

Generalized Douglas-Weyl Finsler Metrics

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ABSTRACT. In this paper, we study generalized Douglas-Weyl Finsler metrics. We find some conditions under which the class of generalized Douglas-Weyl (α, β) -metric with vanishing S-curvature reduce to the class of Berwald metrics.

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1. INTRODUCTION

Let (M, F) be a Finsler manifold. In local coordinates, a curve $c(t)$ is a geodesic if and only if its coordinates $(c^i(t))$ satisfy $\ddot{c}^i + 2G^i(\dot{c}) = 0$, where the local functions $G^i = G^i(x, y)$ are called the spray coefficients [10]. F is called a Berwald metric, if G^i are quadratic in $y \in T_x M$ for any $x \in M$ or equivalently $G^i = \frac{1}{2}\Gamma_{jk}^i(x)y^j y^k$. As a generalization of Berwald curvature, Bácsó-Matsumoto introduced the notion of Douglas metrics which are projective invariants in Finsler geometry [2]. F is called a Douglas metric if $G^i = \frac{1}{2}\Gamma_{jk}^i(x)y^j y^k + P(x, y)y^i$.

A Finsler metric F is called generalized Douglas-Weyl metric (briefly, GDW-metric) if $D_{jkl||m}^i y^m = T_{jkl} y^i$ holds for some tensor T_{jkl} , where $D_{jkl||m}^i$ denotes the horizontal covariant derivatives of D_{jkl}^i with respect to the Berwald

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connection of F [8][18]. For a manifold M , let $\mathcal{GDW}(M)$ denotes the class of all Finsler metrics satisfying in above relation for some tensor T_{jkl} . In [3], Bácsó-Papp showed that $\mathcal{GDW}(M)$ is closed under projective changes. Then, Najafi-Shen-Tayebi characterized generalized Douglas-Weyl Randers metrics [8]. In [18], it is proved that all generalized Douglas-Weyl spaces with vanishing Landsberg curvature have vanishing the quantity $\mathbf{H}$. For other works, see [12] and [13].

The notion of S-curvature is originally introduced by Shen for the volume comparison theorem [9]. The Finsler metric F is said to be of isotropic S-curvature if $\mathbf{S} = (n+1)cF$, where $c = c(x)$ is a scalar function on M . In [14], it is shown that every isotropic Berwald metric has isotropic S-curvature. In [4], Cheng-Shen show that every (α, β) -metric with constant Killing 1-form has vanishing S-curvature. Then, Bácsó-Cheng-Shen proved that a Finsler metric $F = \alpha \pm \beta^2/\alpha + \epsilon\beta$ has vanishing S-curvature if and only if β is a constant Killing 1-form [1]. Therefore, the Finsler metrics with vanishing S-curvature are of some important geometric structures which deserve to be studied deeply.

An (α, β) -metric is a Finsler metric on M defined by $F := \alpha\phi(s)$, $s = \beta/\alpha$, where $\phi = \phi(s)$ is a C^∞ function on the $(-b_0, b_0)$ with certain regularity, $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ is a Riemannian metric and $\beta(y) = b_i(x)y^i$ is a 1-form on M [6]. In this paper, we are going to study generalized Douglas-Weyl (α, β) -metrics with vanishing S-curvature.

Theorem 1.1. *Let $F = \alpha\phi(s)$, $s = \beta/\alpha$, be an (α, β) -metric on a manifold M of dimension $n \geq 3$. Suppose that*

$$F \neq c_3 \alpha \left(\frac{\beta}{\alpha} \right)^{\frac{c_2}{1+c_2}} \left(c_1 \frac{\beta}{\alpha} + c_2 + 1 \right)^{\frac{1}{1+c_2}} \quad \text{and} \quad F \neq d_1 \sqrt{\alpha^2 + d_2 \beta^2} + d_3 \beta.$$

where c_1, c_2, c_3, d_1, d_2 and d_3 are real constants. Let F has vanishing S-curvature. Then F is a GDW-metric if and only if it is a Berwald metric.

2. PRELIMINARY

Given a Finsler manifold (M, F) , then a global vector field $\mathbf{G}$ is induced by F on TM_0 , which in a standard coordinate (x^i, y^i) for TM_0 is given by $\mathbf{G} = y^i \frac{\partial}{\partial x^i} - 2G^i(x, y) \frac{\partial}{\partial y^i}$, where

$$G^i := \frac{1}{4} g^{il} \left\{ [F^2]_{x^k y^l} y^k - [F^2]_{x^l} \right\}, \quad y \in T_x M.$$

The $\mathbf{G}$ is called the spray associated to F .

Define $\mathbf{B}_y : T_x M \otimes T_x M \otimes T_x M \rightarrow T_x M$ and $\mathbf{E}_y : T_x M \otimes T_x M \rightarrow \mathbb{R}$ by $\mathbf{B}_y(u, v, w) := B^i_{jkl}(y) u^j v^k w^l \frac{\partial}{\partial x^i} |_x$ and $\mathbf{E}_y(u, v) := E_{jk}(y) u^j v^k$ where

$$B^i_{jkl} := \frac{\partial^3 G^i}{\partial y^j \partial y^k \partial y^l}, \quad E_{jk} := \frac{1}{2} B^m_{jkm}.$$

$\mathbf{B}$ and $\mathbf{E}$ are called the Berwald curvature and mean Berwald curvature, respectively. F is called a Berwald and weakly Berwald if $\mathbf{B} = \mathbf{0}$ and $\mathbf{E} = 0$, respectively [5][7].

Let

$$D_j^i{}_{kl} := \frac{\partial^3}{\partial y^j \partial y^k \partial y^l} \left(G^i - \frac{1}{n+1} \frac{\partial G^m}{\partial y^m} y^i \right).$$

It is easy to verify that $\mathcal{D} := D_j^i{}_{kl} dx^j \otimes \partial_i \otimes dx^k \otimes dx^l$ is a well-defined tensor on slit tangent bundle TM_0 . We call $\mathcal{D}$ the Douglas tensor. A Finsler metric with $\mathcal{D} = 0$ is called a Douglas metric. The notion of Douglas metrics was proposed by Bácsó-Matsumoto as a generalization of Berwald metrics [2]. The Douglas tensor $\mathcal{D}$ is a non-Riemannian projective invariant, namely, if two Finsler metrics F and $\bar{F}$ are projectively equivalent, $G^i = \bar{G}^i + P y^i$, where $P = P(x, y)$ is positively y -homogeneous of degree one, then the Douglas tensor of F is same as that of $\bar{F}$. Finsler metrics with vanishing Douglas tensor are called Douglas metrics [11].

For a Finsler metric F on an n -dimensional manifold M , the Busemann-Hausdorff volume form $dV_F = \sigma_F(x) dx^1 \cdots dx^n$ is defined by

$$\sigma_F(x) := \frac{\text{Vol}(\mathbb{B}^n(1))}{\text{Vol}\left[\left(y^i \in \mathbb{R}^n \mid F\left(y^i \frac{\partial}{\partial x^i} \Big|_x\right) < 1\right)\right]}.$$

Let G^i denote the geodesic coefficients of F in the same local coordinate system. The S-curvature is defined by

$$\mathbf{S}(\mathbf{y}) := \frac{\partial G^i}{\partial y^i}(x, y) - y^i \frac{\partial}{\partial x^i} \left[\ln \sigma_F(x) \right],$$

where $\mathbf{y} = y^i \frac{\partial}{\partial x^i} \Big|_x \in T_x M$. $\mathbf{S}$ is said to be isotropic if there is a scalar functions $c = c(x)$ on M such that $\mathbf{S} = (n+1)cF$.

For an (α, β) -metric $F = \alpha\phi(s)$, $s = \beta/\alpha$, put

$$\Phi := -(q - sq')[n\Delta + 1 + sq] - (b^2 - s^2)(1 + sq)q'',$$

where

$$q := \frac{\phi'}{\phi - s\phi'}, \quad \Delta := 1 + sq + (b^2 - s^2)q'.$$

In [4], Cheng-Shen characterize (α, β) -metrics with isotropic S-curvature.

Lemma 2.1. ([4]) Let $F = \alpha\phi(s)$, $s = \beta/\alpha$, be a non-Riemannian (α, β) -metric on a manifold M of dimension $n \geq 3$. Suppose that $\phi \neq c_1\sqrt{1 + c_2s^2} + c_3s$ for any constant $c_1 > 0$, c_2 and c_3 . Then F is of isotropic S-curvature $\mathbf{S} = (n+1)cF$ if and only if one of the following holds

(a) β satisfies

$$r_{ij} = \varepsilon(b^2 a_{ij} - b_i b_j), \quad s_j = 0, \quad (2.1)$$

where $\varepsilon = \varepsilon(x)$ is a scalar function, $b := \|\beta_x\|_\alpha$ and $\phi = \phi(s)$ satisfies

$$\Phi = -2(n+1)k \frac{\phi \Delta^2}{b^2 - s^2}, \quad (2.2)$$

where k is a constant. In this case, $\mathbf{S} = (n+1)cF$ with $c = k\varepsilon$.

(b) β satisfies

$$r_{ij} = 0, \quad s_j = 0 \quad (2.3)$$

In this case, $\mathbf{S} = 0$.

The characterization of Finsler metrics with isotropic S-curvature in Cheng-Shen's paper is not complete [4]. Their result is correct for dimension $n \geq 3$. For the case $\dim(M) = 2$, see [16].

3. PROOF OF MAIN RESULTS

Let $F := \alpha\phi(s)$, $s = \beta/\alpha$, be an (α, β) -metric on a manifold M , where $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ and $\beta(y) = b_i(x)y^i$. Define $b_{i|j}$ by $b_{i|j}\theta^j := db_i - b_j\theta_i^j$, where $\theta^i := dx^i$ and $\theta_i^j := \tilde{\Gamma}_{ik}^j dx^k$ denote the Levi-Civita connection forms of α . Let

$$\begin{aligned} r_{ij} &:= \frac{1}{2} [b_{i|j} + b_{j|i}], & s_{ij} &:= \frac{1}{2} [b_{i|j} - b_{j|i}], \\ r_{i0} &:= r_{ij}y^j, & r_{00} &:= r_{ij}y^i y^j, & r_j &:= b^i r_{ij}, & t_j^i &:= s^i_m s_j^m \\ s_{i0} &:= s_{ij}y^j, & s_j &:= b^i s_{ij}, & r_0 &:= r_j y^j, & s_0 &:= s_j y^j. \end{aligned}$$

Then $\beta = b_i(x)y^i$ is a constant Killing one-form on M if $r_{ij} = s_j = 0$ hold. By definition, we have

$$b_{i|j} = s_{ij} + r_{ij}.$$

Since $y^i|_s = 0$, then for a constant Killing 1-form β we have

$$r_{00} = 0, \quad r_i + s_i = 0.$$

For an (α, β) -metric $F = \alpha\phi(s)$, $s = \beta/\alpha$, the following hold.

Proposition 3.1. *Let $F = \alpha\phi(s)$, $s = \beta/\alpha$, be an (α, β) -metric on an n -dimensional manifold M of dimension $n \geq 3$, where $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a one-form on M . Suppose that F is of vanishing S-curvature. Then F is a GDW-metric if and only if the following holds*

$$\begin{aligned} C_1 s_{j0|0} y^i - (C_2 y_j + C_3 b_j) y^i t_{00} &= C_4 y_j s^i_{0|0} + C_5 (b_j s^i_{0|0} + s_{j0} s^i_0) \\ &+ C_6 s^i_{j|0} + C_7 (y_j t^i_0 + s_{j0} s^i_0) + C_8 b_j t^i_0, \end{aligned} \quad (3.1)$$

where

$$\begin{aligned}
C_1 &:= -\left[(n+1)Q_\alpha + 2\beta Q_{\alpha\beta}\right]\alpha^{-3} - \left[Q_{\alpha\alpha} + b^2 Q_{\beta\beta}\right]\alpha^{-2}, \\
C_2 &:= (n+1)\left[Q_\alpha^2 + QQ_{\alpha\alpha} - \alpha^{-1}QQ_\alpha\right]\alpha^{-4} - 2\left[Q_\alpha Q_\beta + QQ_{\alpha\beta}\right]\beta\alpha^{-5} \\
&\quad + 2\left[2Q_\alpha Q_{\alpha\beta} + Q_{\alpha\alpha}Q_\beta + QQ_{\alpha\alpha\beta}\right]\beta\alpha^{-4} + b^2\left[2Q_{\alpha\beta}Q_\beta + Q_\alpha Q_{\beta\beta}\right]\alpha^{-3} \\
&\quad + \left[b^2 QQ_{\alpha\beta\beta} + 3Q_\alpha Q_{\alpha\alpha} + QQ_{\alpha\alpha\alpha}\right]\alpha^{-3}, \\
C_3 &:= (n+3)\left[Q_\alpha Q_\beta + QQ_{\alpha\beta}\right]\alpha^{-3} + 2\left[Q_\alpha Q_{\beta\beta} + QQ_{\alpha\beta\beta}\right]\beta\alpha^{-3} \\
&\quad + \left[2Q_\alpha Q_{\alpha\beta} + Q_\beta Q_{\alpha\alpha} + QQ_{\alpha\alpha\beta} + 4\beta\alpha^{-1}Q_\beta Q_{\alpha\beta}\right]\alpha^{-2} \\
&\quad + b^2\left[3Q_\beta Q_{\beta\beta} + QQ_{\beta\beta\beta}\right]\alpha^{-2}, \\
C_4 &:= -\left[(n+1)Q_\alpha + 2\beta Q_{\alpha\beta}\right]\alpha^{-3} + 2\left[\beta Q_{\alpha\alpha\beta} + Q_{\alpha\alpha}\right]\alpha^{-2} \\
&\quad + \left[b^2 Q_{\alpha\beta\beta} + Q_{\alpha\alpha\alpha}\right]\alpha^{-1}, \\
C_5 &:= (n+3)\alpha^{-1}Q_{\alpha\beta} + Q_{\alpha\alpha\beta} + 2\beta\alpha^{-1}Q_{\alpha\beta\beta} + b^2 Q_{\beta\beta\beta}, \\
C_6 &:= (n+1)\alpha^{-1}Q_\alpha + Q_{\alpha\alpha} + 2\beta\alpha^{-1}Q_{\alpha\beta} + b^2 Q_{\beta\beta}, \\
C_7 &:= (n+1)\alpha^{-3}QQ_\alpha - (n+1)\alpha^{-2}(Q_\alpha^2 + QQ_{\alpha\alpha}) - 2\beta\alpha^{-2}QQ_{\alpha\alpha\beta} \\
&\quad + 2\left[QQ_{\alpha\beta} + Q_\alpha Q_\beta\right]\beta\alpha^{-3} - b^2\left[QQ_{\alpha\beta\beta} + 2Q_{\alpha\beta}Q_\beta\right]\alpha^{-1} \\
&\quad - 2\left[2Q_\alpha Q_{\alpha\beta} + Q_\beta Q_{\alpha\alpha}\right]\beta\alpha^{-2} \\
&\quad - b^2\alpha^{-1}Q_\alpha Q_{\beta\beta} - 3\alpha^{-1}Q_\alpha Q_{\alpha\alpha} - 2\alpha^{-1}QQ_{\alpha\alpha\alpha}, \\
C_8 &:= -(n+3)\left[QQ_{\alpha\beta} + Q_\alpha Q_\beta\right]\alpha^{-1} - 2\left[2Q_\beta Q_{\alpha\beta} + QQ_{\alpha\beta\beta} + Q_\alpha Q_{\beta\beta}\right]\beta\alpha^{-1} \\
&\quad - b^2\left[QQ_{\beta\beta\beta} + 3Q_\beta Q_{\beta\beta}\right] - Q_\beta Q_{\alpha\alpha} - QQ_{\alpha\alpha\beta} - 2Q_\alpha Q_{\alpha\beta}.
\end{aligned}$$

Proof. Let G^i and G_α^i denote the spray coefficients of F and α , respectively, in the same coordinate system. Then, we have

$$G^i = G_\alpha^i + Py^i + Q^i, \quad (3.2)$$

where

$$\begin{aligned}
Q &:= \alpha q = \frac{\alpha\phi'}{\phi - s\phi'}, \\
P &:= \alpha^{-1}\Theta(r_{00} - 2Qs_0), \quad Q^i := Qs^i_0 + \Psi(r_{00} - 2Qs_0)b^i, \\
\Theta &= \frac{q - sq'}{2\Delta} = \frac{\phi\phi' - s(\phi\phi'' + \phi'\phi')}{2\phi[(\phi - s\phi') + (b^2 - s^2)\phi'']} \\
\Psi &:= \frac{q'}{2\Delta} = \frac{1}{2} \frac{\phi''}{(\phi - s\phi') + (b^2 - s^2)\phi''}.
\end{aligned}$$

By Lemma 2.1, we have $r_{00} = s_0 = 0$. Then (3.2) reduces to following

$$G^i = G_\alpha^i + Qs_0^i. \quad (3.3)$$

Let “ $\parallel$ ” and “ $\mid$ ” denote the covariant differentiations with respect to G^i and G_α^i respectively. Then by (3.3), we have

$$\begin{aligned} D_{jkl\parallel m}^i y^m &= D_{jkl\mid m}^i y^m - 2Qs_0^p \frac{\partial D_{jkl}^i}{\partial y^p} + D_{jkl}^p \tilde{N}_p^i - D_{pkl}^i \tilde{N}_j^p \\ &\quad - D_{jpl}^i \tilde{N}_k^p - D_{jkp}^i \tilde{N}_l^p, \end{aligned} \quad (3.4)$$

where

$$\begin{aligned} D_{jkl\mid m}^i y^m &= \alpha^{-4}(Q_{\alpha\alpha} - \alpha^{-1}Q_\alpha)(A_{jk}y_l + A_{kl}y_j + A_{jl}y_k)s_{0\mid 0}^i \\ &\quad + \alpha^{-3}Q_\alpha(A_{jk}s_{l\mid 0}^i + A_{kl}s_{j\mid 0}^i + A_{jl}s_{k\mid 0}^i) \\ &\quad + \alpha^{-3}Q_{\alpha\beta}\left[(A_{jk}b_l + A_{kl}b_j + A_{jl}b_k)s_{0\mid 0}^i\right. \\ &\quad \left.+ (A_{jk}s_{l0} + A_{kl}s_{j0} + A_{jl}s_{k0})s_{0\mid 0}^i\right] \\ &\quad + \alpha^{-2}Q_{\alpha\alpha\beta}\left[(y_jy_kb_l + y_ky_lb_j + y_jy_lb_k)s_{0\mid 0}^i\right. \\ &\quad \left.+ (y_jy_ks_{l0} + y_ky_ls_{j0} + y_jy_ls_{k0})s_{0\mid 0}^i\right] \\ &\quad + \alpha^{-1}Q_{\alpha\beta\beta}\left[(y_jb_kb_l + y_kb_jb_l + y_lb_jb_k)s_{0\mid 0}^i\right. \\ &\quad \left.+ ((y_jb_l + y_lb_j)s_{k0} + (y_jb_k + y_kb_j)s_{l0}\right. \\ &\quad \left.+ (y_kb_l + y_lb_k)s_{j0})s_{0\mid 0}^i\right] + \alpha^{-2}Q_{\alpha\alpha}(y_jy_ks_{l\mid 0}^i + y_ky_ls_{j\mid 0}^i + y_jy_ls_{k\mid 0}^i) \\ &\quad + Q_{\beta\beta\beta}(b_kb_ls_{j0} + b_jb_ls_{k0} + b_jb_ks_{l0})s_{0\mid 0}^i + \alpha^{-3}Q_{\alpha\alpha\alpha}y_jy_ky_ls_{0\mid 0}^i \\ &\quad + \alpha^{-1}Q_{\alpha\beta}\left[(y_jb_k + y_kb_j)s_{l\mid 0}^i + (y_kb_l + y_lb_k)s_{j\mid 0}^i + (y_lb_j + y_jb_l)s_{k\mid 0}^i\right. \\ &\quad \left.+ (y_js_{k0} + y_ks_{j0})s_{l\mid 0}^i + (y_ks_{l0} + y_ls_{k0})s_{j\mid 0}^i + (y_ls_{j0} + y_js_{l0})s_{k\mid 0}^i\right] \\ &\quad + Q_{\beta\beta}\left[b_jb_ks_{l\mid 0}^i + b_kb_ls_{j\mid 0}^i + b_jb_ls_{k\mid 0}^i + (s_{j0}b_k + b_js_{k0})s_{l\mid 0}^i\right. \\ &\quad \left.+ (s_{k0}b_l + b_ks_{l0})s_{j\mid 0}^i + (b_ls_{j0} + b_js_{l0})s_{k\mid 0}^i\right] + Q_{\beta\beta\beta}b_jb_kb_ls_{0\mid 0}^i \end{aligned} \quad (3.5)$$

and

$$A_{ij} = \alpha^2 a_{ij} - y_i y_j, \quad (3.6)$$

$$\tilde{N}_p^i = Qs_p^i + \left[\alpha^{-1}Q_\alpha y_p + Q_\beta b_p\right]s_{0\mid 0}^i, \quad (3.7)$$

$$\frac{\partial D_{jkl}^i}{\partial y^p} = Q_{jklp}s_{0\mid 0}^i + Q_{jklp}s_{\mid 0}^i + Q_{jklp}s_{\mid 0}^i + Q_{jlp}s_{k\mid 0}^i + Q_{klp}s_{j\mid 0}^i. \quad (3.8)$$

Let F is a GDW -metric. Then there exists a tensor D_{jkl} such that

$$D_{jkl\parallel m}^i y^m = D_{jkl}^i y^i.$$

By (3.4), we have

$$D_{jkl}y^i = D_{jkl|m}y^m - 2Q \frac{\partial D_{jkl}^i}{\partial y^p} s^p_0 + D_{jkl}^p \tilde{N}_p^i - D_{pkl}^i \tilde{N}_j^p - D_{jpl}^i \tilde{N}_k^p - D_{jkp}^i \tilde{N}_l^p. \quad (3.9)$$

By contracting (3.9) with y_i and using (3.5), (3.7) and (3.8) we get the following

$$\begin{aligned} D_{jkl} = & D_1 [A_{jk}s_{l0|0} + A_{kl}s_{j0|0} + A_{jl}s_{k0|0}] \\ & + D_2 [y_j y_k s_{l0|0} + y_k y_l s_{j0|0} + y_j y_l s_{k0|0}] \\ & + D_3 [(y_j b_k + y_k b_j)s_{l0|0} + (y_k b_l + y_l b_k)s_{j0|0} + (y_j b_l + y_l b_j)s_{k0|0}] \\ & + D_4 [b_j b_k s_{l0|0} + b_k b_l s_{j0|0} + b_j b_l s_{k0|0}] \\ & + D_5 [A_{jk}y_l + A_{kl}y_j + A_{jl}y_k] t_{00} \\ & + D_6 [A_{jk}b_l + A_{kl}b_j + A_{jl}b_k] t_{00} \\ & + D_7 [y_j y_k b_l + y_k y_l b_j + y_j y_l b_k] t_{00} \\ & + D_8 [y_l b_j b_k + y_j b_k b_l + y_k b_j b_l] t_{00} \\ & + D_9 y_j y_k y_l t_{00} + D_{10} b_j b_k b_l t_{00} \\ & + D_{11} [y_l s_{j0}s_{k0} + y_j s_{k0}s_{l0} + y_k s_{j0}s_{l0}] \\ & + D_{12} [b_l s_{j0}s_{k0} + b_j s_{k0}s_{l0} + b_k s_{j0}s_{l0}], \end{aligned} \quad (3.10)$$

where

$$\begin{aligned} D_1 &:= -\alpha^{-5}Q_\alpha, \\ D_2 &:= -\alpha^{-4}Q_{\alpha\alpha}, \\ D_3 &:= -\alpha^{-3}Q_{\alpha\beta}, \\ D_4 &:= -\alpha^{-2}Q_{\beta\beta}, \\ D_5 &:= -\alpha^{-6}Q_\alpha^2 - \alpha^{-6}QQ_{\alpha\alpha} + \alpha^{-7}QQ_\alpha, \\ D_6 &:= -\alpha^{-5}Q_\alpha Q_\beta - \alpha^{-5}QQ_{\alpha\beta}, \\ D_7 &:= -\alpha^{-4}Q_{\alpha\alpha}Q_\beta - 2\alpha^{-4}Q_{\alpha\beta}Q_\alpha - \alpha^{-4}QQ_{\alpha\alpha\beta}, \\ D_8 &:= -\alpha^{-3}Q_{\beta\beta}Q_\alpha - 2\alpha^{-3}Q_{\alpha\beta}Q_\beta - \alpha^{-3}QQ_{\alpha\beta\beta}, \\ D_9 &:= -3\alpha^{-5}Q_{\alpha\alpha}Q_\alpha - \alpha^{-5}QQ_{\alpha\alpha\alpha}, \\ D_{10} &:= -3\alpha^{-2}Q_{\beta\beta}Q_\beta - \alpha^{-2}QQ_{\beta\beta\beta}, \\ D_{11} &:= -2\alpha^{-3}Q_{\alpha\beta} + 2\alpha^{-4}Q_\alpha^2 + 2\alpha^{-4}QQ_{\alpha\alpha} - 2\alpha^{-5}QQ_\alpha, \\ D_{12} &:= -2\alpha^{-2}Q_{\beta\beta} + 2\alpha^{-3}QQ_{\alpha\beta} + 2\alpha^{-3}Q_\alpha Q_\beta. \end{aligned}$$

Now, by plugging (3.10) into (3.9), and contracting the obtained result with a^{kl} , we get (3.1). $\square$

Proof of Theorem 1.1: Let $F = \alpha\phi(s)$, $s = \beta/\alpha$, be an (α, β) -metric on an n -dimensional manifold M . By multiplying (3.1) with y_i and y^j , we get

$$-\alpha Q Q_{\alpha\alpha} t_{00} = 0. \quad (3.11)$$

If $Q_{\alpha\alpha} = 0$ then

$$Q = c_1\alpha + c_2\frac{\alpha^2}{\beta},$$

where c_1 and c_2 are real constants. Thus, we get

$$F = c_3\alpha\left(\frac{\beta}{\alpha}\right)^{\frac{c_2}{1+c_2}}\left(c_1\frac{\beta}{\alpha} + c_2 + 1\right)^{\frac{1}{1+c_2}},$$

where c_3 is a real constant. This is a contradiction with our assumption. Then by (3.11), we get $t_{00} = 0$ which results that $s_{i0} = 0$. This means that β is a closed one-form. By assumption, β is parallel one-form and then F reduces to a Berwald metric. $\square$

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