\varepsilon$ reduces to the following generalized vector equilibrium problem (GVEP) introduced and studied by Li and Zhao [16]:
Find $x_0 \in X$ such that

$$h(x_0, x) + f(x) - f(x_0) \not<_K 0, \quad \forall x \in X.$$

If $f \equiv 0$ and $\varepsilon = 0$, then $(\text{GVEP})_\varepsilon$ reduces to the following vector equilibrium problem (VEP):

Find $x_0 \in X$ such that

$$h(x_0, x) \not<_K 0, \quad \forall x \in X.$$

For further details on (VEP), we refer [3, 2, 4, 7, 8] and the references therein. If $h(x, y) = \langle T(x), y - x\rangle$ where $T : X \to X_m^\star$, then $(\mathrm{GVEP})_\varepsilon$ reduces to the following generalized ε-vector variational inequality $(\mathrm{GVVI})_\varepsilon$:
Find $x_0 \in X$ such that

$$\langle T(x_0), x - x_0\rangle + f(x) - f(x_0) + \varepsilon\|x - x_0\|1_{\mathbb{R}^m} \not<_K 0, \quad \forall x \in X.$$

For further details on (GVEP), we refer [13, 14, 15] and the references therein. In this paper, we consider the generalized ε-vector equilibrium problem $(\mathrm{GVEP})_\varepsilon$ and establish some existence theorems for solutions of $(\mathrm{GVEP})_\varepsilon$.

Definition 2.1. ([10, 9].) A vector-valued mapping $f : X \to \mathbb{R}^m$ is said to be K-convex if

$$f(tx_1 + (1-t)x_2) \leq_K tf(x_1) + (1-t)f(x_2),$$

for any $x_1, x_2 \in X$ and $t \in [0, 1]$. Furthermore, f is said to be K-concave, if $-f$ is K-convex.

Definition 2.2. A vector-valued mapping $f : X \to \mathbb{R}^m$ is said to be $\varepsilon - K$-convex if for any $x, y \in X$ and any $t \in [0, 1]$

$$f(tx + (1-t)y) \leq_K tf(x) + (1-t)f(y) - \varepsilon t(1-t)\|x - y\|1_{\mathbb{R}^m}.$$

Definition 2.3. [17] Let $f : X \to \mathbb{R}^m$ be a map. A subdifferential of f at $x_0 \in X$ is defined as

$$\partial f(x_0) = \{x^\star \in X_m^\star : \langle x^\star, x - x_0\rangle \leq_K f(x) - f(x_0), \forall x \in X\}.$$

Definition 2.4. [17] Let $f : X \to \mathbb{R}^m$ be a map. An ε-subdifferential of f at $x_0 \in X$ is defined as

$$\partial_\varepsilon f(x_0) = \{x^\star \in X_m^\star : \langle x^\star, x - x_0\rangle - \varepsilon\|x - x_0\|1_{\mathbb{R}^m} \leq_K f(x) - f(x_0), \quad \forall x \in X\}.$$

Letting $\varepsilon = 0$ we follow $\partial_\varepsilon f(x_0) = \partial f(x_0)$. It is worth observing that the following subdifferential equation also holds: $\partial_\varepsilon f(x_0) = \partial(f(.) + \varepsilon\|. - x_0\|)(x_0)$.

The definition of ε-subdifferential of a scalar function can be extended to the vector-valued functions through the following trick. Suppose that $f_i : X \to \bar{\mathbb{R}}, i = 1, 2, \ldots, m$ are the components of $f : X \to \mathbb{R}^m$. The Generalized ε-subdifferential of f at $x \in X$ is defined by the set

$$\partial_\varepsilon f(x) = \partial_\varepsilon f_1(x) \times \partial_\varepsilon f_2(x) \times \cdots \times \partial_\varepsilon f_m(x).$$

Let us introduce the KKM theorem needed for the proof of the existence results of the paper. Indeed, we use the original Fan's theorem for our purposes with the following wording:

Definition 2.5. ([7, 21].) A set-valued mapping $G : X \rightrightarrows X$ is said to be a KKM mapping if for each finite subset $\{x_1, x_2, \ldots, x_n\}$ of X, we have

$$\mathrm{co}\{x_1, x_2, \ldots, x_n\} \subset \bigcup_{i=1}^{n} G(x_i),$$

where co$A$ denotes the convex hull of the set A.

The following well-known Fan-KKM theorem will be used in the sequel.

Theorem 2.6 (Fan-KKM Theorem [7, 21])**.** *Let $G : X \rightrightarrows X$ be a KKM mapping. If for each $x \in X$, $G(x)$ is closed and $G(x_0)$ is compact for some $x_0 \in X$, then*

$$\bigcap_{x \in X} G(x) \neq \emptyset.$$

3. Existence Theorems for $(GVEP)_\varepsilon$ and $(GVVI)_\varepsilon$

In this section, we establish some existence results for solutions of generalized ε-vector equilibrium problem (GVEP)$_\varepsilon$ by using Fan-KKM theorem. As particular cases, we derive some existence results for solutions of generalized ε-vector variational inequality problem (GVVIP)$_\varepsilon$.

Theorem 3.1. *Let the following assumptions hold:*

(i) *The mappings $h : X \times X \to \mathbb{R}^m$ and $f : X \to \mathbb{R}^m$ are continuous;*
(ii) *For any $y \in X$, the set $B_y = \{x \in X : h(x, y) + f(y) - f(x) + \varepsilon\|x - y\|1_{\mathbb{R}^m} <_K 0\}$ is convex;*
(iii) *There exist the nonempty compact subset C and $z \in C$ such that for any $y \in X \setminus C$,*

$$h(z, y) + f(y) - f(z) + \varepsilon\|z - y\|1_{\mathbb{R}^m} <_K 0.$$

Then $(GVEP)_\varepsilon$ is solvable.

Proof. Define a set-valued mapping $\Gamma : X \to X$ by

$$\Gamma(x) = \{y \in X : h(x, y) + f(y) - f(x) + \varepsilon\|x - y\|1_{\mathbb{R}^m} \not<_K 0\}, \quad \forall x \in X.$$

Clearly, $x \in \Gamma(x)$ for all $x \in X$, and thus $\Gamma(x) \neq \emptyset$ for all $x \in X$. Also, x_0 solves (GVEP)$_\varepsilon$ if and only if $x_0 \in \bigcap_{x \in X} \Gamma(x)$. Thus, if we show $\bigcap_{x \in X} \Gamma(x) \neq \emptyset$, then the desired conclusion follows.To prove this, it only suffices to show that Γ is a KKM mapping and satisfies the conditions of Fan-KKM theorem. Suppose, on the contrary, that $\Gamma(x)$ is not a KKM mapping. Then there exists a finite subset $\{y_1, y_2, \ldots, y_n\}$ of X such that

$$\mathrm{co}\{y_1, y_2, \ldots, y_n\} \not\subset \bigcup_{i=1}^{n} \Gamma(y_i).$$

Hence, there exists $y \in \text{co}\{y_1, y_2, \ldots, y_n\}$ such that

$$y \notin \bigcup_{i=1}^{n} \Gamma(y_i).$$

So, for any $i \in \{1, 2, \ldots, n\}$, we have

$$h(y_i, y) + f(y) - f(y_i) + \varepsilon\|y_i - y\|1_{\mathbb{R}^m} <_K 0.$$

Hence, $\{y_1, y_2, \ldots, y_n\} \subset B_y$. Since B_y is convex, we deduce that

$$\text{co}\{y_1, y_2, \ldots, y_n\} \subset B_y.$$

Since $y \in \text{co}\{y_1, y_2, \ldots, y_n\}$, we have $y \in B_y$. This implies that

$$h(y, y) + f(y) - f(y) + \varepsilon\|y - y\|1_{\mathbb{R}^m} <_K 0,$$

which is absurd. Therefore, Γ is a KKM mapping.

The closedness of $\Gamma(x)$ is a straightforward conclusion of the continuity of the two mappings h and f and the fact that $\text{int}K$ is open.

Assumption (iii) implies that $\Gamma(z)$ is contained in a compact set. Being closed, $\Gamma(z)$ is compact by it's own right. Therefore, by Theorem 2.6, we have

$$\bigcap_{x \in X} \Gamma(x) \neq \emptyset.$$

This completes the proof. □

Remark 3.2. If the function f is K-concave, the mapping $x \mapsto h(x, y)$ is K-convex for all $y \in X$ and $K = \mathbb{R}^m_+$, then the condition (ii) of Theorem 3.1 holds.

To see this, let $x_1, x_2 \in B_y$ and $t \in [0, 1]$. Then, we have

$$h(x_1, y) + f(y) - f(x_1) + \varepsilon\|x_1 - y\|1_{\mathbb{R}^m} \in -\text{int}K,$$

and

$$h(x_2, y) + f(y) - f(x_2) + \varepsilon\|x_2 - y\|1_{\mathbb{R}^m} \in -\text{int}K.$$

Since f is K-concave and $h(., y)$ is K-convex, we have (see [10]: page 22, Lemma 2.3.4)

$$\begin{aligned}
& h(tx_1 + (1-t)x_2, y) + f(y) - f(tx_1 + (1-t)x_2) \\
& +\varepsilon\|tx_1 + (1-t)x_2 - y\|1_{\mathbb{R}^m} \\
\in\ & t[h(x_1, y) + f(y) - f(x_1) + \varepsilon\|x_1 - y\|1_{\mathbb{R}^m}] \\
& +(1-t)[h(x_2, y) + f(y) - f(x_2) + \varepsilon\|x_2 - y\|1_{\mathbb{R}^m}] - K - K - K \\
\subseteq\ & -\text{int}K - intK - K - K - K \\
\subseteq\ & -\text{int}K.
\end{aligned}$$

So, B_y is convex.

Remark 3.2 readily implies the following result.

Corollary 3.3. *Let $K = \mathbb{R}^m_+$. Suppose that the following conditions are satisfied:*

(i) *The function $y \mapsto h(x, y)$ is continuous for all $x \in X$ and the function $x \mapsto h(x, y)$ is K-convex for all $y \in X$;*
(ii) *$f : X \to \mathbb{R}^m$ is a continuous and K-concave mapping;*
(iii) *There exist the nonempty compact subset C and $z \in C$ such that for any $y \in X \setminus C$,*

$$h(z, y) + f(y) - f(z) + \varepsilon\|z - y\|1_{\mathbb{R}^m} <_K 0.$$

Then (GVEP)$_\varepsilon$ is solvable.

Example 3.4. Let $X = \mathbb{R}$ and $m = 2$. Let $K = \mathbb{R}^2_+$. Define the two mappings $f : X \to \mathbb{R}^2$ and $h : X \times X \to \mathbb{R}^2$ respectively by $x \mapsto -(x^2, x^2)$ and $(x, y) \mapsto (x-y, x-y)$. One can easily verify that the two first conditions of the Corollary 3.3 are satisfied. Furthermore, letting $z = 0$ we see that the set

$$\{y \in X : h(z, y) + f(y) - f(z) + \varepsilon|z - y|1_{\mathbb{R}^m} \not<_K 0\},$$

is compact and thus the last condition of the Corollary 3.3 is satisfied too. By Corollary 3.3 we conclude that the associated (GVEP)$_\varepsilon$ problem has a solution.

Theorem 3.1 and Corollary 3.3 yield the following existence results for solutions of (GVVIP)$_\varepsilon$.

Theorem 3.5. *Suppose that the following conditions are satisfied:*

(i) *The mappings $T : X \to X^\star_m$ and $f : X \to \mathbb{R}^m$ are continuous;*
(ii) *For any $y \in X$, the set*

$$B_y = \{x \in X : \langle T(y), x - y\rangle + f(x) - f(y) + \varepsilon\|x - y\|1_{\mathbb{R}^m} <_K 0\},$$

is convex;
(iii) *There exists a nonempty compact subset C and $z \in C$ such that for any $y \in X \setminus C$,*

$$\langle T(y), z - y\rangle + f(z) - f(y) + \varepsilon\|z - y\|1_{\mathbb{R}^m} <_K 0.$$

Then (GVVIP)$_\varepsilon$ is solvable.

Corollary 3.6. *Let $K = \mathbb{R}^m_+$. Suppose that the following conditions are satisfied:*

(i) *$T : X \to X^\star_m$ is a continuous mapping and $f : X \to \mathbb{R}^m$ is a continuous and K-concave mapping;*
(ii) *There exists a nonempty compact subset C and $z \in C$ such that for any $y \in X \setminus C$,*

$$\langle T(y), z - y\rangle + f(z) - f(y) + \varepsilon\|z - y\|1_{\mathbb{R}^m} <_K 0.$$

Then (GVVIP)$_\varepsilon$ is solvable.

Remark 3.7. Theorem 3.5 and Corollary 3.6 can be viewed as extensions of Theorem 1 in Yang [22].

The following theorems using weak topology yields some similar results. The details are as follows.

Theorem 3.8. *Let $K = \mathbb{R}^m_+$. Assume that the following assumptions hold.*

(i) *$y \mapsto h(x, y)$ is weakly continuous for all $x \in X$ and $f : X \to \mathbb{R}^m$ is weakly continuous;*
(ii) *For any $y \in X$, the set $B_y = \{x \in X : h(x, y) + f(y) - f(x) - \varepsilon\|x - y\|1_{\mathbb{R}^m} <_K 0\}$ is convex;*
(iii) *there exists a nonempty weakly compact subset C and $z \in C$ such that for any $y \in X \setminus C$,*

$$h(z, y) + f(y) - f(z) - \varepsilon\|z - y\|1_{\mathbb{R}^m} <_K 0.$$

Then (GVEP)$_\varepsilon$ is solvable.

Proof. Define a set-valued mapping $\Gamma_\varepsilon : X \to X$ by

$$\Gamma_\varepsilon(x) := \{y \in X : h(x, y) + f(y) - f(x) - \varepsilon\|x - y\|1_{\mathbb{R}^m} \not<_K 0\}, \quad \forall x \in X.$$

Obviously, $x \in \Gamma_\varepsilon(x)$ for all $x \in X$, and thus $\Gamma_\varepsilon(x) \neq \emptyset$ for all $x \in X$. Clearly, x_0 solves (GVEP)$_\varepsilon$ if and only if $x_0 \in \bigcap_{x\in X} \Gamma_{-\varepsilon}(x)$. Thus if we show $\bigcap_{x\in X} \Gamma_{-\varepsilon}(x) \neq \emptyset$, then the desired conclusion immediately follows. We proceed to show that Γ_ε is a KKM mapping and satisfies the conditions of Fan-KKM theorem. One can easily, by using an argument analogous to that of Theorem 3.1, verify that Γ_ε is a KKM mapping. Let us verify that each $\Gamma_\varepsilon(x)$ is weakly closed. This is a conclusion of the weak continuity of the mappings $y \mapsto h(x, y)$ and f, the openness of int$K$, the fact that the vector-valued norm function $x \mapsto \|x\|1_{\mathbb{R}^m}$ is weakly lower semicontinuous and finally a simple application of a *net-argument* discussion. Let us verify this item more precisely. Let $(y_\gamma)_{\gamma\in\Gamma}$ be a net in $\Gamma_\varepsilon(x)$ converging weakly to some $y \in X$. Notice first that the weak lower semicontinuity of the function $x \mapsto \|x\|$ implies that

$$\liminf_\gamma \|y_\gamma - x\| \geq \|y - x\|,$$

from which we deduce that

$$\liminf_\gamma \|y_\gamma - x\|1_{\mathbb{R}^m} \geq_K \|y - x\|1_{\mathbb{R}^m}. \tag{3.1}$$

By way of contradiction, we assume that $y \notin \Gamma_\varepsilon(x)$. Let $w_x = h(x, y) + f(y) - f(x) - \varepsilon\|x - y\|1_{\mathbb{R}^m}$. Thus $w_x \in -\text{int}K$. On the other hand by the first

hypothesis of theorem and (3.1) we deduce that

$$\begin{aligned} & \liminf_{\gamma} \{h(x, y_\gamma) + f(y_\gamma) - f(x) - \varepsilon \|y_\gamma - x\| 1_{\mathbb{R}^m}\} \\ = \; & h(x, y) + f(y) - f(x) - \varepsilon \liminf_{\gamma} \|y_\gamma - x\| 1_{\mathbb{R}^m} \\ = \; & w_x + (\varepsilon \|x - y\| 1_{\mathbb{R}^m} - \varepsilon \liminf_{\gamma} \|y_\gamma - x\| 1_{\mathbb{R}^m}) \\ \in \; & -\text{int}K - K \\ \subseteq \; & -\text{int}K. \end{aligned}$$

The openness of int$K$ now implies that there exists some $\gamma_0 \in \Gamma$ so that

$$h(x, y_{\gamma_0}) + f(y_{\gamma_0}) - f(x) - \varepsilon \|y_{\gamma_0} - x\| 1_{\mathbb{R}^m} \in -\text{int}K,$$

which contradicts the fact that each $y_\gamma \in \Gamma_\varepsilon(x)$. This completes the proof of weak closedness of each $\Gamma_\varepsilon(x)$. Obviously condition (iii) implies that $\Gamma(x')$ is contained in a weakly compact set. Being weakly closed, $\Gamma(x')$ is weakly compact by it's own right. Therefore, by Theorem 2.6 we have

$$\bigcap_{x \in X} \Gamma_\varepsilon(x) \neq \emptyset.$$

But notice that $\Gamma_\varepsilon(x) \subseteq \Gamma_{-\varepsilon}(x)$ for all $x \in X$, from which the desired result follows. □

Theorem 3.9. *Let $K = \mathbb{R}^m_+$. Assume that the following conditions are satisfied:*

(i) *the function $y \mapsto h(x, y) + \varepsilon \|y - x\|$ is weakly upper semicontinuous for all $x \in X$ and $f : X \to \mathbb{R}^m$ is weakly upper semicontinuous;*

(ii) *for any $y \in X$, the set $B_y = \{x \in X : h(x, y) + f(y) - f(x) + \varepsilon \|x - y\| 1_{\mathbb{R}^m} <_K 0\}$ is convex;*

(iii) *there exists a nonempty weakly compact subset C and $x' \in C$ such that for any $y \in X \setminus C$ one has*

$$h(x', y) + f(y) - f(x') + \varepsilon \|x' - y\| 1_{\mathbb{R}^m} <_K 0.$$

Then, the generalized ε-vector equilibrium problem $(GVEP)_\varepsilon$ is solvable.

Proof. The proof is analogous with the proof of the previous theorem, we therefore omit it. □

The following theorem yields a better result. Indeed, when X is infinite dimensional, the function $x \to \|x\|$ is not weakly upper semicontinuous, since otherwise the weak and strong topologies coincide. Hence the first condition of the previous theorem may be rarely satisfied.

Theorem 3.10. *Let $K = \mathbb{R}^m_+$. Assume that the following conditions are satisfied:*

(i) *the function $y \mapsto h(x, y)$ is weakly upper semicontinuous for all $x \in X$ and $f : X \to \mathbb{R}^m$ is weakly upper semicontinuous;*

(ii) *for any* $y \in X$, *the set* $B_y = \{x \in X : h(x,y) + f(y) - f(x) - \varepsilon\|x - y\|1_{\mathbb{R}^m} <_K 0\}$ *is convex;*

(iii) *there exist a nonempty weakly compact subset* C, $x' \in C$ *and* $\bar{x}^\star \in X^\star$ *such that for any* $y \in X \setminus C$ *one has*

$$h(x', y) + f(y) - f(x') - \varepsilon|\bar{x}^\star(x' - y)|1_{\mathbb{R}^m} <_K 0.$$

Then, the generalized ε*-vector equilibrium problem* $(GVEP)_\varepsilon$ *is solvable.*

Proof. Again, we aim to use the Fan's theorem to achieve the desired result. Let

$$\Gamma(x, x^\star) := \{y \in X : h(x,y) + f(y) - f(x) - \varepsilon|x^\star(x - y)|1_{\mathbb{R}^m} \not<_K 0\}, \quad \forall x \in X,$$

$$\Gamma(x) := \{y \in X : h(x,y) + f(y) - f(x) - \varepsilon\|x - y\|1_{\mathbb{R}^m} \not<_K 0\}, \quad \forall x \in X.$$

Using a net-argument one can verify that $\Gamma(x, x^\star)$ is weakly closed for each $x \in X$ and $x^\star \in X^\star$. This implies that the intersection $\bigcap_{x^\star \in U_{X^\star}} \Gamma(x, x^\star)$ is weakly closed for each $x \in X$, where $U_{X^\star}$ denotes the set of all norm-one functionals belonging to $X^\star$. The following equality holds:

$$\Gamma(x) = \bigcap_{x^\star \in U_{X^\star}} \Gamma(x, x^\star). \tag{3.2}$$

To see this let $y \in \Gamma(x)$. Thus $h(x,y)+f(y)-f(x)-\varepsilon\|x-y\|1_{\mathbb{R}^m} \not<_K 0$. If now $y \notin \bigcap_{x^\star \in U_{X^\star}} \Gamma(x, x^\star)$, thus there exists some $x^\star \in U_{X^\star}$ so that $y \notin \Gamma(x, x^\star)$ and therefore $h(x,y)+f(y)-f(x)-\varepsilon|x^\star(x-y)|1_{\mathbb{R}^m} <_K 0$. On the other hand $|x^\star(x-y)| \leq \|x-y\|$ from which we deduce that $\varepsilon|x^\star(x-y)|1_{\mathbb{R}^m} - \varepsilon\|x-y\|1_{\mathbb{R}^m} \in -K$. These two last inequalities imply $h(x,y)+f(y)-f(x)-\varepsilon\|x-y\|1_{\mathbb{R}^m} <_K 0$ which is absurd. Conversely let $y \in \bigcap_{x^\star \in U_{X^\star}} \Gamma(x, x^\star)$. This implies $h(x,y) + f(y) - f(x) - \varepsilon|x^\star(x-y)|1_{\mathbb{R}^m} \not<_K$ for all $x^\star \in U_{X^\star}$. By Hahn-Banach theorem there exists some $x^\star \in U_{X^\star}$ so that $x^\star(x - y) = \|x - y\|$, from which we deduce that $h(x,y) + f(y) - f(x) - \varepsilon\|x - y\|1_{\mathbb{R}^m} \not<_K 0$. Thus equality (3.2) holds. On the other hand the last hypothesis of theorem implies that $\Gamma(x', \bar{x}^\star)$ is weakly compact and thus $\bigcap_{x^\star \in U_{X^\star}} \Gamma(x', x^\star)$ is weakly compact too. Hence $\Gamma(x')$ is weakly compact. By our discussion above (before the equality (3.2)), and using the mentioned equality we know that $\Gamma(x)$ is weakly closed for all $x \in X$. It is not difficult to prove that $\Gamma(.)$ is a KKM map and we see that the whole conditions of Fan's theorem hold. The reminder of the proof is easy. □

Theorem 3.11. *Let* K *be a closed convex pointed cone in* $\mathbb{R}^m$. *Assume that the following conditions are satisfied:*

(i) *the function* $y \mapsto h(x,y)$ *is weakly upper semicontinuous for all* $x \in X$ *and* $f : X \to \mathbb{R}^m$ *is weakly upper semicontinuous;*

(ii) *for any* $y \in X$ *and* $y^\star \in S_{X^\star}$, *the set* $B_{y,y^\star} = \{x \in X : h(x,y) + f(y) - f(x) + \varepsilon y^\star(x - y)1_{\mathbb{R}^m} <_K 0\}$ *is convex, where* $S_{X^\star}$ *denotes the closed unite ball in* $X^\star$*;*

(iii) *there exist a nonempty weakly compact subset* C, $x' \in C$ *and* $\bar{x}^\star \in X^\star$ *such that for any* $y \in X \setminus C$ *one has*

$$h(x', y) + f(y) - f(x') + \varepsilon \bar{x}^\star(x' - y)1_{\mathbb{R}^m} <_K 0.$$

Then, the generalized ε*-vector equilibrium problem* $(GVEP)_\varepsilon$ *is solvable.*

Proof. The proof is somewhat similar to that of Theorem 3.10 and we therefor give only a sketch of the proof. Let $\Theta = X \times S_{X^\star}$. Equip Θ with the product topology, being X equipped with the $weak(\sigma(X, X^\star))$ topology and $S_{X^\star}$ with the relative $weak^\star(\sigma(X^\star, X))$ topology. Define the set-valued map $\Gamma : \Theta \rightrightarrows \Theta$ by

$$\Gamma(x, x^\star) := \{y \in X : h(x, y) + f(y) - f(x) + \varepsilon x^\star(x - y)1_{\mathbb{R}^m} \not<_K 0\} \times S_{X^\star},$$

$\forall (x, x^\star) \in \Theta$. One can easily verify that Γ is a KKM map. By Banach-Alaoglu theorem and in virtue of the last condition of theorem we know that Γ satisfies the conditions of Fan's theorem entirely. It follows that there exists some $(y, y^\star) \in \Theta$ so that $(y, y^\star) \in \Gamma(x, x^\star)$ for all $(x, x^\star) \in \Theta$. Thus

$$h(x, y) + f(y) - f(x) + \varepsilon x^\star(x - y)1_{\mathbb{R}^m} \not<_K 0, \tag{3.3}$$

for all $x \in X$ and $x^\star \in S_{X^\star}$. By Hahn-Banach theorem we deduce that for any $x \in X$ there exists some $x_x^\star \in S_{X^\star}$ satisfying $x_x^\star(x - y) = \|x - y\|$. By (3.3) we have

$$h(x, y) + f(y) - f(x) + \varepsilon x_x^\star(x - y)1_{\mathbb{R}^m} \not<_K 0,$$

for all $x \in X$, from which the desired conclusion follows. □

The following corollaries shed a little more light on the preceding theorems:

Corollary 3.12. *Let* $K = \mathbb{R}^m_+$. *Suppose that the following conditions are satisfied:*

(i) *the function* $y \mapsto h(x, y)$ *is weakly continuous for all* $x \in X$ *and the function* $x \mapsto h(x, y) - \varepsilon\|x - y\|1_{\mathbb{R}^m}$ *is* K*-convex for all* $y \in X$*;*
(ii) *the function* $f : X \to \mathbb{R}^m$ *is weakly continuous and* K*-concave;*
(iii) *there exists a nonempty weakly compact subset* C *and an* $x' \in C$ *such that for any* $y \in X \setminus C$ *one has*

$$h(x', y) + f(y) - f(x') - \varepsilon\|x' - y\|1_{\mathbb{R}^m} <_K 0.$$

Then, the generalized ε*-vector equilibrium problem* $(GVEP)_\varepsilon$ *has a solution.*

Corollary 3.13. *Let* $K = \mathbb{R}^m_+$. *Suppose that the following conditions are satisfied:*

(i) *the function* $y \mapsto h(x, y) + \varepsilon\|x - y\|1_{\mathbb{R}^m}$ *is weakly upper semicontinuous for all* $x \in X$ *and the function* $x \mapsto h(x, y)$ *is* K*-convex for all* $y \in X$*;*
(ii) *the function* $f : X \to \mathbb{R}^m$ *is weakly upper semicontinuous and* K*-concave;*

(iii) *there exists a nonempty weakly compact subset* C *and an* $x' \in C$ *such that for any* $y \in X \setminus C$ *one has*

$$h(x', y) + f(y) - f(x') - \varepsilon\|x' - y\|1_{\mathbb{R}^m} <_K 0.$$

Then, the generalized ε*-vector equilibrium problem* $(GVEP)_\varepsilon$ *has a solution.*

Corollary 3.14. *Let* K *be a closed convex pointed cone in* $\mathbb{R}^m$*. Assume that the following conditions are satisfied:*

(i) *the function* $y \mapsto h(x, y)$ *is weakly upper semicontinuous for all* $x \in X$ *and* $f : X \to \mathbb{R}^m$ *is weakly upper semicontinuous;*

(ii) *the function* $x \mapsto h(x, y)$ *is* K*-convex for all* $y \in X$ *and* $f : X \to \mathbb{R}^m$ *is* K*-concave;*

(iii) *there exist a nonempty weakly compact subset* C*,* $x' \in C$ *and* $\bar{x}^\star \in X^\star$ *such that for any* $y \in X \setminus C$ *one has*

$$h(x', y) + f(y) - f(x') + \varepsilon\bar{x}^\star(x' - y)1_{\mathbb{R}^m} <_K 0.$$

Then, the generalized ε*-vector equilibrium problem* $(GVEP)_\varepsilon$ *has a solution.*

Proof. Notice that the condition (ii) guarantees that the second condition of Theorem 3.11 holds. This completes the proof. □

Remark 3.15. Corollary 3.14 relaxes the condition $K = \mathbb{R}^m_+$ used in the previous results into a general one, by letting K to be an arbitrary closed convex pointed cone in $\mathbb{R}^m$.

Example 3.16. Let $(X, \|.\|)$ be an infinite dimensional Banach space. Let $m = 2$ and $K = \mathbb{R}^2_+$. Define the two mappings $f : X \to \mathbb{R}^2$ and $h : X \times X \to \mathbb{R}^2$ respectively by $x \mapsto -(\|x\|, \|x\|)$ and $(x, y) \mapsto (\|x\| - \|y\|, \|x\| - \|y\|)$. One can easily verify that the two first conditions of the Corollary 3.14 are satisfied. Let $x^\star \in X^\star$ be a functional satisfying $\|x^\star\| < \frac{2}{\varepsilon}$. Having these we can see that the set

$$\{y \in X : h(\acute{x}, y) + f(y) - f(\acute{x}) + \varepsilon x^\star(\acute{x} - y)1_{\mathbb{R}^m} \not<_K 0\},$$

at $\acute{x} = 0$ is weakly compact and thus the last condition of the Corollary 3.14 is also fulfilled. Corollary 3.14 now implies that the $(\mathrm{GVEP})_\varepsilon$ problem associated with these definitions has a solution. Of course if we assume that $\varepsilon < 2$, then using again Corollary 3.3 one may prove the solvability of this problem as well.

Example 3.17. Let $X = \mathbb{R}$. Let $m = 2$ and $K = \mathbb{R}^2_+$. Define the two mappings $f : X \to \mathbb{R}^2$ and $h : X \times X \to \mathbb{R}^2$ respectively by $x \mapsto -(|x|, |x|)$ and

$$(x, y) \mapsto \begin{cases} (|x| - |y|, |x| - |y|) & y \geq 1; \\ \\ (|\frac{x}{2}| - |\frac{y}{2}|, |\frac{x}{2}| - |\frac{y}{2}|) & y > 1. \end{cases}$$

One can easily verify that the conditions of the Corollary 3.14 are entirely satisfied. Corollary 3.14 implies that the $(\mathrm{GVEP})_\varepsilon$ problem associated with

these data has a solution. Notice that in this example the function $y \mapsto h(x, y)$ is not continuous at $y = 1$ for any $x \neq \pm 1$ and thus Corollary 3.3 fails to respond.

In the following theorem we state and prove an existence result for an $(GVVI)_\varepsilon$ problem. Let $T : X \rightrightarrows X_m^\star$ be a set-valued map. Given a map $f : X \to \mathbb{R}^m$, the following generalized $(GVVI)_\varepsilon$ problem is discussed here: Find $x \in X$ such that for any $y \in X$ there exists $x^\star \in T(x)$ satisfying:

$$\langle x^\star, y - x\rangle \not<_K f(x) - f(y) - \varepsilon\|y - x)\|1_{\mathbb{R}^m}.$$

For this problem we have the following existence result. We first recall the notion of Painlevé-Kuratowski set-convergence [19]. Let X and Y be normed linear spaces. For a sequence of sets (S_n) in X, we set the notations

$$\begin{aligned} LiS_n &:= \{x \in X : x = \lim_{n\to\infty} x_n,\ x_n \in S_n, \quad \text{for sufficiently large } n\},\\ LsS_n &:= \{x \in X : x = \lim_{k\to\infty} x_{n_k},\ x_{n_k} \in S_{n_k}, \quad (n_k) \text{ a subsequence of } (n)\}. \end{aligned}$$

We say that the sequence of sets (S_n) converges to a set S in the sense of *Painlevé-Kuratowski* if and only if

$$LsS_n \subseteq S \subseteq LiS_n.$$

For easy reference consider the following definition.

Definition 3.18. Let X and Y be normed linear spaces. A set valued map $T : X \rightrightarrows Y$ is said to be compactly-sequentially upper continuous if

(i) $x_n \to x$ implies $T(x_n) \to T(x)$ in the sense of Painlevé-Kuratowski described above;

(ii) for all $x \in X$ there exists a compact set N_x containing $T(x)$ for which the following holds:
if (U_n) is a sequence of neighborhoods so that each U_n contains N_x, then there exists a neighborhood V containing x so that $T(V) \subseteq U_n$ for all $n \in \mathbb{N}$.

We now are completely ready to state the desired existence result.

Theorem 3.19. *Assume that $f : X \to \mathbb{R}^m$ is a lower semicontinuous ε-convex function and $K \subset \mathbb{R}^m$ be a convex cone with a nonempty interior. Suppose that the set-valued map $T : X \rightrightarrows X_m^\star$ is compactly-sequentially upper continuous. Suppose further that $\partial_\varepsilon f(x) \neq \emptyset$ implies $\partial_\varepsilon f(x) \bigcap T(x) \neq \emptyset$. If there exists a nonempty compact subset D of X and $x_0 \in X$ such that for every $x \in X \setminus D$ and $x^\star \in T(x)$ one has*

$$\langle x^\star, x_0 - x\rangle <_K f(x) - f(x_0) - \varepsilon\|x - x_0\|1_{\mathbb{R}^m},$$

then the problem $(GVVI)_\varepsilon$ has a solution.

Proof. For each $x \in X$ define the set-valued map $\Gamma : X \rightrightarrows X$ by

$$\Gamma(x) = \{y \in X : \exists y^\star \in T(y) \text{ such that } \langle y^\star, x-y\rangle \not<_K f(y)-f(x)-\varepsilon\|x-y\|1_{\mathbb{R}^m}\}.$$

We will prove that $\Gamma(x)$ is a KKM map on X. Suppose, on the contrary, that $\Gamma(x)$ is not a KKM map. Then there exists $t_i \in [0,1]$ and $x_i \in X, i = 1, ..., n$ with $\sum_{i=1}^n t_i = 1$ such that $\sum_{i=1}^n t_i x_i = x \notin \bigcup_{i=1}^n \Gamma(x_i)$. Then

$$\langle x^\star, x_i - x\rangle <_K f(x) - f(x_i) - \varepsilon\|x_i - x\|1_{\mathbb{R}^m},$$

for every $x^\star \in T(x)$ and $i = 1, 2, ..., n$. The lower semicontinuity and ε-convexity of f at x, implies that f is locally Lipschitz at x and hence the Clarke subdifferential of f at x is a nonempty set. This, of course, forces $\partial_\varepsilon f(x)$ to be nonempty as well, as it contains Clarke subdifferential of f at x (Proposition 4.3[6]). Consequently for every $x^\star \in \partial_\varepsilon f(x) \bigcap T(x)$ we have

$$\langle x^\star, x_i - x\rangle - \varepsilon\|x_i - x\|1_{\mathbb{R}^m} \leq_K f(x_i) - f(x).$$

These two last inequalities yield

$$\langle x^\star, x_i - x\rangle <_K 0.$$

Multiplying both sides of this result by t_i and then summing over $i = 1, ..., n$ we deduce that

$$\sum_{i=1}^n \langle x^\star, t_i(x_i - x)\rangle <_K 0,$$

for every $x^\star \in \partial_\varepsilon f(x) \bigcap T(x)$. This leads to a contradiction, since $\sum_{i=1}^n t_i x_i = x$ and $\sum_{i=1}^n t_i = 1$. Therefore Γ is a KKM mapping. Let us now show that $\Gamma(u)$ is closed for each $u \in X$. For an arbitrary $u \in X$, let (x_n) be a sequence in $\Gamma(u)$ converging to some $x \in X$. Then there exists $x_n^\star \in T(x_n)$ such that

$$\langle x_n^\star, u - x_n\rangle \not<_K f(x_n) - f(u) - \varepsilon\|u - x_n\|1_{\mathbb{R}^m}. \tag{3.4}$$

Let N_x be the compact set satisfying the item (ii) of Definition 3.18. For $\delta > 0$, let

$$G_\delta = \bigcup_{w^\star \in N_x} \{x^\star : \|x^\star - w^\star\| < \delta\}.$$

Obviously for any $\delta > 0$, the set G_δ is an open neighborhood in $X_m^\star$ containing N_x. Now the compactly-sequentially upper continuity of T at x implies that there exists a neighborhood U of x such that $T(U) \subset G_{\frac{1}{n}}$ for all $n \in \mathbb{N}$. For sufficiently large n, we observe that $x_n \in U$ and hence $T(x_n) \subseteq G_{\frac{1}{n}}$ for n sufficiently large. Since $x_n^\star \in T(x_n)$ it follows that there exists a sequence $(w_n^\star)$ in N_x so that $\|x_n^\star - w_n^\star\| < \frac{1}{n}$ for n sufficiently large. The compactness of N_x guarantees that there exists a subsequence $\{w_{n_k}^\star\}$ of the sequence $\{w_n^\star\}$

such that $w^\star_{n_k}$ converges to some $x^\star \in N_x$. Thus the subsequence $(x^\star_{n_k})$ of the sequence $(x^\star_n)$ satisfies

$$\begin{aligned} \|x^\star_{n_k} - x^\star\| &\leq \|x^\star_{n_k} - w^\star_{n_k}\| + \|w^\star_{n_k} - x^\star\| \\ &< \frac{1}{n_k} + \|w^\star_{n_k} - x^\star\|. \end{aligned}$$

Letting $k \to \infty$ it follows that $x^\star_{n_k} \to x^\star$. The first item in Definition 3.18 guarantees that $x^\star \in T(x)$. Notice that (3.4) implies

$$\langle x^\star_{n_k}, u - x_{n_k}\rangle \not< _K\ f(x_{n_k}) - f(u) - \varepsilon\|u - x_{n_k}\|1_{\mathbb{R}^m}. \tag{3.5}$$

Letting $k \to \infty$ in (3.5) it follows that $x \in \Gamma(u)$. This guarantees the closedness of $\Gamma(u)$. Moreover, the last condition of theorem implies that $\Gamma(x_0)$ is contained in the compact set D. As a consequence $\Gamma(x_0)$ is compact too, as a closed subset of the compact set D. Using the KKM theorem it follows that $\bigcap_{x\in X}\Gamma(x) \neq \emptyset$. This completes the proof. □

Acknowledgments

The authors are grateful to the office of Graduate Studies of the University of Isfahan for their supports. The authors are also grateful to the referee for valuable remarks and suggestions.

References

1. M. Abbasi, Kh. Pourbarat, The System of vector variational-like inequalities with weakly relaxed $\{\eta_\gamma - \alpha_\gamma\}_{\gamma\in\Gamma}$ pseudomonotone mappings in Banach spaces, *Iranian Journal of Mathematical Sciences and Informatics*, **2**, (2010), 1-11.
2. Q. H. Ansari, I. V. Konnov, J. C. Yao, Existence of a solution and variational principles for vector equilibrium problems. *Journal of Optimization Theory and Applications*, **110**, (2001), 481-492.
3. Q. H. Ansari, I. V. Konnov, J. C. Yao, Characterizations of solutions for vector equilibrium problems. *Journal of Optimization Theory and Applications*, **113**, (2000), 435-447.
4. Q. H. Ansari, X.Q. Yang, J. C. Yao, Existence and duality of implicit vector variational problems, *Numerical Functional Analysis and Optimization*, **22**, (2001), 815-829.
5. R. I. Bo, S. M. Grad, G. Wanka, *Duality in Vector Optimization*, Springer-Verlag, Berlin, Heidelberg, 2009.
6. F. H. Clarke, *Optimization and Nonsmooth Analysis, Wiley-Interscience*, 1983.
7. K. Fan, A generalization of Tychonoffs fixed point theorem, *Mathematische Annalen*, **142**, (1991), 305310.
8. F. Giannessi, *Vector Variational Inequalities and Vector Equilibria: Mathematical Theories, Nonconvex Optimization and its Applications,* Kluwer Academic Publishers, Dordrecht, Holland, 2000.
9. A. Jofre, D.T. Luc, M. Thera, ε-subdifferential and ε-monotonicity, *Nonlinear Anal.*, **33**, (1998), 71-90.
10. A. A. Khan, Ch. Tammer, C. Zalinescu, *set-valued optimization*, Springer-Verlag, 2015.
11. S. Kum, G. S. Kim, G. M. Lee, Duality for ε-variational inequality, *Journal of Optimization Theory and Applications*, **139**, (2008), 649-655.
12. C. S. Lalitha, G. Bhatia, Duality in ε-variational inequality problems, *J. Math. Anal. Appl.*, **356**, (2009), 168-178.

13. G. M. Lee, D. S. Kim, B. S. Lee, Generalized vector variational inequality, *Applied Mathematics Letters*, **9**, (1996), 39-42.
14. G. M. Lee, D. S. Kim, B. S. Lee, G. Y. Chen, Generalized vector variational inequality and its duality for set-valued maps, *Applied Mathematics Letters*, **11**, (1998), 21-26.
15. G. M. Lee, D. S. Kim, B. S. Lee, G. Y. Chen, Generalized vector variational inequality and its duality for set-valued maps, *Applied Mathematics Letters*, **11**, (1998), 21-26.
16. S. J. Li, P. Zhao, A method of duality for a mixed vector equilibrium problem, *Optimization Letters*, **4**, (2010), 85-96.
17. H. V. Ngai, D. T. Luc, M. Théra, Approximate convex functions, *J. Nonlinear Convex Anal.*, **1**, (2000), 155-176.
18. Kh. Pourbarat, M. Abbasi, On the Vector variational-like inequalities with relaxed $\eta - \alpha$ pseudomonotone mappings, *Iranian Journal of Mathematical Sciences and Informatics*, **1**, (2009), 37-42.
19. R. T. Rockafellar, R. J. B. Wets, Variational Analysis, Grundlehren der Mathematischen Wissenschaften, **317**, Springer-Verlag, Berlin 1998.
20. P. H. Sach, D. S. Kim, L. A. Tuan, G. M. Lee, Duality results for generalized vector variational inequalities with set-valued maps, *Journal of Optimization Theory and Applications*, **136**, (2006), 105-123.
21. E. U. Tarafdar, M.S.R. Chowdhury, *Topological methods for Set-Valued nonlinear Analysis*, World Scientific Publishing Co. Pte. Ltd, 2008.
22. X. Q. Yang, Vector variational inequality and its duality, *Nonlinear Analysis*, **21**, (1993), 869-877.