

Generalized Approximate Amenability of Direct Sum of Banach Algebras

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ABSTRACT. In the present paper for two \mathfrak{A} -module Banach algebras A and B , we investigate relations between φ - \mathfrak{A} -module approximate amenability of A , ψ - \mathfrak{A} -module approximate amenability of B , and $\varphi \oplus \psi$ - \mathfrak{A} -module approximate amenability of $A \oplus B$ (l^1 -direct sum of A and B), where $\varphi \in \text{Hom}_{\mathfrak{A}}(A)$ and $\psi \in \text{Hom}_{\mathfrak{A}}(B)$.

Keywords: Banach algebra, Module derivation, Module approximate amenability.

2000 Mathematics subject classification: 46H25.

1. INTRODUCTION

The notion of approximate amenable Banach algebras was introduced and extensively studied by Ghahramani and Loy in [5]. They showed in [6] that if A and B are approximately amenable Banach algebras and one of A or B has a bounded approximate identity, then $A \oplus B$ is approximately amenable, but in general the direct sum of two approximately amenable Banach algebras need not be approximately amenable (see [7]).

The concept of module amenable Banach algebras was introduced by Amini in [1], and the notion of module approximate amenable Banach algebras was studied by Pourmahmood and Bodaghi in [15]. Recently, some authors have

studied φ -derivations, and φ -amenability of Banach algebra A , whenever φ is a continuous homomorphism on A (see [8, 9, 10, 11, 12]).

The aim of the present paper is to investigate generalized approximate amenability of $A \oplus B$.

The organization of this paper is as follows:

Section 2 is devoted to the notations and definitions which are needed throughout the paper.

In section 3 for \mathfrak{A} -module Banach algebras A and B where each has a bounded approximate identity we show that A is φ - $\mathfrak{A}^\#$ -module approximately amenable and B is ψ - $\mathfrak{A}^\#$ -module approximately amenable if and only if $A \oplus B$ is $\varphi \oplus \psi$ - $\mathfrak{A}^\#$ -module approximately amenable.

In section 4 we show that if \mathfrak{A} has a bounded approximate identity and $\frac{A}{J_{A, \mathfrak{A}}}$ and $\frac{B}{J_{B, \mathfrak{A}}}$ are unital, then A is φ - \mathfrak{A} -module approximately amenable and B is ψ - \mathfrak{A} -module approximately amenable if and only if $A \oplus B$ is $\varphi \oplus \psi$ - \mathfrak{A} -module approximately amenable.

2. PRELIMINARIES

Let \mathfrak{A} and A be Banach algebras such that A is a Banach \mathfrak{A} -bimodule with compatible actions given by

$$\alpha.(ab) = (\alpha.a)b, \quad (ab).\alpha = a(b.\alpha) \quad (a, b \in A, \alpha \in \mathfrak{A}).$$

Let X be a Banach A -bimodule and a Banach \mathfrak{A} -bimodule with compatible left actions defined by

$$\alpha.(a.x) = (\alpha.a).x, \quad a.(\alpha.x) = (a.\alpha).x, \quad (\alpha.x).a = \alpha.(x.a)$$

$$(a \in A, \alpha \in \mathfrak{A}, x \in X), \quad (2.1)$$

and similar for the right or two-sided actions. Then we say that X is a Banach A - \mathfrak{A} -module. A Banach A - \mathfrak{A} -module X is called commutative A - \mathfrak{A} -module, if $\alpha.x = x.\alpha$ ($\alpha \in \mathfrak{A}, x \in X$). Note that in general, A does not satisfy the compatibility condition $a.(\alpha.b) = (a.\alpha).b$ ($a, b \in A, \alpha \in \mathfrak{A}$).

If X is a commutative Banach A - \mathfrak{A} -module, then so is X^* , where the actions of A and \mathfrak{A} on X^* are defined as follows

$$\langle \alpha.f, x \rangle = \langle f, x.\alpha \rangle, \quad \langle a.f, x \rangle = \langle f, x.a \rangle \quad (a \in A, \alpha \in \mathfrak{A}, x \in X, f \in X^*),$$

and similar for the right actions.

Let A and B be Banach \mathfrak{A} -bimodules. Then a \mathfrak{A} -module morphism from A to B is a norm continuous map $h : A \rightarrow B$ with $h(a \pm b) = h(a) \pm h(b)$ which is multiplicative, that is

$$h(\alpha.a) = \alpha.h(a), \quad h(a.\alpha) = h(a).\alpha, \quad h(ab) = h(a)h(b) \quad (a \in A, b \in B, \alpha \in \mathfrak{A}).$$

We denote by $\text{Hom}_{\mathfrak{A}}(A, B)$, the space of all such morphism and denote $\text{Hom}_{\mathfrak{A}}(A, A)$ by $\text{Hom}_{\mathfrak{A}}(A)$. In the case that $\mathfrak{A} = \mathbb{C}$, we denote $\text{Hom}_{\mathbb{C}}(A, B)$ by $\text{Hom}(A, B)$ and denote $\text{Hom}_{\mathbb{C}}(A, A)$ by $\text{Hom}(A)$.

Let X be a Banach A -bimodule and let $\varphi \in \text{Hom}_{\mathfrak{A}}(A)$. A bounded map $D : A \rightarrow X$ is called a φ - \mathfrak{A} -module derivation if

$$D(a \pm b) = D(a) \pm D(b), \quad D(ab) = D(a).\varphi(b) + \varphi(a).D(b) \quad (a, b \in A), \quad (2.2)$$

and

$$D(\alpha.a) = \alpha.D(a), \quad D(a.\alpha) = D(a).\alpha \quad (a \in A, \alpha \in \mathfrak{A}). \quad (2.3)$$

Although D in general is not linear, but still its boundedness implies its norm continuity.

Let X be a commutative Banach A - \mathfrak{A} -module. For every $x \in X$ define ad_x^φ by $ad_x^\varphi(a) = \varphi(a).x - x.\varphi(a)$ ($a \in A$). It is easily seen that ad_x^φ is a φ - \mathfrak{A} -module derivation. A φ - \mathfrak{A} -module derivation D is called φ -inner if there is $x \in X$ such that $D(a) = ad_x^\varphi(a)$ ($a \in A$) and is called approximately φ -inner if there exists a net $(x_\alpha)_\alpha \subseteq X$ such that $D(a) = \lim_\alpha ad_{x_\alpha}^\varphi(a)$ ($a \in A$). A Banach algebra A is called φ - \mathfrak{A} -module amenable if for any commutative Banach A - \mathfrak{A} -module X , each φ - \mathfrak{A} -module derivation $D : A \rightarrow X^*$ is φ -inner, and A is called φ - \mathfrak{A} -module approximately amenable if each φ - \mathfrak{A} -module derivation $D : A \rightarrow X^*$ is approximately φ -inner (see [1, 15]).

In the case that $\mathfrak{A} = \mathbb{C}$, φ - \mathfrak{A} -module derivations (resp. φ - \mathfrak{A} -module amenable Banach algebras, φ - \mathfrak{A} -module approximately amenable Banach algebras) are called φ -derivation (resp. φ -amenable, φ -approximately amenable) (see [9, 10]).

3. $\varphi \oplus \psi$ -MODULE APPROXIMATE AMENABILITY OF THE DIRECT SUM OF BANACH ALGEBRAS

We commence this section with the following remark from [1]:

Remark 3.1. Assume that A has a bounded approximate identity $(e_\alpha)_\alpha$, and let $M_{\mathfrak{A}}(A)$ denotes the algebra of \mathfrak{A} -multipliers of A , that is $M_{\mathfrak{A}}(A) = \{(T_1, T_2) : T_1, T_2 \in L_{\mathfrak{A}}(A) : T_1(ab) = T_1(a)b, T_2(ab) = aT_2(b) (a, b \in A)\}$, where $L_{\mathfrak{A}}(A)$ is the space of all \mathfrak{A} -module morphisms on A . Then $M_{\mathfrak{A}}(A)$ is an A - \mathfrak{A} -module and A embeds in $M_{\mathfrak{A}}(A)$ via $a \mapsto (L_a, R_a)$, where $L_a(b) = ab, R_a(b) = ba$ ($a, b \in A$). For any element $T = (T_1, T_2)$ of $M_{\mathfrak{A}}(A)$ it is easy to see that $\|T_1\| = \|T_2\|$ and if we put $\|T\|$ equal to this common value, then $M_{\mathfrak{A}}(A)$ becomes a Banach A - \mathfrak{A} -module, and A is dense in $M_{\mathfrak{A}}(A)$ in the strict topology.

Before proving our next proposition we note that if $\varphi \in \text{Hom}_{\mathfrak{A}}(A)$, then by continuity of φ in the strict topology, it can be extended to an \mathfrak{A} -homomorphism $\tilde{\varphi} : M_{\mathfrak{A}}(A) \rightarrow M_{\mathfrak{A}}(A)$ defined by $\tilde{\varphi}(L_a, R_a) = (L_{\varphi(a)}, R_{\varphi(a)})$.

Proposition 3.2. *Let A be an \mathfrak{A} -module Banach algebra with a bounded approximate identity $(e_\alpha)_\alpha$, and let $\varphi \in \text{Hom}_{\mathfrak{A}}(A)$. Then A is φ - \mathfrak{A} -module approximately amenable if and only if $M_{\mathfrak{A}}(A)$ is $\tilde{\varphi}$ - \mathfrak{A} -module approximately amenable.*

Proof. Let $M_{\mathfrak{A}}(A)$ be $\tilde{\varphi}$ - \mathfrak{A} -module approximately amenable and let $D : A \rightarrow X^*$ be a φ - \mathfrak{A} -module derivation for some commutative Banach A - \mathfrak{A} -module X . Then by the following actions

$$T.x = \lim_{\alpha} T_1(e_\alpha).x, \quad x.T = \lim_{\alpha} x.T_2(e_\alpha) \quad (x \in X, T = (T_1, T_2) \in M_{\mathfrak{A}}(A)),$$

X is a commutative Banach $M_{\mathfrak{A}}(A)$ - \mathfrak{A} -module and by continuity of D in the strict topology, it can be extended to a bounded $\tilde{\varphi}$ - \mathfrak{A} -derivation $\tilde{D} : M_{\mathfrak{A}}(A) \rightarrow X^*$, defined by $\tilde{D}(L_a, R_a) = D(a)$. From the $\tilde{\varphi}$ - \mathfrak{A} -module approximate amenability of $M_{\mathfrak{A}}(A)$, it follows that there exists a net $(x_\beta^*)_\beta \subset X^*$ such that

$$\tilde{D}(T) = \lim_{\beta} (\tilde{\varphi}(T).x_\beta^* - x_\beta^*.\tilde{\varphi}(T)).$$

Hence for every $a \in A$ we have

$$\begin{aligned} D(a) &= \tilde{D}(L_a, R_a) = \lim_{\beta} (\tilde{\varphi}(L_a, R_a).x_\beta^* - x_\beta^*.\tilde{\varphi}(L_a, R_a)) \\ &= \lim_{\beta} ((L_{\varphi(a)}, R_{\varphi(a)}).x_\beta^* - x_\beta^*.(L_{\varphi(a)}, R_{\varphi(a)})) \\ &= \lim_{\beta} (\lim_{\alpha} L_{\varphi(a)}(e_\alpha).x_\beta^* - \lim_{\alpha} x_\beta^*.R_{\varphi(a)}(e_\alpha)) \\ &= \lim_{\beta} (\varphi(a).x_\beta^* - x_\beta^*.\varphi(a)). \end{aligned}$$

This means that D is approximately φ -inner and so A is φ - \mathfrak{A} -module approximately amenable.

Conversely, Suppose that A is φ - \mathfrak{A} -module approximately amenable. Let X be a commutative Banach $M_{\mathfrak{A}}(A)$ - \mathfrak{A} -module and let $D : M_{\mathfrak{A}}(A) \rightarrow X^*$ be a $\tilde{\varphi}$ - \mathfrak{A} -module derivation. We consider the module actions of A on X by

$$a.x = (L_a, R_a).x, \quad x.a = x.(L_a, R_a) \quad (a \in A, x \in X). \quad (3.1)$$

Thus X is a commutative Banach A - \mathfrak{A} -module. Define $\tilde{D} : A \rightarrow X^*$ by $\tilde{D}(a) = D(L_a, R_a)$ ($a \in A$). It is easy to see that \tilde{D} is a φ - \mathfrak{A} -module derivation and from the φ - \mathfrak{A} -module approximate amenability of A , it follows that there exists a net $(x_\beta^*)_\beta \subset X^*$ such that

$$\tilde{D}(a) = \lim_{\beta} (\varphi(a).x_\beta^* - x_\beta^*.\varphi(a)) \quad (a \in A).$$

Then $D(L_a, R_a) = \lim_{\beta} (\tilde{\varphi}(L_a, R_a).x_\beta^* - x_\beta^*.\tilde{\varphi}(L_a, R_a))$. Now by the continuity of D and $\tilde{\varphi}$, and density of A in $M_{\mathfrak{A}}(A)$ in the strict topology, we conclude that

$$D(T) = \lim_{\beta} (\tilde{\varphi}(T).x_\beta^* - x_\beta^*.\tilde{\varphi}(T)) \quad (T \in M_{\mathfrak{A}}(A)).$$

So D is a approximately $\tilde{\varphi}$ -inner. Therefore $M_{\mathfrak{A}}(A)$ is $\tilde{\varphi}$ - \mathfrak{A} -module approximately amenable. \square

Let I be a closed ideal of a Banach algebra A with a bounded approximate identity $(e_\alpha)_\alpha$, and let X be a commutative Banach I - \mathfrak{A} -module. Let $\varphi \in \text{Hom}_{\mathfrak{A}}(A)$ be such that $\varphi|_I \subset I$, then X is a commutative Banach A - \mathfrak{A} -module with the following actions

$$a.x = \lim_{\alpha} \varphi(e_\alpha)a.x, \quad x.a = \lim_{\alpha} x.\varphi(e_\alpha)a \quad (a \in A, x \in X). \quad (3.2)$$

Proposition 3.3. *Let I be a closed ideal of an \mathfrak{A} -module Banach algebra A which has a bounded approximate identity $\{e_\alpha\}$, and let I be \mathfrak{A} -invariant, i.e. $\mathfrak{A}.I \subseteq I$. Let $\varphi \in \text{Hom}_{\mathfrak{A}}(A)$ be such that $\varphi|_I \subset I$. If A is φ - \mathfrak{A} -module approximately amenable, then I is $\varphi|_I$ - \mathfrak{A} -module approximately amenable.*

Proof. Let X be a commutative Banach $M_{\mathfrak{A}}(I)$ - \mathfrak{A} -module, and $D : M_{\mathfrak{A}}(I) \rightarrow X^*$ be a $\tilde{\varphi}$ - \mathfrak{A} -module derivation. By the same actions as (3.1), we can consider X as a commutative Banach I - \mathfrak{A} -module. So, by (3.2), X is a commutative Banach A - \mathfrak{A} -module. By definition of $M_{\mathfrak{A}}(I)$, there is an \mathfrak{A} -module morphism $h : A \rightarrow M_{\mathfrak{A}}(I)$ and $D \circ h$ is a module derivation on A , so it is approximately φ -inner. Hence D is approximately $\tilde{\varphi}$ -inner. Since I has a bounded approximate identity, by Proposition 3.2, I is $\varphi|_I$ - \mathfrak{A} -module approximately amenable. \square

Let A and B be \mathfrak{A} -module Banach algebras. It is well known that $A \oplus B$, the l^1 -direct sum of A and B , is a Banach algebra with respect to the canonical multiplication defined by $(a, b)(c, d) := (ac, bd)$, and is a Banach \mathfrak{A} -bimodule by the following actions

$$\alpha.(a, b) := (\alpha.a, \alpha.b), \quad (a, b).\alpha := (a.\alpha, b.\alpha) \quad (\alpha \in \mathfrak{A}, a \in A, b \in B).$$

We note that if $\varphi \in \text{Hom}_{\mathfrak{A}}(A)$ and $\psi \in \text{Hom}_{\mathfrak{A}}(B)$, then $\varphi \oplus \psi : A \oplus B \rightarrow A \oplus B$ defined by $\varphi \oplus \psi(a, b) = (\varphi(a), \psi(b))$ is an \mathfrak{A} -morphism on $A \oplus B$.

Lemma 3.4. *Let A be a unital \mathfrak{A} -module Banach algebra, $\varphi \in \text{Hom}_{\mathfrak{A}}(A)$, and let $D : A \rightarrow X^*$ be a φ - \mathfrak{A} -module derivation for some commutative Banach A - \mathfrak{A} -module X . If the left (resp. right, two-sided) action of $\varphi(A)$ on X^* is zero, then D is φ -inner.*

Proof. Let e_A be the identity of A and let the left (resp. right, two-sided) action of $\varphi(A)$ on X^* is zero. We can easily show that $D = ad_{-D(e)}^{\varphi}$ (resp. $D = ad_{D(e)}^{\varphi}, D = 0$). So D is φ -inner. \square

The proof of the following proposition is adopted from that of Proposition 2.7 of [5].

Proposition 3.5. *Let A and B be unital \mathfrak{A} -module Banach algebras with identities e_A and e_B , respectively, and let $\varphi \in \text{Hom}_{\mathfrak{A}}(A)$ and $\psi \in \text{Hom}_{\mathfrak{A}}(B)$ such that $\varphi(e_A).\alpha = \alpha.\varphi(e_A)$, and $\psi(e_B).\alpha = \alpha.\psi(e_B)$ ($\alpha \in \mathfrak{A}$). If A is φ - \mathfrak{A} -module approximately amenable and B is ψ - \mathfrak{A} -module approximately amenable, then $A \oplus B$ is $\varphi \oplus \psi$ - \mathfrak{A} -module approximately amenable.*

Proof. Let X be a commutative Banach $A \oplus B$ - \mathfrak{A} -module and let $D : A \oplus B \rightarrow X^*$ be a $\varphi \oplus \psi$ - \mathfrak{A} -module derivation. Write $Y_1 = \varphi(e_A).X^*.\varphi(e_A)$, $Y_2 = \psi(e_B).X^*.\psi(e_B)$, $Y_3 = \varphi(e_A).X^*.\psi(e_B)$, $Y_4 = \psi(e_B).X^*.\varphi(e_A)$, $Y_5 = (1 - \varphi(e_A))(1 - \psi(e_B)).X^*.\varphi(e_A)$, $Y_6 = (1 - \varphi(e_A))(1 - \psi(e_B)).X^*.\psi(e_B)$, $Y_7 = \varphi(e_A).X^*.(1 - \varphi(e_A))(1 - \psi(e_B))$, $Y_8 = \psi(e_B).X^*.(1 - \varphi(e_A))(1 - \psi(e_B))$, $Y_9 = (1 - \varphi(e_A))(1 - \psi(e_B)).X^*.(1 - \varphi(e_A))(1 - \psi(e_B))$ and let $\pi_j : X^* \rightarrow Y_j$ be the associated projections. Thus $X^* = Y_1 \oplus Y_2 \oplus Y_3 \oplus Y_4 \oplus Y_5 \oplus Y_6 \oplus Y_7 \oplus Y_8 \oplus Y_9$. Consider the derivations $D_j = \pi_j \circ D$, so $D = D_1 + D_2 + D_3 + D_4 + D_5 + D_6 + D_7 + D_8 + D_9$. From the fact that $\varphi(e_A).\alpha = \alpha.\varphi(e_A)$ ($\alpha \in \mathfrak{A}$), and $\psi(e_B).\alpha = \alpha.\psi(e_B)$ ($\alpha \in \mathfrak{A}$), one can easily check that Y_j for $j = 1, \dots, 9$ is a commutative Banach $A \oplus B$ - \mathfrak{A} -module. Since the action of $\varphi(A) \oplus \psi(B)$ on (at least) one side on Y_5 (resp. Y_6, Y_7, Y_8, Y_9) is zero, by Lemma 3.4, we conclude that D_5 (resp. D_6, D_7, D_8, D_9) is approximately $\varphi \oplus \psi$ -inner.

From the φ - \mathfrak{A} -module approximate amenability of A , it follows that the $\varphi \oplus \psi$ - \mathfrak{A} -module derivation $A \oplus 0 \rightarrow \varphi(e_A).X^*.\varphi(e_A)$ is approximately $\varphi \oplus \psi$ -inner and since the action of $0 \oplus \psi(B)$ on $\varphi(e_A).X^*.\varphi(e_A)$ is zero, we conclude that D_1 is approximately $\varphi \oplus \psi$ -inner. Similarly, the $\varphi \oplus \psi$ - \mathfrak{A} -module derivation $D_2 : A \oplus B \rightarrow \psi(e_B).X^*.\psi(e_B)$ is approximately $\varphi \oplus \psi$ -inner.

The right action of $\varphi(A) \oplus 0$ on $\varphi(e_A).X^*.\psi(e_B)$ is zero. Hence, by Lemma 3.4, $D_3|_{A \oplus 0}$ is $\varphi \oplus \psi$ -inner. So there exists $\xi \in \varphi(e_A).X^*.\psi(e_B)$ such that

$$D_3|_{A \oplus 0}(a, 0) = \varphi(a).\xi - \xi.\varphi(a) = (\varphi(a), \psi(b))\varphi(e_A).\xi.\psi(e_B),$$

for every $a \in A$ and $b \in B$. Similarly, there exists $\eta \in \varphi(e_A).X^*.\psi(e_B)$ such that

$$D_3|_{0 \oplus B}(0, b) = \psi(b).\eta - \eta.\psi(b) = -\varphi(e_A).\eta.\psi(e_B)(\varphi(a), \psi(b)),$$

for every $a \in A$ and $b \in B$. Hence

$$D_3(a, b) = (\varphi(a), \psi(b))\varphi(e_A).\xi.\psi(e_B) - \varphi(e_A).\eta.\psi(e_B)(\varphi(a), \psi(b)).$$

Since $D_3(e_A, e_B) = 0$, it follows that

$$0 = D_3(e_A, e_B) = \varphi(e_A).\xi.\psi(e_B) - \varphi(e_A).\eta.\psi(e_B).$$

Then for every $a \in A$ and $b \in B$, we have

$$D_3(a, b) = (\varphi(a), \psi(b))\varphi(e_A).\xi.\psi(e_B) - \varphi(e_A).\xi.\psi(e_B)(\varphi(a), \psi(b)).$$

Thus D_3 is $\varphi \oplus \psi$ -inner. The same argument holds for the $\varphi \oplus \psi$ - \mathfrak{A} -module derivation $D_4 : A \oplus B \rightarrow \psi(e_B).X^*.\varphi(e_A)$. Therefore D is approximately $\varphi \oplus \psi$ -inner, and so $A \oplus B$ is $\varphi \oplus \psi$ - \mathfrak{A} -module approximately amenable. \square

Lemma 3.6. *Let A and B be \mathfrak{A} -module Banach algebras, $\varphi \in \text{Hom}_{\mathfrak{A}}(A)$ and $\psi \in \text{Hom}_{\mathfrak{A}}(B)$. If there is a h in $\text{Hom}_{\mathfrak{A}}(A, B)$ such that $h \circ \varphi = \psi \circ h$ and the range of h is a dense subset of B , then φ - \mathfrak{A} -module approximate amenability of A implies ψ - \mathfrak{A} -module approximate amenability of B .*

Proof. Let $D : B \rightarrow X^*$ be a ψ - \mathfrak{A} -module derivation for some commutative Banach B - \mathfrak{A} -module X . Then by the following actions

$$a \bullet x = h(a).x, \quad x \bullet a = x.h(a) \quad (a \in A, x \in X),$$

X is a commutative Banach A - \mathfrak{A} -module. Let $\tilde{D} = D \circ h : A \rightarrow X^*$. One can easily prove that \tilde{D} is a φ - \mathfrak{A} -module derivation. From the φ - \mathfrak{A} -module approximate amenability of A , it follows that there exists a net $(x_\alpha^*)_\alpha$ in X^* such that $\tilde{D}(a) = \lim_\alpha (\varphi(a) \bullet x_\alpha^* - x_\alpha^* \bullet \varphi(a))$ ($a \in A$). Now continuity and density of $h(A)$ in B , imply that D is approximately ψ -inner. Therefore B is ψ - \mathfrak{A} -module approximately amenable. \square

Proposition 3.7. *Let A and B be \mathfrak{A} -module Banach algebras, $\varphi \in \text{Hom}_{\mathfrak{A}}(A)$ and $\psi \in \text{Hom}_{\mathfrak{A}}(B)$. If A is not φ - \mathfrak{A} -module approximately amenable or B is not ψ - \mathfrak{A} -module approximately amenable, then $A \oplus B$ is not $\varphi \oplus \psi$ - \mathfrak{A} -module approximately amenable.*

Proof. Suppose that A is not φ - \mathfrak{A} -module approximately amenable. The projection map $\pi : A \oplus B \rightarrow A$ determines an \mathfrak{A} -module epimorphism of $A \oplus B$ onto A such that $\pi \circ (\varphi \oplus \psi) = \varphi \circ \pi$. So, if $A \oplus B$ is $\varphi \oplus \psi$ - \mathfrak{A} -module approximately amenable, then by Lemma 3.6, A is φ - \mathfrak{A} -module approximately amenable. This contradicts the fact that A is not φ - \mathfrak{A} -module approximately amenable. Therefore $A \oplus B$ is not $\varphi \oplus \psi$ - \mathfrak{A} -module approximately amenable. Similarly, we can prove the result for B . \square

Let \mathfrak{A} be a non-unital Banach algebra. Then $\mathfrak{A}^\# = \mathfrak{A} \oplus \mathbb{C}$, the unitization of \mathfrak{A} is a unital Banach algebra which contains \mathfrak{A} as a closed ideal. Let A be a Banach \mathfrak{A} -bimodule. Then A is a Banach $\mathfrak{A}^\#$ -module with the following module actions:

$$(\alpha, \lambda).a = \alpha.a + \lambda a, \quad a.(\alpha, \lambda) = a.\alpha + \lambda a \quad (\lambda \in \mathbb{C}, \alpha \in \mathfrak{A}, a \in A).$$

Let $A^\# = (A \oplus \mathfrak{A}^\#, \bullet)$, where the multiplication \bullet is defined through

$$(a, u) \bullet (b, v) = (ab + a.v + u.b, uv) \quad (a, b \in A, u, v \in \mathfrak{A}^\#).$$

Then with the actions defined by

$$u.(a, v) = (u.a, uv), \quad (a, v).u = (a.u, vu) \quad (a \in A, u, v \in \mathfrak{A}^\#),$$

$A^\#$ is a unital $\mathfrak{A}^\#$ -module Banach algebra with the identity $1_{A^\#} = (0, 1_{\mathfrak{A}^\#})$ (see [4]).

Before we turn to our next result we note that if for every $\varphi \in \text{Hom}_{\mathfrak{A}^\#}(A)$, one defines $\varphi^\# : A^\# \rightarrow A^\#$ by $\varphi^\#(a, u) = (\varphi(a), u)$ ($(a, u) \in A^\#$), then $\varphi^\# \in \text{Hom}_{\mathfrak{A}^\#}(A^\#)$.

The following proposition generalizes Proposition 2.7 of [5].

Theorem 3.8. *Let A and B be \mathfrak{A} -module Banach algebras and each has a bounded approximate identity. Let $\varphi \in \text{Hom}_{\mathfrak{A}^\#}(A)$ and $\psi \in \text{Hom}_{\mathfrak{A}^\#}(B)$. Then A*

is $\varphi\text{-}\mathfrak{A}^\#$ -module approximately amenable and B is $\psi\text{-}\mathfrak{A}^\#$ -module approximately amenable if and only if $A \oplus B$ is $\varphi \oplus \psi\text{-}\mathfrak{A}^\#$ -module approximately amenable.

Proof. Suppose that A is $\varphi\text{-}\mathfrak{A}^\#$ -module approximately amenable and B is $\psi\text{-}\mathfrak{A}^\#$ -module approximately amenable. By Proposition 12 of [13], $A^\#$ is $\varphi^\#\text{-}\mathfrak{A}^\#$ -module approximately amenable and $B^\#$ is $\psi^\#\text{-}\mathfrak{A}^\#$ -module approximately amenable, so by Proposition 3.5, $A^\# \oplus B^\#$ is $\varphi^\# \oplus \psi^\#\text{-}\mathfrak{A}^\#$ -module approximately amenable. Since $A \oplus B$ is a closed $\mathfrak{A}^\#$ -invariant ideal in $A^\# \oplus B^\#$, the result follows from Proposition 3.3.

For the converse, suppose that $A \oplus B$ is $\varphi \oplus \psi\text{-}\mathfrak{A}^\#$ -module approximately amenable. Then by Proposition 3.7, A is $\varphi\text{-}\mathfrak{A}^\#$ -module approximately amenable and B is $\psi\text{-}\mathfrak{A}^\#$ -module approximately amenable. \square

4. $\varphi \oplus \psi$ -MODULE APPROXIMATE AMENABILITY AND $\varphi \oplus \psi$ -AMENABILITY OF DIRECT SUM OF BANACH ALGEBRAS

We start this section with the following definition:

Definition 4.1. We say the Banach algebra \mathfrak{A} acts trivially on A from the left (right) if for every $\alpha \in \mathfrak{A}$ and $a \in A$, $\alpha.a = f(\alpha)a$ (resp. $a.\alpha = f(\alpha)a$), where f is a multiplicative linear functional on \mathfrak{A} .

We assume that $J_{A,\mathfrak{A}}$ is the closed linear span of

$$\{(a.\alpha)b - a(\alpha.b) \mid \alpha \in \mathfrak{A}, a, b \in A\},$$

in A . It follows immediately that $J_{A,\mathfrak{A}}$ is both A -submodule and \mathfrak{A} -submodule of A . So $\frac{A}{J_{A,\mathfrak{A}}}$ is both Banach A -module and \mathfrak{A} -module (see page 346 of [14]).

To prove our next result we need to quote the following lemma from [2].

Lemma 4.2. Let A be a Banach algebra and Banach \mathfrak{A} -module with compatible actions, and J_0 be a closed ideal of A such that $J_{A,\mathfrak{A}} \subseteq J_0$. If $\frac{A}{J_0}$ has a left or right identity $e + J_0$, then for each $\alpha \in \mathfrak{A}$ and $a \in A$ we have $a.\alpha - \alpha.a \in J_0$, i.e., $\frac{A}{J_0}$ is commutative Banach \mathfrak{A} -module.

Before we turn to our next result we note that if for every $\varphi \in \text{Hom}_{\mathfrak{A}}(A)$, one defines $\bar{\varphi} : \frac{A}{J_{A,\mathfrak{A}}} \rightarrow \frac{A}{J_{A,\mathfrak{A}}}$ by $\bar{\varphi}(a + J_{A,\mathfrak{A}}) = \varphi(a) + J_{A,\mathfrak{A}}$, then $\bar{\varphi} \in \text{Hom}_{\mathfrak{A}}(\frac{A}{J_{A,\mathfrak{A}}})$.

Theorem 4.3. Let A and B be \mathfrak{A} -module Banach algebras and let $\varphi \in \text{Hom}_{\mathfrak{A}}(A)$ and $\psi \in \text{Hom}_{\mathfrak{A}}(B)$. Then the following statements are valid:

- (i) $A \oplus B$ is $\varphi \oplus \psi\text{-}\mathfrak{A}$ -module amenable (resp. $\varphi \oplus \psi\text{-}\mathfrak{A}$ -module approximately amenable) if and only if $\frac{A}{J_{A,\mathfrak{A}}} \oplus \frac{B}{J_{B,\mathfrak{A}}}$ is $\bar{\varphi} \oplus \bar{\psi}\text{-}\mathfrak{A}$ -module amenable (resp. $\bar{\varphi} \oplus \bar{\psi}\text{-}\mathfrak{A}$ -module approximately amenable).
- (ii) Let \mathfrak{A} acts on A and B trivially from the left by $f \in \text{Hom}_{\mathbb{C}}(\mathfrak{A})$. Suppose that $\frac{A}{J_{A,\mathfrak{A}}}$ and $\frac{B}{J_{B,\mathfrak{A}}}$ are unital, and $A \oplus B$ is $\varphi \oplus \psi\text{-}\mathfrak{A}$ -module amenable (resp. $\varphi \oplus \psi\text{-}\mathfrak{A}$ -module approximately amenable), then $\frac{A}{J_{A,\mathfrak{A}}} \oplus \frac{B}{J_{B,\mathfrak{A}}}$ is $\bar{\varphi} \oplus \bar{\psi}$ -amenable (resp. $\bar{\varphi} \oplus \bar{\psi}$ -approximately amenable).

(iii) Let \mathfrak{A} have a bounded approximately identity and $\frac{A}{J_{A,\mathfrak{A}}} \oplus \frac{B}{J_{B,\mathfrak{A}}}$ is $\overline{\varphi} \oplus \overline{\psi}$ -amenable (resp. $\overline{\varphi} \oplus \overline{\psi}$ -approximately amenable). Then $A \oplus B$ is $\varphi \oplus \psi$ - \mathfrak{A} -module amenable (resp. $\varphi \oplus \psi$ - \mathfrak{A} -module approximately amenable).

Proof. (i) Let $A \oplus B$ be $\varphi \oplus \psi$ - \mathfrak{A} -module amenable, and let $D : \frac{A}{J_{A,\mathfrak{A}}} \oplus \frac{B}{J_{B,\mathfrak{A}}} \rightarrow X^*$ be $\overline{\varphi} \oplus \overline{\psi}$ - \mathfrak{A} -module derivation for some commutative Banach $\frac{A}{J_{A,\mathfrak{A}}} \oplus \frac{B}{J_{B,\mathfrak{A}}}$ - \mathfrak{A} -module X . Then X becomes a $A \oplus B$ -bimodule through the following actions

$$(a, b).x := (a + J_{A,\mathfrak{A}}, b + J_{B,\mathfrak{A}}).x \quad (a \in A, b \in B, x \in X), \tag{4.1}$$

and

$$x.(a, b) := x.(a + J_{A,\mathfrak{A}}, b + J_{B,\mathfrak{A}}) \quad (a \in A, b \in B, x \in X). \tag{4.2}$$

Hence X is a commutative Banach $A \oplus B$ - \mathfrak{A} -module. Define $\tilde{D} : A \oplus B \rightarrow X^*$ by

$$\tilde{D}(a, b) = D(a + J_{A,\mathfrak{A}}, b + J_{B,\mathfrak{A}}) \quad (a \in A, b \in B).$$

It is easy to check that, \tilde{D} is a $\varphi \oplus \psi$ - \mathfrak{A} -module derivation. From the $\varphi \oplus \psi$ - \mathfrak{A} -module amenability of $A \oplus B$, it follows that there exists $x^* \in X^*$ such that

$$\tilde{D}(a, b) = \varphi \oplus \psi(a, b).x^* - x^*.\varphi \oplus \psi(a, b) \quad (a \in A, b \in B).$$

Thus

$$\begin{aligned} D(a + J_{A,\mathfrak{A}}, b + J_{B,\mathfrak{A}}) &= \overline{\varphi} \oplus \overline{\psi}(a + J_{A,\mathfrak{A}}, b + J_{B,\mathfrak{A}}).x^* \\ &\quad - x^*.\overline{\varphi} \oplus \overline{\psi}(a + J_{A,\mathfrak{A}}, b + J_{B,\mathfrak{A}}). \end{aligned}$$

This means that D is $\overline{\varphi} \oplus \overline{\psi}$ -inner. Therefore $\frac{A}{J_{A,\mathfrak{A}}} \oplus \frac{B}{J_{B,\mathfrak{A}}}$ is $\overline{\varphi} \oplus \overline{\psi}$ - \mathfrak{A} -module amenable.

Conversely, suppose that $\frac{A}{J_{A,\mathfrak{A}}} \oplus \frac{B}{J_{B,\mathfrak{A}}}$ is $\overline{\varphi} \oplus \overline{\psi}$ - \mathfrak{A} -module amenable. Let $D : A \oplus B \rightarrow X^*$ be a $\varphi \oplus \psi$ - \mathfrak{A} -module derivation for some commutative Banach $A \oplus B$ - \mathfrak{A} -module X . We consider the following module actions of $\frac{A}{J_{A,\mathfrak{A}}} \oplus \frac{B}{J_{B,\mathfrak{A}}}$ on X ,

$$(a + J_{A,\mathfrak{A}}, b + J_{B,\mathfrak{A}}).x := (a, b).x, \quad x.(a + J_{A,\mathfrak{A}}, b + J_{B,\mathfrak{A}}) := x.(a, b),$$

for all $a \in A, b \in B$ and $x \in X$. Using (2.1) and the commutativity of X , we have $J_{A,\mathfrak{A}}X = J_{B,\mathfrak{A}}X = XJ_{A,\mathfrak{A}} = XJ_{B,\mathfrak{A}} = 0$. Thus $(J_{A,\mathfrak{A}} \oplus J_{B,\mathfrak{A}})X = X(J_{A,\mathfrak{A}} \oplus J_{B,\mathfrak{A}}) = 0$. So X is a commutative Banach $\frac{A}{J_{A,\mathfrak{A}}} \oplus \frac{B}{J_{B,\mathfrak{A}}}$ - \mathfrak{A} -module. Define $\tilde{D} : \frac{A}{J_{A,\mathfrak{A}}} \oplus \frac{B}{J_{B,\mathfrak{A}}} \rightarrow X^*$ by

$$\tilde{D}(a + J_{A,\mathfrak{A}}, b + J_{B,\mathfrak{A}}) = D(a, b) \quad (a \in A, b \in B).$$

Also using (2.2) and (2.3) we see that D vanishes on $J_{A,\mathfrak{A}} \oplus J_{B,\mathfrak{A}}$. Hence \tilde{D} is well defined. One can easily check that \tilde{D} is a $\overline{\varphi} \oplus \overline{\psi}$ - \mathfrak{A} -module derivation.

Now from the $\bar{\varphi} \oplus \bar{\psi}$ - \mathfrak{A} -module amenability of $\frac{A}{J_{A,\mathfrak{A}}} \oplus \frac{B}{J_{B,\mathfrak{A}}}$, it follows that there exists $x^* \in X^*$ such that

$$\begin{aligned} \tilde{D}(a + J_{A,\mathfrak{A}}, b + J_{B,\mathfrak{A}}) &= \bar{\varphi} \oplus \bar{\psi}(a + J_{A,\mathfrak{A}}, b + J_{B,\mathfrak{A}}).x^* \\ &\quad - x^*.\bar{\varphi} \oplus \bar{\psi}(a + J_{A,\mathfrak{A}}, b + J_{B,\mathfrak{A}}) \quad (a \in A, b \in B). \end{aligned}$$

It follows that

$$D(a, b) = \varphi \oplus \psi(a, b).x^* - x^*.\varphi \oplus \psi(a, b) \quad (a \in A, b \in B).$$

Thus D is $\varphi \oplus \psi$ -inner. So $A \oplus B$ is $\varphi \oplus \psi$ - \mathfrak{A} -module amenable.

Similarly, we can show that $A \oplus B$ is $\varphi \oplus \psi$ - \mathfrak{A} -module approximately amenable if and only if $\frac{A}{J_{A,\mathfrak{A}}} \oplus \frac{B}{J_{B,\mathfrak{A}}}$ is $\bar{\varphi} \oplus \bar{\psi}$ - \mathfrak{A} -module approximately amenable.

(ii) Let $A \oplus B$ be $\varphi \oplus \psi$ - \mathfrak{A} -module amenable and let $D : \frac{A}{J_{A,\mathfrak{A}}} \oplus \frac{B}{J_{B,\mathfrak{A}}} \rightarrow X^*$ be a derivation for some Banach $\frac{A}{J_{A,\mathfrak{A}}} \oplus \frac{B}{J_{B,\mathfrak{A}}}$ -bimodule X . Then X becomes a $A \oplus B$ -bimodule through the actions as (4.1) and (4.2). Also X is an \mathfrak{A} -bimodule with f -trivial actions, that is

$$\alpha.x = x.\alpha = f(\alpha)x \quad (\alpha \in \mathfrak{A}, x \in X).$$

Then X is a commutative Banach $A \oplus B$ - \mathfrak{A} -module. Define

$$\Gamma : \frac{A \oplus B}{I} \rightarrow \frac{A}{J_{A,\mathfrak{A}}} \oplus \frac{B}{J_{B,\mathfrak{A}}}, \quad (a, b) + I \mapsto (a + J_{A,\mathfrak{A}}, b + J_{B,\mathfrak{A}}),$$

where $I = J_{A,\mathfrak{A}} \oplus J_{B,\mathfrak{A}}$. It is routinely checked that Γ defines an \mathfrak{A} -bimodule morphism. Let $\Pi : A \oplus B \rightarrow \frac{A \oplus B}{I}$ be the quotient map, and let $\tilde{D} := D \circ \Gamma \circ \Pi : A \oplus B \rightarrow X^*$. For every $(a, b), (a', b') \in A \oplus B$, we may easily prove that

$$\tilde{D}((a, b)(a', b')) = \tilde{D}(a, b).\varphi \oplus \psi(a', b') + \varphi \oplus \psi(a, b).\tilde{D}(a', b'),$$

and for every $(a, b) \in A \oplus B$, and $\alpha \in \mathfrak{A}$, we have

$$\begin{aligned} \tilde{D}(\alpha.(a, b)) &= \tilde{D}((\alpha.a, \alpha.b)) = \tilde{D}((f(\alpha)a, f(\alpha)b)) \\ &= D((f(\alpha)a + J_{A,\mathfrak{A}}, f(\alpha)b + J_{B,\mathfrak{A}})) \\ &= D(f(\alpha)(a + J_{A,\mathfrak{A}}, b + J_{B,\mathfrak{A}})) \\ &= f(\alpha)D((a + J_{A,\mathfrak{A}}, b + J_{B,\mathfrak{A}})) \\ &= \alpha.D((a + J_{A,\mathfrak{A}}, b + J_{B,\mathfrak{A}})) \\ &= \alpha.\tilde{D}(a, b), \end{aligned}$$

and using Lemma 4.2, we have

$$\begin{aligned} \tilde{D}\left((a, b).\alpha\right) &= \tilde{D}\left((a.\alpha, b.\alpha)\right) = D\left((a.\alpha + J_{A,\mathfrak{A}}, b.\alpha + J_{B,\mathfrak{A}})\right) \\ &= D\left((\alpha.a + J_{A,\mathfrak{A}}, \alpha.b + J_{B,\mathfrak{A}})\right) \\ &= D\left(f(\alpha)(a + J_{A,\mathfrak{A}}, b + J_{B,\mathfrak{A}})\right) \\ &= f(\alpha)D\left((a + J_{A,\mathfrak{A}}, b + J_{B,\mathfrak{A}})\right) \\ &= D\left((a + J_{A,\mathfrak{A}}, b + J_{B,\mathfrak{A}})\right).\alpha \\ &= \tilde{D}(a, b).\alpha. \end{aligned}$$

Thus \tilde{D} is a $\varphi \oplus \psi$ - \mathfrak{A} -module derivation and from the $\varphi \oplus \psi$ - \mathfrak{A} -module amenability of $A \oplus B$, it follows that there exists $x^* \in X^*$ such that

$$\tilde{D}(a, b) = \varphi \oplus \psi(a, b).x^* - x^*.\varphi \oplus \psi(a, b) \quad (a \in A, b \in B).$$

It follows that

$$\begin{aligned} D(a + J_{A,\mathfrak{A}}, b + J_{B,\mathfrak{A}}) &= \bar{\varphi} \oplus \bar{\psi}(a + J_{A,\mathfrak{A}}, b + J_{B,\mathfrak{A}}).x^* \\ &\quad - x^*.\bar{\varphi} \oplus \bar{\psi}(a + J_{A,\mathfrak{A}}, b + J_{B,\mathfrak{A}}). \end{aligned}$$

So D is $\bar{\varphi} \oplus \bar{\psi}$ -inner. Therefore $\frac{A}{J_{A,\mathfrak{A}}} \oplus \frac{B}{J_{B,\mathfrak{A}}}$ is $\bar{\varphi} \oplus \bar{\psi}$ -amenable.

(iii) Suppose that $\frac{A}{J_{A,\mathfrak{A}}} \oplus \frac{B}{J_{B,\mathfrak{A}}}$ is $\bar{\varphi} \oplus \bar{\psi}$ -amenable. Since \mathfrak{A} has a bounded approximate identity, by Proposition 2.1 of [1], we conclude that $\frac{A}{J_{A,\mathfrak{A}}} \oplus \frac{B}{J_{B,\mathfrak{A}}}$ is $\bar{\varphi} \oplus \bar{\psi}$ - \mathfrak{A} -module amenable. So by (i), $A \oplus B$ is $\varphi \oplus \psi$ - \mathfrak{A} -module amenable.

Similar relations can be obtained between the $\varphi \oplus \psi$ - \mathfrak{A} -module approximate amenability of $A \oplus B$ and $\bar{\varphi} \oplus \bar{\psi}$ -approximate amenability of $\frac{A}{J_{A,\mathfrak{A}}} \oplus \frac{B}{J_{B,\mathfrak{A}}}$. \square

Proposition 4.4. *Let A be an \mathfrak{A} -module Banach algebra, where \mathfrak{A} acts on A trivially from the left by $f \in \text{Hom}_{\mathbb{C}}(\mathfrak{A})$. Let $\varphi \in \text{Hom}_{\mathfrak{A}}(A)$ and $\frac{A}{J_{A,\mathfrak{A}}}$ be unital. If A is φ - \mathfrak{A} -module approximately amenable, then $\frac{A}{J_{A,\mathfrak{A}}}$ is $\bar{\varphi}$ -approximately amenable.*

Proof. Let X be a Banach $\frac{A}{J_{A,\mathfrak{A}}}$ -bimodule and $D : \frac{A}{J_{A,\mathfrak{A}}} \rightarrow X^*$ be a $\bar{\varphi}$ -derivation. Then X becomes a A -bimodule through the following actions

$$a.x = (a + J_{A,\mathfrak{A}}).x, \quad x.a = x.(a + J_{A,\mathfrak{A}}) \quad (a \in A, x \in X),$$

and X is an \mathfrak{A} -bimodule with f -trivial actions, that is $\alpha.x = x.\alpha = f(\alpha)x$ ($\alpha \in \mathfrak{A}, x \in X$). By Lemma 4.2, $f(\alpha)a - a.\alpha \in J_{A,\mathfrak{A}}$ ($\alpha \in \mathfrak{A}, a \in A$). So, $f(\alpha)a + J_{A,\mathfrak{A}} = a.\alpha + J_{A,\mathfrak{A}}$ ($\alpha \in \mathfrak{A}, a \in A$), and the actions of \mathfrak{A} and A on X are compatible. Thus X is a commutative Banach A - \mathfrak{A} -module. Let $\tilde{D} : A \rightarrow X^*$ be defined by $\tilde{D}(a) = D(a + J_{A,\mathfrak{A}})$ ($a \in A$). A similar argument as in the proof of Theorem 3.2 of [2], shows that \tilde{D} is approximately φ -inner. So, D is approximately $\bar{\varphi}$ -inner. Therefore $\frac{A}{J_{A,\mathfrak{A}}}$ is $\bar{\varphi}$ -approximately amenable. \square

Theorem 4.5. *Let \mathfrak{A} have a bounded approximate identity, and let A and B be \mathfrak{A} -module Banach algebras, where \mathfrak{A} acts on A and B trivially from the left. Let $\varphi \in \text{Hom}_{\mathfrak{A}}(A)$, $\psi \in \text{Hom}_{\mathfrak{A}}(B)$, and let $\frac{A}{J_{A,\mathfrak{A}}}$ and $\frac{B}{J_{B,\mathfrak{A}}}$ be unital. Then A is φ - \mathfrak{A} -module approximately amenable and B is ψ - \mathfrak{A} -module approximately amenable if and only if $A \oplus B$ is $\varphi \oplus \psi$ - \mathfrak{A} -module approximately amenable.*

Proof. Suppose that A is φ - \mathfrak{A} -module approximately amenable and B is ψ - \mathfrak{A} -module approximately amenable. By Proposition 4.4, $\frac{A}{J_{A,\mathfrak{A}}}$ and $\frac{B}{J_{B,\mathfrak{A}}}$ are $\bar{\varphi}$ -approximately amenable and $\bar{\psi}$ -approximately amenable, respectively. Now by using Proposition 3.5 for $\mathfrak{A} = \mathbb{C}$, we conclude that $\frac{A}{J_{A,\mathfrak{A}}} \oplus \frac{B}{J_{B,\mathfrak{A}}}$ is $\bar{\varphi} \oplus \bar{\psi}$ -approximately amenable. So, Theorem 4.3, implies that $A \oplus B$ is $\varphi \oplus \psi$ - \mathfrak{A} -module approximately amenable.

Conversely, suppose that $A \oplus B$ is $\varphi \oplus \psi$ - \mathfrak{A} -module approximately amenable. Then by Proposition 3.7, A is φ - \mathfrak{A} -module approximately amenable and B is ψ - \mathfrak{A} -module approximately amenable. \square

ACKNOWLEDGMENTS

The author is thankful to the referees for their valuable suggestions and comments and is grateful to the office of Graduate studies of the university of Isfahan for their support.

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