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On the 2-absorbing Submodules

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ABSTRACT. Let R be a commutative ring and M be an R-module. In this paper, we investigate some properties of 2-absorbing submodules of M. It is shown that N is a 2-absorbing submodule of M if and only if whenever $IJL \subseteq N$ for some ideals I, J of R and a submodule L of M, then $IL \subseteq N$ or $JL \subseteq N$ or $IJ \subseteq N :_R M$. Also, if N is a 2-absorbing submodule of M and M/N is Noetherian, then a chain of 2-absorbing submodules of M is constructed. Furthermore, the annihilation of $E(R/\mathfrak{p})$ is studied whenever 0 is a 2-absorbing submodule of $E(R/\mathfrak{p})$, where \mathfrak{p} is a prime ideal of R and $E(R/\mathfrak{p})$ is an injective envelope of R/\mathfrak{p} .

Keywords: 2-absorbing ideal, 2-absorbing submodule, A chain of 2-absorbing submodule.

2000 Mathematics subject classification: 13C99.

1. Introduction

Throughout this paper R is a commutative ring with non-zero identity and M is an unitary R-module. We defined a submodule N of M is 2-absorbing whenever $abm \in N$ for some $a,b \in R$, $m \in M$, then $am \in N$ or $bm \in N$ or $ab \in N :_R M$, see for instance [1, 3, 4, 6, 7, 9, 10]. It is well known that, a submodule N of M is prime if and only if $IL \subseteq N$ for an ideal I of R and

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a submodule L of M, then either $L \subseteq N$ or $I \subseteq N :_R M$. This statement persuaded us to prove that, a submodule N of M is 2-absorbing if and only if $IJL \subseteq N$ for some ideals I,J of R and a submodule L of M, then $IL \subseteq N$ or $JL \subseteq N$ or $IJ \subseteq N :_R M$. As a corollary of this theorem, it is shown that $L = \{m \in M : \mathfrak{p} \subseteq r(N : m)\}$ is a 2-absorbing submodule of M, where N is a 2-absorbing submodule of M with $r(N :_R M) = \mathfrak{p} \cap \mathfrak{q}$ for some prime ideals $\mathfrak{p}, \mathfrak{q}$ of R. Also, it is shown that if M/N is Noetherian, then there exists a chain of 2-absorbing submodules of M that begins with N. Assume that $E(R/\mathfrak{p})$ is an injective envelope of R/\mathfrak{p} , it is shown that if 0 is a 2-absorbing submodule of $E(R/\mathfrak{p})$, then $F(0 :_R E(R/\mathfrak{p})) = \mathfrak{p}$ and $F(0 :_R E(R/\mathfrak{p}))$.

Now, we define the concepts that we will use later. For a submodule L of M let $L:_R M$ denote the ideal $\{r \in R: rM \subseteq L\}$. Similarly, for an element $m \in M$ let $L:_R m$ denote the ideal $\{r \in R: rm \in L\}$. If I is an ideal of R, then r(I) denotes the radical of I. We say that $\mathfrak{p} \in \operatorname{Spec}(R)$ is an associated prime ideal of M if there exists $m \in M$ with $0:_R m = \mathfrak{p}$. The set of associated prime ideals of M is denoted by $\operatorname{Ass}_R(M)$, the set of integers is denoted by \mathbb{Z} .

2. 2-Absorbing Submodules

Let N be a proper submodule of M. We say that N is a 2-absorbing submodule of M if whenever $a, b \in R$, $m \in M$ and $abm \in N$, then $am \in N$ or $bm \in N$ or $ab \in N :_R M$.

Lemma 2.1. Let I be an ideal of R and N be a 2-absorbing submodule of M. If $a \in R$, $m \in M$ and $Iam \subseteq N$, then $am \in N$ or $Im \subseteq N$ or $Ia \subseteq N :_R M$.

Proof. Let $am \notin N$ and $Ia \nsubseteq N :_R M$. Then there exists $b \in I$ such that $ba \notin N :_R M$. Now, $bam \in N$ implies that $bm \in N$, since N is a 2-absorbing submodule of M. We have to show that $Im \subseteq N$. Let c be an arbitrary element of I. Thus $(b+c)am \in N$. Hence, either $(b+c)m \in N$ or $(b+c)a \in N :_R M$. If $(b+c)m \in N$, then by $bm \in N$ it follows that $cm \in N$. If $(b+c)a \in N :_R M$, then $ca \notin N :_R M$, but $cam \in N$. Thus $cm \in N$. Hence, we conclude that $Im \subseteq N$.

Lemma 2.2. Let I, J be ideals of R and N be a 2-absorbing submodule of M. If $m \in M$ and $IJm \subseteq N$, then $Im \subseteq N$ or $Jm \subseteq N$ or $IJ \subseteq N :_R M$.

Proof. Let $I \subseteq N :_R m$ and $J \subseteq N :_R m$. We have to show that $IJ \subseteq N :_R M$. Assume that $c \in I$ and $d \in J$. By assumption there exists $a \in I$ such that $am \notin N$ but $aJm \subseteq N$. Now, Lemma 2.1 shows that $aJ \subseteq N :_R M$ and so $(I \setminus N :_R m)J \subseteq N :_R M$, similarly there exists $b \in J \setminus N :_R m$ such that $Ib \subseteq N :_R M$ and also $I(J \setminus N :_R m) \subseteq N :_R M$. Thus we have $ab \in N :_R M$, $ad \in N :_R M$ and $cb \in N :_R M$. By $a + c \in I$ and $b + d \in J$ it follows that $(a + c)(b + d)m \in N$. Therefore, $(a + c)m \in N$ or $(b + d)m \in N$ or

 $(a+c)(b+d) \in N :_R M$. If $(a+c)m \in N$, then $cm \notin N$ hence, $c \in I \setminus N :_R m$ which implies that $cd \in N :_R M$. Similarly by $(b+d)m \in N$, we can deduce that $cd \in N :_R M$. If $(a+c)(b+d) \in N :_R M$, then $ab+ad+cb+cd \in N :_R M$ and so $cd \in N :_R M$. Therefore, $IJ \subseteq N :_R M$.

Theorem 2.3. Let N be a proper submodule of M. The following statement are equivalent:

- (i) N is a 2-absorbing submodule of M;
- (ii) If $IJL \subseteq N$ for some ideals I, J of R and a submodule L of M, then $IL \subseteq N$ or $JL \subseteq N$ or $IJ \subseteq N :_R M$.

Proof. (ii) ⇒ (i) is obvious. To prove (i) ⇒ (ii), assume that $IJL \subseteq N$ for some ideals I,J of R and a submodule L of M and $IJ \not\subseteq N :_R M$. Then by Lemma 2.2 for all $m \in L$ either $Im \subseteq N$ or $Jm \subseteq N$. If $Im \subseteq N$, for all $m \in L$ we are done. Similarly if $Jm \subseteq N$, for all $m \in L$ we are done. Suppose that $m,m' \in L$ are such that $Im \not\subseteq N$ and $Jm' \not\subseteq N$. Thus $Jm \subseteq N$ and $Im' \subseteq N$. Since $IJ(m+m') \subseteq N$ we have either $I(m+m') \subseteq N$ or $J(m+m') \subseteq N$. By $I(m+m') \subseteq N$, it follows that $Im \subseteq N$ which is a contradiction, similarly by $J(m+m') \subseteq N$ we get a contradiction. Therefore either $IL \subseteq N$ or $JL \subseteq N$.

A submodule N of M is called strongly 2-absorbing if it satisfies in condition (ii), see [5]. Therefore, Theorem 2.3 shows that N is a 2-absorbing submodule of M if and only if N is a strongly 2-absorbing submodule of M.

Corollary 2.4. Let M be an R-module and N be a 2-absorbing submodule of M. Then $N:_M I = \{m \in M : Im \subseteq N\}$ is a 2-absorbing submodules of M for all ideal I of R. Furthermore $N:_M I^n = N:_M I^{n+1}$, for all $n \ge 2$.

Proof. Let I be an ideal of R, $a,b \in R$, $m \in M$ and $abm \in N :_M I$. Thus $Iabm \subseteq N$. Hence, $Im \subseteq N$ or $Iab \subseteq N :_R M$ or $abm \in N$, by Lemma 2.2. If $Im \subseteq N$ we are done. If $Iab \subseteq N :_R M$, then $ab \in (N :_R M) :_R I = (N :_M I) :_R M$. If $abm \in N$, then $am \in N$ or $bm \in N$ or $ab \in N :_R M$. Thus $Iam \subseteq N$ or $Iab \subseteq N :_R M$ which complete the proof.

For the second statement, it is enough to show that $N:_M I^2 = N:_M I^3$. It is clear that $N:_M I^2 \subseteq N:_M I^3$. Let $m \in N:_M I^3$. Then $I^3m \subseteq N$. Now, by Lemma 2.2, we have $I^2m \subseteq N$ or $Im \subseteq N$ or $I^3 \subseteq N:_R M$. If $I^2m \subseteq N$ or $Im \subseteq N$, we are done. If $I^3 \subseteq N:_R M$, then $I^2 \subseteq N:_R M$ since $N:_R M$ is a 2-absorbing ideal of R by [9, Theorem 2.3].

It is clear that, $n\mathbb{Z}$ is a 2-absorbing ideal of \mathbb{Z} if and only if $n=0, p, p^2, pq$, where p,q are distinct prime integers. It is easy to see that $4\mathbb{Z} :_{\mathbb{Z}} 6\mathbb{Z} = 2\mathbb{Z}$ but $4\mathbb{Z} :_{\mathbb{Z}} 36\mathbb{Z} = \mathbb{Z}$. Hence, the equality mentioned in the Corollary 2.4, is not necessarily true for n=1.

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Theorem 2.5. Let N be a 2-absorbing submodule of M such that $r(N:_R M) = \mathfrak{p} \cap \mathfrak{q}$ where \mathfrak{p} and \mathfrak{q} are the only distinct prime ideals of R that are minimal over $N:_R M$. Then $L = \{m \in M : \mathfrak{p} \subseteq r(N:_R m)\}$ is a 2-absorbing submodule of M containing N. Also, either $r(L:_R M) = \mathfrak{q}$ or $r(L:_R M) = \mathfrak{p}' \cap \mathfrak{q}$, where \mathfrak{p}' is a prime ideal of R containing \mathfrak{p} .

Proof. It is clear that L is a submodule of M containing N. Assume that $a,b\in R,\ m\in M$ and $abm\in L$. We have to show that $am\in L$ or $bm\in L$ or $ab\in L:_RM$. Since $\mathfrak{p}\subseteq r(N:_Rabm)$, thus $\mathfrak{p}^2abm\subseteq N$, by [9, Theorem 2.4] and [2, Theorem 2.4]. Therefore, by Lemma 2.1, we have $abm\in N$ or $\mathfrak{p}^2m\subseteq N$ or $\mathfrak{p}^2ab\subseteq N:_RM$. If $abm\in N$, then $am\in N$ or $bm\in N$ or $ab\in N:_RM$ which implies that $am\in L$ or $bm\in L$ or $ab\in L:M$. If $\mathfrak{p}^2m\subseteq N$, then $\mathfrak{p}^2\subseteq N:_Rm$ and so $\mathfrak{p}\subseteq r(N:_Rm)$ thus $m\in L$ and we are done. If $\mathfrak{p}^2ab\subseteq N:_RM$, then by [2, Theorem 2.13], we have $\mathfrak{p}^2a\subseteq N:_RM$ or $\mathfrak{p}^2b\subseteq N:_RM$ or $ab\in N:_RM$. In the first case we conclude that $\mathfrak{p}^2\subseteq N:_Rm$ and so $am\in L$. By a similar argument in the second case we can deduced that $am\in L$. If $ab\in N:_RM$, then $ab\in L:_RM$. Therefore, the result follows.

For the second statement, first we show that $r(N:_R M) = r(L:_R M) \cap \mathfrak{p}$. It is clear $r(N:_R M) \subseteq r(L:_R M) \cap \mathfrak{p}$. Assume that $a \in (L:_R M) \cap \mathfrak{p}$. Thus $aM \subseteq L$ and so, by definition of L, $\mathfrak{p} \subseteq r(N:_R am)$, for all $m \in M$. Hence, [2, Theorem 2.4] shows that $\mathfrak{p}^2 \subseteq N:_R am$, for all $m \in M$. Therefore, $a^3 \in N:_R m$, for all $m \in M$. So that $a^3 \in N:_R M$ and then $a \in r(N:_R M)$. Thus $r(L:_R M) \cap \mathfrak{p} \subseteq r(N:_R M)$. Now, $L:_R M$ is a 2-absorbing ideal of R, therefore either $r(L:_R M) = \mathfrak{p}'$ or $r(L:_R M) = \mathfrak{p}' \cap \mathfrak{q}'$, for some prime ideals \mathfrak{p}' , \mathfrak{q}' of R. In the first case we have $r(N:_R M) = \mathfrak{p} \cap \mathfrak{p}' \cap \mathfrak{q}'$ which implies that $\mathfrak{p}' = \mathfrak{q}$ and in the second case we have $r(N:_R M) = \mathfrak{p} \cap \mathfrak{p}' \cap \mathfrak{q}'$ which implies that either $\mathfrak{p}' = \mathfrak{q}$ or $\mathfrak{q}' = \mathfrak{q}$.

Corollary 2.6. Let N be a 2-absorbing submodule of M such that $r(N:_R M) = \mathfrak{p} \cap \mathfrak{q}$ where \mathfrak{p} and \mathfrak{q} are the only distinct prime ideals of R that are minimal over $N:_R M$. If M/N is a Noetherian R-module, then

- (i) there exists a chain $N = L_0 \subseteq L_1 \subseteq \cdots \subseteq L_{n-1} \subseteq L_n = M$ of 2-absorbing submodules of M. Furthermore, $\operatorname{Ass}(M) \subseteq \operatorname{Ass}(M/L_{n-1}) \cup \operatorname{Ass}(L_{n-1}/L_{n-2}) \cup \operatorname{Ass}(L_{n-2}/L_{n-3}) \cup \cdots \cup \operatorname{Ass}(L_1/N)$, where $\operatorname{Ass}(L_i/N)$ is the union of at most two totally ordered set, for all i.
- (ii) there exists a chain $N \subseteq L_n \subseteq L_{n-1} \subseteq \cdots \subseteq L_1 \subseteq L_0 = M$ of submodules of M such that L_i is a 2-absorbing submodule of L_{i+1} , for all $0 \le i \le n-1$.

Proof. (i) Let $L_1 = \{m \in M : \mathfrak{p} \subseteq r(N :_R m)\}$. Then by Corollary 2.4, L_1 is a 2-absorbing submodule of M and so either $r(L_1 :_R M) = \mathfrak{q}$ or $r(L_1 :_R M) = \mathfrak{p}_1 \cap \mathfrak{q}$, where \mathfrak{p}_1 is a prime ideal of R containing \mathfrak{p} . If $r(L_1 :_R M) = \mathfrak{q}$, then choose $L_2 = \{m \in M : \mathfrak{q} \subseteq r(L_1 :_R m)\} = M$. Hence, $N \subseteq L_1 \subseteq L_2 = M$ is requested chain. If $r(L_1 :_R M) = \mathfrak{p}_1 \cap \mathfrak{q}$, set $L_2 = \{m \in M : \mathfrak{p}_1 \subseteq r(L_1 :_R m)\}$

and so either $r(L_2:_R M) = \mathfrak{q}$ or $r(L_2:_R M) = \mathfrak{p}_2 \cap \mathfrak{q}$, where \mathfrak{p}_2 is a prime ideal of R containing \mathfrak{p}_1 . Proceeding in this way, we can achieve $N \subseteq L_0 \subseteq L_1 \subseteq \cdots \subseteq L_{n-1} \subseteq L_n = M$ of 2-absorbing submodules of M. The last statement is obvious, by [9, Theorem 2.6].

(ii) Let $L_1 = \{m \in M : \mathfrak{p} \subseteq r(N :_R m)\}$. Then N is a 2-absorbing submodule of L_1 . So that either $r(N :_R L_1) = \mathfrak{p}_1$ or $r(N :_R L_1) = \mathfrak{p}_1 \cap \mathfrak{q}_1$, for some prime ideals $\mathfrak{p}_1, \mathfrak{q}_1$ of R. If $r(N :_R L_1) = \mathfrak{p}_1$, then choose $L_2 = \{x \in L_1 : \mathfrak{p}_1 \subseteq r(N :_R x)\} = N$. Hence, in this case $N \subseteq L_1 \subseteq L_0 = M$ is the requested chain. If $r(N :_R L_1) = \mathfrak{p}_1 \cap \mathfrak{q}_1$, then set $L_2 = \{x \in L_1 : \mathfrak{p}_1 \subseteq r(N :_R x)\}$ and continue the same way to achieve the chain $N \subseteq L_n \subseteq L_{n-1} \subseteq \cdots \subseteq L_1 \subseteq L_0 = M$ of 2-absorbing submodules of M.

Theorem 2.7. Let N be a 2-absorbing submodule of M. Then $N :_R M$ is a prime ideal of R if and only if $N :_R m$ is a prime ideal of R for all $m \in M \setminus N$.

Proof. Assume that $a,b \in R, m \in M \setminus N$ and $ab \in N :_R m$. Then $abm \subseteq N$. We have $am \in N$ or $bm \in N$ or $ab \in N :_R M$ since N is a 2-absorbing submodule of M. If $am \in N$ or $bm \in N$ we are done. If $ab \in N :_R M$, then by assumption either $a \in N :_R M$ or $b \in N :_R M$. Thus either $a \in N :_R m$ or $b \in N :_R m$. So $N :_R m$ is a prime ideal.

Conversely, suppose that $ab \in N :_R M$ for some $a,b \in R$ and assume that there exist $m,m' \in M$ such that $am \notin N$ and $bm' \notin N$. By $abm,abm' \in N$ it follows that $bm \in N$ and $am' \in N$ since $N :_R m$ and $N :_R m'$ are prime ideals of R. If $m+m' \in N$, then $am \in N$ which is a contradiction. Thus $m+m' \notin N$. Now by $ab(m'+m'') \in N$ we have $a(m'+m'') \in N$ or $b(m'+m'') \in N$ which is a contradiction. Thus $aM \subseteq N$ or $bM \subseteq N$ which implies that $N :_R M$ is prime.

Corollary 2.8. Let N be a 2-absorbing submodule of M. Then $N:_R M$ is a prime ideal of R if and only if $N:_R K$ is a prime ideal of R for all submodules K of M containing N.

Proof. By Theorem 2.7 and [9, Theorem 2.6] it follows that $\{N :_R x : x \in K \setminus N\}$ is a totally ordered set of prime ideals of R. Hence, $N :_R K = \cap_{x \in K} N :_R x$ is a prime ideal of R.

Theorem 2.9. Let \mathfrak{p} be a prime ideal of R and $E(R/\mathfrak{p})$ be an injective envelop of R/\mathfrak{p} . If 0 is a 2-absorbing submodule of $E(R/\mathfrak{p})$, then

- (i) $\mathfrak{p}^2 \subseteq 0 :_R E(R/\mathfrak{p}) \subseteq \mathfrak{p}$ so that $r(0 :_R E(R/\mathfrak{p})) = \mathfrak{p}$.
- (ii) $\mathfrak{p}^2 \subseteq 0 :_R x \subset 0 :_R ax = \mathfrak{p}$, for all non-zero element x of $E(R/\mathfrak{p})$ and all $a \in \mathfrak{p} \setminus 0 :_R x$.
- (iii) $\mathfrak{p}^2 \subseteq 0 :_R x = 0 :_R a^n x \subseteq \mathfrak{p}$, for all $a \notin \mathfrak{p}$.

Proof. (i) We have $r(0:_R x) = \mathfrak{p}$ for all non-zero element x of $E(R/\mathfrak{p})$, by [8, Theorem 18.4]. Also it is obvious $0:_R E(R/\mathfrak{p}) \subseteq 0:_R x$. Thus $0:_R E(R/\mathfrak{p}) \subseteq \mathfrak{p}$.

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Now, assume that $a \in \mathfrak{p}^2$ and x is a non-zero element of $E(R/\mathfrak{p})$. Since 0 is a 2-absorbing submodule of M, $0:_R x$ is a 2-absorbing ideal of R, by [9, Theorem 2.4]. Therefore we have \mathfrak{p}^2 is a subset of $0:_R x$, by [2, Theorem 2.4]. Hence, ax = 0 and therefore $aE(R/\mathfrak{p}) = 0$ and $\mathfrak{p}^2 \subseteq 0 :_R E(R/\mathfrak{p})$.

- (ii) Let x be a non-zero element of $E(R/\mathfrak{p})$. Then we have $\mathfrak{p}^2 \subseteq 0 :_R x \subseteq \mathfrak{p}$. Assume that $a \in \mathfrak{p} \setminus 0 :_R x$. Thus $ax \neq 0$ but $a^2x = 0$ which shows that $0:_R x \subset 0:_R ax$. If $b \in \mathfrak{p}$, then $ab \in \mathfrak{p}^2$ and abx = 0. Thus $b \in 0:_R ax$ and so $\mathfrak{p}\subseteq 0:_R ax.$
- (iii) Assume that $a \notin \mathfrak{p}$. It is obvious that $0:_R x \subseteq 0:_R a^n x$, for all $n \in \mathbb{N}$. Let $b \in \operatorname{Ann}_R(a^n x)$. Thus $ba^n x = 0$. But multiplication by a^n is an automorphism on $E(R/\mathfrak{p})$, so that bx = 0 and $b \in 0$:_R x. Therefore, $0:_R x = 0:_R a^n x.$

Corollary 2.10. Let R be a principal ideal domain and \mathfrak{p} is a prime ideal of R. If 0 is a 2-absorbing submodule of $E(R/\mathfrak{p})$, then for all non-zero element x of $E(R/\mathfrak{p})$ either $0:_R x = \mathfrak{p}^2$ or $0:_R x = \mathfrak{p}$.

Proof. Let $\mathfrak{p}=(a)$. Then $\mathfrak{p}^2=(a^2)$. Let x be a non-zero element of $E(R/\mathfrak{p})$. Then $\mathfrak{p}^2 \subseteq 0:_R x = (b) \subseteq \mathfrak{p}$ by Theorem 2.9(ii). Thus $a^2 = bc$ and b = ae for some $c, e \in R$. Hence, $a^2 = aec$. So $a = ec \in \mathfrak{p}$. Therefore, either $c \in \mathfrak{p}$ or $e \in \mathfrak{p}$. If $c \in \mathfrak{p}$, then c = ac' and so a = eac' which implies that 1 = ec' and a = bc' thus $0:_R x = \mathfrak{p}$. If $e \in \mathfrak{p}$, then e = ae' and so a = ae'c which implies that 1 = e'c and $b = a^2e'$ thus $0 :_R x = \mathfrak{p}^2$.

The following example shows that the condition "0 is a 2-absorbing submodule of $E(R/\mathfrak{p})$ " is essential. It is well-known that $E(\mathbb{Z}/p\mathbb{Z}) = \mathbb{Z}(p^{\infty}) =$ $\{m/n + \mathbb{Z} : m, n \in \mathbb{Z}, n \neq 0\}$, where p is a prime integer. But neither $p^2\mathbb{Z}=0:_{\mathbb{Z}}1/p^3+\mathbb{Z}$ nor $0:_{\mathbb{Z}}1/p^3+\mathbb{Z}=p\mathbb{Z}$. Hence, 0 is not a 2-absorbing submodule of $E(\mathbb{Z}/p\mathbb{Z})$. Also, this example shows that if 0 is a 2-absorbing submodule of M, then it is not necessarily a 2-absorbing submodule of E(M).

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References

- 1. D. F. Anderson, A. Badawi, On n-absorbing ideals of commutative rings, Comm. Algebra, 39, (2011), 1646-1672.
- 2. A. Badawi, On 2-absorbing ideals of commutavie rings, Bull. Austral. Math. Soc., 75, (2007), 417-429.
- 3. A. Badawi, A. Y. Darani, On weakly 2-absorbing ideals of commutative rings, Houston J. Math., 39 (2), (2013), 441-452.
- 4. A. Y. Darani, E. R. Puczylowski, On 2-absorbing commutative semigroups and their applications to rings, Semigroup Forum, 86, (2013), 83-91.

- A. Y. Darani, F. Soheilnia, On n-absorning submodules, Math. Commun., 17, (2012), 547-557.
- A. Y. Darani, F. Soheilnia, 2-Absorning and weakly 2-Absorbing submodules, Thai J. of Math., 9, (2011), 577-584.
- 7. R. Jahani-Nezhad, The Dual of a Strongly Prime Ideal, *Iranian Journal of Mathematical Sciences and Informatics*, **5**, (2010), 19-26.
- 8. H. Matsumura, Commutative Rings Theory, Second edition, Cambridge Studies in Advanced Mathematics, 8, Cambridge University Press, Cambridge, 1989.
- 9. Sh. Payrovi, S. Babaei, On 2-absorbing submodules, Algebra Collq., 19, (2012), 913-920.
- 10. Sh. Payrovi, S. Babaei, On the 2-absorbing ideals, Int. Math. Forum, 7, (2012), 265-271.