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Distance-Balanced Closure of Some Graphs

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ABSTRACT. In this paper we prove that any distance-balanced graph G with $\Delta(G) \geq |V(G)| - 3$ is regular. Also we define notion of distance-balanced closure of a graph and we find distance-balanced closures of trees T with $\Delta(T) \geq |V(T)| - 3$.

Keywords: Distances in graphs, Distance-balanced graphs, Distance-balanced closure.

2000 Mathematics subject classification: 05B20, 05E30.

1. Introduction

Let G be a graph with vertex set V(G) and edge set E(G). We denote |V(G)| by n. The set of neighbors of a vertex $v \in V(G)$ is denoted by $N_G(v)$, and $N_G[v] = N_G(v) \cup \{v\}$. The degree of a vertex v is denoted by $\deg_G(v)$ and minimum degree and maximum degree of G denoted by $\delta(G)$ and $\Delta(G)$,

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respectively. The distance $d_G(u, v)$ between vertices u and v is the length of a shortest path between u and v in G. The diameter $\operatorname{diam}(G)$ of graph G is defined as $\max\{d_G(u, v) : u, v \in V(G)\}$. The notion of distance is studied in several works in graph theory (See [2] and the references therein) and many research works are based on the concepts related to this notion (See for instance [8] and [10]).

For an edge xy of a graph G, W_{xy}^G is the set of vertices which are closer to x than y, more formally

$$W_{xy}^G = \{ u \in V(G) | d_G(u, x) < d_G(u, y) \}.$$

Moreover, $_xW_y^G$ is the set of vertices of G that have equal distances to x and y, that is

$$_{x}W_{u}^{G} = \{u \in V(G) | d_{G}(u, x) = d_{G}(u, y)\}.$$

These sets play important roles in metric graph theory, see for instance [1, 3, 4, 5]. Since x always belongs to W_{xy}^G , for convenience we let $U_{xy}^G = W_{xy}^G \setminus \{x\}$. Distance-balanced graphs are introduced in [9] as graphs for which $|W_{xy}^G| = |W_{yx}^G|$ (or equivalently $|U_{xy}^G| = |U_{yx}^G|$) for every pair of adjacent vertices $x, y \in V(G)$.

In [9], the parameter b(G) of a graph G is introduced as the smallest number of the edges which can be added to G such that the obtained graph is distance-balanced. Since the complete graph is distance-balanced, this parameter is well-defined. We call graph G a distance-balanced closure of H if G is distance-balanced and H is a spanning subgraph of G with |E(G)| = b(H) + |E(H)|; in other words, a distance-balanced closure of H is a distance-balanced graph G which contains H as a spanning subgraph and has minimum number of edges. As mentioned in [9], the computation of b(G) is quite hard in general but it might be interesting in some special cases. In this paper we compute b(G) for all trees T with $\Delta(T) \geq |V(T)| - 3$. In Section 2, we compute that distance-balanced closure of graphs G with $\Delta(G) = n - 1$. In Section 3, and Section 4, we concern graphs G with $\Delta(G) = n - 2$ and $\Delta(G) = n - 3$, respectively. Then we compute b(T) for all trees T with $\Delta(T) = n - 2$ and $\Delta(T) = n - 3$.

Here we mention some more definitions and notations about trees. Let P_n denoted the path with n vertices. A tree which has exactly one vertex of degree greater than two is said to be starlike. The vertex of maximum degree is called the $central\ vertex$. We denote by $S(n_1,n_2,\ldots,n_k)$ a starlike tree in which removing the central vertex leaves disjoint paths $P_{n_1},P_{n_2},\ldots,P_{n_k}$. We say that $S(n_1,n_2,\ldots,n_k)$ has branches of length n_1,n_2,\ldots,n_k . It is obvious that $S(n_1,n_2,\ldots,n_k)$ has $n_1+n_2+\ldots+n_k+1$ vertices. For simplicity a starlike with α_i branches of length n_i $(1 \le i \le k)$ is denoted by $S(n_1^{\alpha_1},n_2^{\alpha_2},\ldots,n_k^{\alpha_k})$.

2. Distance-Balanced Graphs with Maximum Degree n-1

In this section we prove that for any graph G with $\Delta(G) = n - 1$, the only distance-balanced closure of G is the complete graph K_n . The following result is very useful in this paper. It is in fact a slight modification of Corollary 2.3 of [9].

Theorem 2.1. Let G be a graph with diameter at most 2 and H be a distance-balanced graph such that G is a spanning subgraph of H. Then H is a regular graph. Moreover, every regular graph with diameter at most 2 is distance-balanced.

Corollary 2.2. For every integer $m \ge 1$, the graph $K_{1,m}$ has a unique distance-balanced closure which is isomorphic to K_{m+1} , hence, $b(K_{1,m}) = {m+1 \choose 2} - m$.

Proof. Let G be a distance-balanced closure of $K_{1,m}$. By Theorem 2.1, G is a regular graph and since $K_{1,m}$ has a vertex of degree m, G should be m-regular, hence $G \cong K_{m+1}$.

The following is an immediate conclusion of Theorem 2.1.

Corollary 2.3. Let G be a graph with n vertices and $\Delta(G) = n - 1$. Then the graph G has a unique distance balanced closure. Moreover, this closure is isomorphic to K_n .

3. Distance-Balanced Graphs with Maximum Degree n-2

In this section, we prove that any distance-balanced graph G with $\Delta(G) = n-2$ is a regular graph using this, we construct a distance-balanced closure of T where T is a tree with this property (that is $\Delta(T) = n-2$) and then compute b(T).

The following Lemma will be used occasionally in this paper and the proof is easily deduced from the definition of U_{xy}^G .

Lemma 3.1. Let x and y be two adjacent vertices of a graph G, then $U_{xy}^G \cap N_G(y) = \emptyset$ (or $U_{xy}^G \subseteq V(G) \setminus N_G[y]$). Furthermore, $N_G[y] \setminus U_{yx}^G \subseteq N_G[x]$.

Theorem 3.2. Let $T = S(2, 1^{m-1})$ be a starlike tree and H be a distance-balanced graph containing T as a spanning subgraph. Then $\operatorname{diam}(H) \leq 2$, hence, H is an r-regular graph for some $m \leq r \leq m+1$.

Proof. Suppose that the vertices of T are labeled as shown in Figure 1. If $oy \in E(H)$, then H contains $K_{1,m+1}$ as a spanning subgraph, so by Corollary 2.2, $H \cong K_{m+2}$.

So, we may assume that $oy \notin E(H)$ (and consequently diam $(H) \neq 1$). We prove diam(H) = 2. For this, it is enough to show that $d(y, x_i) \leq 2$ for $i = 2, \dots, m$. Let i be an integer with $2 \leq i \leq m$; using the fact that y is the only vertex which is not adjacent to o in H and using Lemma 3.1, we

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conclude that $U_{x_io}^H \subseteq \{y\}$; If $U_{x_io}^H = \{y\}$, then $x_iy \in E(H)$ and $d_H(x_i,y) = 1$, hence in this case $d_H(x_i,y) \leq 2$. Otherwise, $U_{x_io}^H = \emptyset$, in this case, for every $z \in V(H) \setminus \{o,x_i\}$ we have $d_H(z,x_i) = d_H(z,o)$, particularly, $d_H(z,x_i) = d_H(y,o) = 2$. Hence, diam(H) = 2, as required. The result is now concluded using Theorem 2.1.

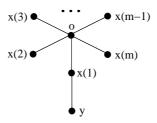


Figure 1

Theorem 3.3. Let $T = S(2, 1^{m-1})$ be a starlike of order m + 2. Then

$$b(T) = \begin{cases} \frac{m^2}{2} - 1 & if m \text{ is even;} \\ \binom{m+1}{2} & otherwise. \end{cases}$$

Proof. Suppose that the vertices of T be labeled as in Figure 1, and let \overline{T} be a distance-balanced closure of T. First, suppose that m is an odd integer; Since there is no m-regular graph of order m+2, by Theorem 3.2, $\overline{T} \cong K_{m+2}$.

Now, suppose that m is an even integer. Let $H=K_{m+2}$ be a complete graph with vertex set V(T) and M be a complete matching of H which contains the edge oy. Then $H \setminus M$, is an m-regular graph with diameter 2 and contains T as a spanning subgraph. Hence, by Theorem 2.1 and Theorem 3.2, $\overline{T} = H \setminus M$, is a distance-balanced closure of T and $b(T) = \frac{m^2}{2} - 1$.

Corollary 3.4. Let G be a connected graph of order n, with $\Delta(G) = n-2$ and H be a distance-balanced graph which contains G as a spanning subgraph. Then H is either an (n-2)-regular graph or the complete graph K_n .

Proof. In this case $S(2,1^{n-3})$ is an spanning subgraph of G. So, by Theorem 3.2, H is either an (n-2)-regular graph or the complete graph K_n .

4. Distance-Balanced Graphs with Maximum Degree n-3

In this section we will prove that every distance-balanced graph with $\Delta(G) = n-3$ is regular. Moreover, by constructing distance-balanced closure of trees with $\Delta(T) = n-3$ we compute b(T) for these trees.

Theorem 4.1. Let $T = S(2^2, 1^{m-2})$ be a starlike of order m+3 and H be a distance-balanced graph which contains T as a spanning subgraph. Then $\operatorname{diam}(H) \leq 2$. Moreover, H is an r-regular graph with $r \geq m$.

Proof. Suppose the vertices of T are labeled as in Figure 2. If either oy or oz be an edge of H, then by Theorem 3.2, H is an r-regular graph with $r \ge m+1$, which proves this theorem. So, suppose that oy, $oz \notin E(H)$.

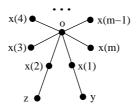


Figure 2

By Lemma 3.1, for each $1 \leq i \leq m$, $U_{x_io}^H \subseteq \{y, z\}$. Now, we prove that $\deg_H(x_i) = m$, for each $i = 1, \dots, m$. For this, we consider three possible cases:

Case 1. $|U_{x_io}^H| = 0$: Then $U_{x_io}^H = \emptyset$ and $U_{ox_i}^H = \emptyset$. Hence, by Lemma 3.1, $N_H(x_i) = \{o, x_1, x_2, \dots, x_m\} \setminus \{x_i\}$ and $\deg_H(x_i) = m$.

 $N_H(x_i) = \{o, x_1, x_2, \dots, x_m\} \setminus \{x_i\}$ and $\deg_H(x_i) = m$. **Case 2.** $|U_{x_io}^H| = 1$: Without loss of generality we can assume that $U_{x_io}^H = \{y\}$. Then there is an integer $1 \leq j \leq m$ such that $U_{ox_i}^H = \{x_j\}$. Since $d_H(o, y) = 2$, $yx_i \in E(H)$. Hence, using Lemma 3.1, $N_H(x_i) = \{o, y, x_1, x_2, \dots, x_m\} \setminus \{x_j\}$ and $\deg_H(x_i) = m$.

 $\{o, y, x_1, x_2, \dots, x_m\} \setminus \{x_j\}$ and $\deg_H(x_i) = m$. **Case 3.** $|U_{x_io}^H| = 2$: We have $U_{x_io}^H = \{y, z\}$. Since $d_H(o, y) = d_H(o, z) = 2$, we conclude that $yx_i, zx_i \in E(H)$. Since $|U_{ox_i}^H| = 2$, there are integers j and k such that $U_{ox_i}^H = \{x_j, x_k\}$. Hence, by Lemma 3.1, we have $N_H(x_i) = \{o, y, z, x_1, x_2, x_3, \dots, x_m\} \setminus \{x_i, x_j, x_k\}$ and $\deg_H(x_i) = m$.

Next, we prove that $\deg_H(y) \geq m-3$ and $\deg_H(z) \geq m-3$. From $\deg_H(x_1) = m$ and Lemma 3.1, it concludes that $|U_{yx_1}^H| \leq 2$, hence, $|U_{x_1y}^H| \leq 2$, which means that there are at most two elements in $N_H(x_1) \setminus N_H[y]$. Using this and Lemma 3.1, we provide $\deg_H(y) \geq m-3$. With a similar argument, the inequality $\deg_H(z) \geq m-3$ is concluded.

Now, by using $\deg_H(y)$, $\deg_H(z) \geq m-3$, $\deg_H(x_i) \geq m$, $(i=1,\cdots,m)$, and $oy, oz \notin E(H)$, hence every two nonadjacent vertices have a common neighbor, provided that $m \geq 7$. This means that $\operatorname{diam}(H) = 2$, which proves the result in case $m \geq 7$, using Theorem 2.1. For the cases, $3 \leq m \leq 6$, through a case by case inspection (by using $\deg_H(x_i) \geq m$, $i=1,\cdots,m$,) the same result is obtained.

Theorem 4.2. For the starlike tree $T = S(2^2, 1^{m-2})$ of order m + 3, $b(G) = \frac{m^2 + m - 4}{2}$.

Proof. Let the vertices of T be labeled as in Figure 2 and \overline{T} be a distance-balanced closure of T. Now, we are going to construct \overline{T} . Let $H=K_{m+3}$ be a complete graph with the same vertex set as H. Omit the edges of cycles $C_1=x_1x_2x_3\ldots x_mx_1$ and $C_2=oyzo$ from H to obtain $\overline{T}=H\setminus (C_1\cup C_2)$. Now, \overline{G} is an m-regular graph with diameter 2, which contains T as a spanning subgraph, so by Theorem 4.1 and Theorem 2.1, \overline{T} is a distance-balanced closure of T and $b(T)=\frac{m^2+m-4}{2}$.

Theorem 4.3. Let T be the tree of Figure 3 and H be a distance-balanced graph which contains T as a spanning subgraph. Then $\operatorname{diam}(H) \leq 2$, hence, H is a regular graph. Moreover, $b(T) = \frac{m^2 + m - 4}{2}$.

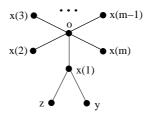


Figure 3

Proof. If either $oy \in E(H)$ or $oz \in E(H)$, then by Theorem 3.2, $\operatorname{diam}(H) \leq 2$ and H is a regular graph. So, suppose that neither oy nor oz is in E(H). Since $|W^H_{x_1y}| = |W^H_{yx_1}|$ and $o \in W^H_{x_1y}$, there exists a vertex $x_i, i \neq 1$, such that $yx_i \in E(H)$. Therefore, graph H contains graph $S(2^2, 1^{m-2})$ as a spanning subgraph and using Theorem 4.1, $\operatorname{diam}(H) \leq 2$ and H is a regular graph. Furthermore, the graph introduced in the proof of Theorem 4.2, is also distance-balanced closure of T. Hence $b(T) = \frac{m^2 + m - 4}{2}$.

Theorem 4.4. Consider the starlike tree $T = S(3, 1^{m-1})$ of order m + 3 and let H be a distance-balanced graph which contains T as a spanning subgraph. Then H is an r-regular graph for some $m \le r \le m + 2$.

Proof. Let the vertices of T be labeled as in Figure 4.

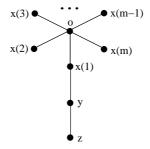


Figure 4

If either $oy \in E(H)$ or $oz \in E(H)$, then by Theorem 3.2, $\operatorname{diam}(H) \leq 2$ and H is a regular graph. If $zx_1 \in E(H)$, then H contains the graph shown in Figure 3 as a spanning subgraph, so by Theorem 4.3, H is a regular graph. So we may assume that $\{oy, oz, x_1z\} \cap E(H) = \emptyset$. Since $|W_{yz}^H| = |W_{zy}^H|$ and $x_1 \in W_{yz}^H$, the vertex z is adjacent to at least one vertex in $\{x_2, x_3, \ldots, x_m\}$ (because otherwise according to the structure of T we have $V \setminus \{y, z\} \subseteq U_{yz}$). Hence, H contains the graph $S(2^2, 1^{m-2})$, as a spanning subgraph. So, by Theorem 4.1, $\operatorname{diam} H \leq 2$ and H is a regular graph, as desired.

Corollary 4.5. Let G be a connected graph of order n with $\Delta(G) = n-3$. Then every distance-balanced graph H which contains G as a spanning subgraph, is regular.

Proof. Since $\Delta(G) = n - 3$, G contains at least one of the graphs $S(2^2, 1^{n-2})$, $S(3, 1^{n-1})$ or the graph shown in Figure 3, as a spanning subgraph. Hence, the result follows from Theorem 4.2, Theorem 4.6 and Theorem 4.3.

Theorem 4.6. For the starlike tree $G = S(3, 1^{m-1})$ of order m+3, $b(G) = \frac{m^2+m-4}{2}$.

Proof. Let the vertices of G be labeled as in Figure 4 and let \overline{G} be a distance-balanced closure of G. Now, we are going to construct \overline{G} . Let $H = K_{m+3}$ be a complete graph with the same vertex set as G. Omit the edges of cycles $C_1 = x_3x_4 \dots x_mx_3$ and $C_2 = oyx_2x_1z_0$ from H to obtain $\overline{G} = H \setminus (C_1 \cup C_2)$. Then the graph \overline{G} is an n-regular graph with diameter 2, which contains G as a spanning subgraph. So by Theorem 4.4, \overline{G} is a distance-balanced closure of G and $b(G) = \frac{m^2 + m - 4}{2}$.

Conclusion. In previous sections, we have proved that any connected distance-balanced graph G with $\Delta(G) \geq |V(G)| - 3$, is a regular graph, moreover, distanced-closure of such a graph G is a smallest regular graph which contains G. This helped us to find a distance-balanced closure of trees T with $\Delta(T) \geq |V(T)| - 3$ and to compute b(T) for such trees.

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References

- H. J. Bandelt, V. Chepoi, Metric graph theory and geometry: a survey, manuscript, 2004.
- 2. F. Buckley, F. Harary, Distance in graphs, Addison-Wesley, 1990.
- V. Chepoi, Isometric subgraphs of hamming graphs and d-convexity, Cybernetics, 24, (1988), 6-10 (Russian, English transl.)

- D. Ž. Dijoković, Dictance preserving subgraphs of hypercubes, J. Combin. Theory Ser B., 14, (1973), 263-267.
- D. Eppstein, The lattice dimention of a graph, European J. Combin., 26, (2005), 585-592.
- A. Graovac, M. Juvan, M.Petkovšek, A. Vasel, J. Žerovnik, The Szged index of fasciagraphs, MATCH Commun. Math. Comput. Chem., 49, (2003), 47-66.
- I. Gutman, L. Popovič, P. V. Khadikar S. Karmarkar, S. Joshi, M. Mandloi, Relations between Wiener and Szeged indices of monocyclic molecules, MATCH Commun. Math. Comput. Chem., 35, (1997), 91-103.
- 8. J. Fathali, N. Jafari Rad, S. Rahimi Sherbaf, The p-median and p-center Problems on Bipartite Graphs, *Iranian Journal of Mathematical Sciences and Informatics*, **9** (2), (2014), 37-43.
- 9. J. Jerebic, S. Klavžar, D. F. Rall, Dictance-balanced graphs, Ann. Combin., 12, (2008), 71-79.
- H. S. Ramane, I. Gutman, A. B. Ganagi, On Diameter of Line Graphs, Iranian Journal of Mathematical Sciences and Informatics, 8 (1), (2013), 105-109.