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OD-characterization of Almost Simple Groups Related to $D_4(4)$

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ABSTRACT. Let G be a finite group and $\pi_e(G)$ be the set of orders of all elements in G. The set $\pi_e(G)$ determines the prime graph (or Grunberg-Kegel graph) $\Gamma(G)$ whose vertex set is $\pi(G)$. The set of primes dividing the order of G, and two vertices p and q are adjacent if and only if $pq \in \pi_e(G)$. The degree deg(p) of a vertex $p \in \pi(G)$, is the number of edges incident on p. Let $\pi(G) = \{p_1, p_2, ..., p_k\}$ with $p_1 < p_2 < ... < p_k$. We define $D(G) := (deg(p_1), deg(p_2), ..., deg(p_k))$, which is called the degree pattern of G. The group G is called k-fold OD-characterizable if there exist exactly k non-isomorphic groups M satisfying conditions |G| = |M| and D(G) = D(M). Usually a 1-fold OD-characterizable group is simply called OD-characterizable. In this paper, we classify all finite groups with the same order and degree pattern as an almost simple groups related to $D_4(4)$.

Keywords: Degree pattern, k-fold OD-characterizable, Almost simple group.

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1. INTRODUCTION

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Let G be a finite group, $\pi(G)$ the set of all prime divisors of |G| and $\pi_e(G)$ be the set of orders of elements in G. The prime graph (or Grunberg-Kegel graph) $\Gamma(G)$ of G is a simple graph with vertex set $\pi(G)$ in which two vertices p and q are joined by an edge (and we write $p \sim q$) if and only if G contains an element of order pq (i.e. $pq \in \pi_e(G)$).

The degree deg(p) of a vertex $p \in \pi(G)$ is the number of edges incident on p. If $\pi(G) = \{p_1, p_2, ..., p_k\}$ with $p_1 < p_2 < ... < p_k$, then we define $D(G):=(deg(p_1), deg(p_2), ..., deg(p_k))$, which is called the degree pattern of G, and leads a following definition.

Definition 1.1. The finite group G is called k-fold OD-characterizable if there exist exactly k non-isomorphic groups H satisfying conditions |G| = |H| and D(G) = D(H). In particular, a 1-fold OD-characterizable group is simply called OD-characterizable.

The interest in characterizing finite groups by their degree patterns started in [7] by M. R. Darafsheh and et. all, in which the authors proved that the following simple groups are uniquely determined by their order and degree patterns: All sporadic simple groups, the alternating groups A_p with p and p-2 primes and some simple groups of Lie type. Also in a series of articles (see [4, 6, 8, 9, 14, 17]), it was shown that many finite simple groups are ODcharacterizable.

Let A and B be two groups then a split extension is denoted by A : B. If L is a finite simple group and $Aut(L) \cong L : A$, then if B is a cyclic subgroup of A of order n we will write L : n for the split extension L : B. Moreover if there are more than one subgroup of orders n in A, then we will denote them by $L : n_1, L : n_2$, etc.

Definition 1.2. A group G is said to be an almost simple group related to S if and only if $S \leq G \lesssim \operatorname{Aut}(S)$, for some non-abelian simple group S.

In many papers (see [2, 3, 10, 13, 15, 16]), it has been proved, up to now, that many finite almost simple groups are OD-characterizable or k-fold OD-characterizable for certain $k \ge 2$.

We denote the socle of G by Soc(G), which is the subgroup generated by the set of all minimal normal subgroups of G. For $p \in \pi(G)$, we denote by G_p and $Syl_p(G)$ a Sylow *p*-subgroup of G and the set of all Sylow *p*-subgroups of G respectively, all further unexplained notation are standard and can be found in [11].

In this article our main aim is to show the recognizability of the almost simple groups related to $L := D_4(4)$ by degree pattern in the prime graph and order of the group. In fact, we will prove the following Theorem.

Main Theorem Let M be an almost simple group related to $L := D_4(4)$. If G is a finite group such that D(G) = D(M) and |G| = |M|, then the following assertions hold:

(a) If M = L, then $G \cong L$.

(b) If $M = L : 2_1$, then $G \cong L : 2_1$ or $L : 2_3$.

(c) If $M = L : 2_2$, then $G \cong L : 2_2$ or $\mathbb{Z}_2 \times L$.

- (d) If $M = L : 2_3$, then $G \cong L : 2_3$ or $L : 2_1$.
- (e) If M = L : 3, then $G \cong L : 3$ or $\mathbb{Z}_3 \times L$.

(f) If $M = L : 2^2$, then $G \cong L : 2^2$, $\mathbb{Z}_2 \times (L : 2_1)$, $\mathbb{Z}_2 \times (L : 2_2)$, $\mathbb{Z}_2 \times (L : 2_3)$, $\mathbb{Z}_4 \times L$ or $(\mathbb{Z}_2 \times \mathbb{Z}_2) \times L$.

(g) If $M = L : (D_6)_1$, then $G \cong L : (D_6)_1$, L : 6, $\mathbb{Z}_3 \times (L : 2_1)$, $\mathbb{Z}_3 \times (L : 2_3)$ or $(\mathbb{Z}_3 \times L) : \mathbb{Z}_2$.

(h) If $M = L : (D_6)_2$, then $G \cong L : (D_6)_2$, $\mathbb{Z}_2 \times (L : 3)$, $\mathbb{Z}_3 \times (L : 2_2)$, $(\mathbb{Z}_3 \times L) . \mathbb{Z}_2$, $\mathbb{Z}_6 \times L$ or $D_6 \times L$.

(i) If M = L : 6, then $G \cong L : 6$, $L : (D_6)_1$, $\mathbb{Z}_3 \times (L : 2_1)$, $\mathbb{Z}_3 \times (L : 2_3)$ or $(\mathbb{Z}_3 \times L) . \mathbb{Z}_2$.

(j) If $M = L : D_{12}$, then $G \cong L : D_{12}$, $\mathbb{Z}_2 \times (L : (D_6)_1)$, $\mathbb{Z}_2 \times (L : (D_6)_2)$, $\mathbb{Z}_2 \times (L : 6)$, $\mathbb{Z}_3 \times (L : 2^2)$, $(\mathbb{Z}_3 \times (L : 2_1)) \cdot \mathbb{Z}_2$, $(\mathbb{Z}_3 \times (L : 2_2)) \cdot \mathbb{Z}_2$, $(\mathbb{Z}_3 \times (L : 2_3)) \cdot \mathbb{Z}_2$, $\mathbb{Z}_4 \times (L : 3)$, $(\mathbb{Z}_2 \times \mathbb{Z}_2) \times (L : 3)$, $(\mathbb{Z}_4 \times L) \cdot \mathbb{Z}_3$, $((\mathbb{Z}_2 \times \mathbb{Z}_2) \times L) \cdot \mathbb{Z}_3$, $\mathbb{Z}_6 \times (L : 2_1)$, $\mathbb{Z}_6 \times (L : 2_2)$, $\mathbb{Z}_6 \times (L : 2_3)$, $(\mathbb{Z}_6 \times L) \cdot \mathbb{Z}_2$, $D_6 \times (L : 2_1)$, $D_6 \times (L : 2_2)$, $D_6 \times (L : 2_3)$, $\mathbb{Z}_{12} \times L$, $(\mathbb{Z}_2 \times \mathbb{Z}_6) \times L$, $(\mathbb{Z}_2 \times L) \cdot D_6$, $\mathbb{A}_4 \times L$, $L \cdot \mathbb{A}_4$, $D_{12} \times L$ or $T \times L$.

2. Preliminary Results

It is well-known that $\operatorname{Aut}(D_4(4)) \cong D_4(4) : D_{12}$ where D_{12} denotes the dihedral group of order 12. We remark that D_{12} has the following non-trivial proper subgroups up to conjugacy: three subgroups of order 2, one cyclic subgroup each of order 3 and 6, two subgroups isomorphic to $D_6 \cong \mathbb{S}_3$ and one subgroup of order 4 isomorphic to the Klein's four group denoted by 2^2 . The field and the duality automorphisms of $D_4(4)$ are denoted by 2_1 and 2_2 respectively, and we set $2_3 = 2_1 \cdot 2_2$ (field*duality which is called the diagonal automorphism). Therefore up to conjugacy we have the following almost simple groups related to $D_4(4)$.

Lemma 2.1. If G is an almost simple group related to $L:=D_4(4)$, then G is isomorphic to one of the following groups: $L, L: 2_1, L: 2_2, L: 2_3, L: 3, L: 2^2, L: (D_6)_1, L: (D_6)_2, L: 6, L: D_{12}.$

Lemma 2.2 ([5]). Let G be a Frobenius group with kernel K and complement H. Then:

- (a) K is a nilpotent group.
- (b) $|K| \equiv 1 \pmod{|H|}$.

Let $p \geq 5$ be a prime. We denote by \mathfrak{S}_p the set of all simple groups with prime divisors at most p. Clearly, if $q \leq p$, then $\mathfrak{S}_q \subseteq \mathfrak{S}_p$. We list all the simple groups in class \mathfrak{S}_{17} with their order and the order of their outer automorphisms in TABLE 1, taken from [12].

S	S	$ \operatorname{Out}(S) $	S	S	$ \operatorname{Out}(S) $
A_5	$2^2 \cdot 3 \cdot 5$	2	$G_{2}(3)$	$2^6\cdot 3^6\cdot 7\cdot 13$	2
A_6	$2^3 \cdot 3^2 \cdot 5$	4	${}^{3}D_{4}(2)$	$2^{12}\cdot 3^4\cdot 7^2\cdot 13$	3
$S_4(3)$	$2^6 \cdot 3^4 \cdot 5$	2	$L_2(64)$	$2^6\cdot 3^2\cdot 5\cdot 7\cdot 13$	6
$L_{2}(7)$	$2^3 \cdot 3 \cdot 7$	2	$U_{4}(5)$	$2^7\cdot 3^4\cdot 5^6\cdot 7\cdot 13$	4
$L_{2}(8)$	$2^3 \cdot 3^2 \cdot 7$	3	$L_{3}(9)$	$2^7\cdot 3^6\cdot 5\cdot 7\cdot 13$	4
$U_{3}(3)$	$2^5 \cdot 3^3 \cdot 7$	2	$S_{6}(3)$	$2^9\cdot 3^9\cdot 5\cdot 7\cdot 13$	2
A_7	$2^3 \cdot 3^2 \cdot 5 \cdot 7$	2	$O_{7}(3)$	$2^9\cdot 3^9\cdot 5\cdot 7\cdot 13$	2
$L_{2}(49)$	$2^4 \cdot 3 \cdot 5^2 \cdot 7^2$	4	$G_{2}(4)$	$2^{12}\cdot 3^3\cdot 5^2\cdot 7\cdot 13$	2
$U_{3}(5)$	$2^4 \cdot 3^2 \cdot 5^3 \cdot 7$	6	$S_4(8)$	$2^{12}\cdot 3^4\cdot 5\cdot 7^2\cdot 13$	6
$L_{3}(4)$	$2^6 \cdot 3^2 \cdot 5 \cdot 7$	12	$O_8^+(3)$	$2^{12} \cdot 3^{12} \cdot 5^2 \cdot 7 \cdot 13$	24
A_8	$2^6 \cdot 3^2 \cdot 5 \cdot 7$	2	$L_{5}(3)$	$2^9\cdot 3^{10}\cdot 5\cdot 11^2\cdot 13$	2
A_9	$2^6 \cdot 3^4 \cdot 5 \cdot 7$	2	A_{13}	$2^9\cdot 3^5\cdot 5^2\cdot 7\cdot 11\cdot 13$	2
J_2	$2^7 \cdot 3^3 \cdot 5^2 \cdot 7$	2	A_{14}	$2^{10} \cdot 3^5 \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 13$	2
A_{10}	$2^7 \cdot 3^4 \cdot 5^2 \cdot 7$	2	A_{15}	$2^{10} \cdot 3^6 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13$	2
$U_{4}(3)$	$2^7 \cdot 3^6 \cdot 5 \cdot 7$	8	$L_{6}(3)$	$2^{11} \cdot 3^{15} \cdot 5 \cdot 7 \cdot 11^2 \cdot 13^2$	4
$S_4(7)$	$2^8 \cdot 3^2 \cdot 5^2 \cdot 7^4$	2	Suz	$2^{13} \cdot 3^7 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$	2
$S_{6}(2)$	$2^9\cdot 3^4\cdot 5\cdot 7$	1	A_{16}	$2^{14} \cdot 3^6 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13$	2
$O_8^+(2)$	$2^{12}\cdot 3^5\cdot 5^2\cdot 7$	6	Fi_{22}	$2^{17} \cdot 3^9 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$	2
$L_2(11)$	$2^2 \cdot 3 \cdot 5 \cdot 11$	2	$L_2(17)$	$2^4 \cdot 3^2 \cdot 17$	2
M_{11}	$2^4 \cdot 3^2 \cdot 5 \cdot 11$	1	$L_2(16)$	$2^4 \cdot 3 \cdot 5 \cdot 17$	4
M_{12}	$2^6 \cdot 3^3 \cdot 5 \cdot 11$	2	$S_{4}(4)$	$2^8 \cdot 3^2 \cdot 5^2 \cdot 17$	4
$U_{5}(2)$	$2^{10} \cdot 3^5 \cdot 5 \cdot 11$	2	He	$2^{10} \cdot 3^3 \cdot 5^2 \cdot 7^3 \cdot 17$	2
M_{22}	$2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$	2	$O_8^{-}(2)$	$2^{12} \cdot 3^4 \cdot 5 \cdot 7 \cdot 17$	2
A_{11}	$2^7 \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 11$	2	$L_{4}(4)$	$2^{12}\cdot 3^4\cdot 5^2\cdot 7\cdot 17$	4
$M^{c}L$	$2^7 \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11$	2	$S_{8}(2)$	$2^{16}\cdot 3^5\cdot 5^2\cdot 7\cdot 17$	1
HS	$2^9 \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 11$	2	$U_{4}(4)$	$2^{12} \cdot 3^2 \cdot 5^3 \cdot 13 \cdot 17$	4
A_{12}	$2^9 \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 11$	2	$U_3(17)$	$2^6 \cdot 3^4 \cdot 7 \cdot 13 \cdot 17^3$	6
$U_{6}(2)$	$2^{15} \cdot 3^6 \cdot 5 \cdot 7 \cdot 11$	6	$O_{10}^{-}(2)$	$2^{20} \cdot 3^6 \cdot 5^2 \cdot 7 \cdot 11 \cdot 17$	2
$L_{3}(3)$	$2^4 \cdot 3^3 \cdot 13$	2	$L_2(13^2)$	$2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 13^2 \cdot 17$	4
$L_2(25)$	$2^3 \cdot 3 \cdot 5^2 \cdot 13$	4	$S_4(13)$	$2^6\cdot 3^2\cdot 5\cdot 7^2\cdot 13^4\cdot 17$	2
$U_{3}(4)$	$2^6 \cdot 3 \cdot 5^2 \cdot 13$	4	$L_3(16)$	$2^{12} \cdot 3^2 \cdot 5^2 \cdot 7 \cdot 13 \cdot 17$	24
$S_4(5)$	$2^6 \cdot 3^2 \cdot 5^4 \cdot 13$	2	$S_{6}(4)$	$2^{18} \cdot 3^4 \cdot 5^3 \cdot 7 \cdot 13 \cdot 17$	2
$L_4(3)$.	$2^7 \cdot 3^6 \cdot 5 \cdot 13$	4	$O_8^+(4)$	$2^{24} \cdot 3^5 \cdot 5^4 \cdot 7 \cdot 13 \cdot 17^2$	12
${}^{2}F_{4}(2)'$	$2^{11} \cdot 3^3 \cdot 5^2 \cdot 13$	2	$F_4(2)$	$2^{24} \cdot 3^6 \cdot 5^2 \cdot 7^2 \cdot 13 \cdot 17$	2
$L_2(13)$	$2^2 \cdot 3 \cdot 7 \cdot 13$	2	A_{17}	$2^{14} \cdot 3^6 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17$	2
$L_{2}(27)$	$2^2 \cdot 3^3 \cdot 7 \cdot 13$	6	A10	$2^{15} \cdot 3^8 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17$	2

TABLE 1: Simple groups in \mathfrak{S}_p , $p \leq 17$.

Definition 2.3. A completely reducible group will be called a CR-group. The center of a CR-group is a direct product of the abelian factor in the decomposition. Hence, a CR-group is centerless, that is, has trivial center, if and only if it is a direct product of non-abelian simple groups. The following Lemma determines the structure of the automorphism group of a centerless CR-group.

Lemma 2.3 ([11]). Let R be a finite centerless CR-group and write $R = R_1 \times R_2 \times ... \times R_k$, where R_i is a direct product of n_i isomorphic copies of a simple group H_i , and H_i and H_j are not isomorphic if $i \neq j$. Then $\operatorname{Aut}(R) = \operatorname{Aut}(R_1) \times \operatorname{Aut}(R_2) \times ... \times \operatorname{Aut}(R_k)$ and $\operatorname{Aut}(R_i) \cong \operatorname{Aut}(H_i) \wr \mathbb{S}_{n_i}$, where in this wreath product $\operatorname{Aut}(H_i)$ appears in its right regular representation and the symmetric group \mathbb{S}_{n_i} in its natural permutation representation. Moreover, these isomorphisms induce isomorphisms $\operatorname{Out}(R) \cong \operatorname{Out}(R_1) \times \operatorname{Out}(R_2) \times ... \times \operatorname{Out}(R_k)$ and $\operatorname{Out}(R_i) \cong \operatorname{Out}(H_i) \wr \mathbb{S}_{n_i}$.

3. OD-Characterization of Almost Simple Groups Related to $D_4(4)$

In this section, we study the problem of characterizing almost simple groups by order and degree pattern. Especially we will focus our attention on almost simple groups related to $L = D_4(4)$, namely, we will prove the Main Theorem of Sec. 1. We break the proof into a number of separate propositions.

By assumption, we depict all possibilities for the prime graph associated with G by use of the variables for some vertices in each proposition. Also, we need to know the structure of $\Gamma(M)$ to determine the possibilities for G in some proposition, therefore we depict the prime graph of all extension of L in pages 18 to 20. Note that the set of order elements in each of the following propositions is calculated using Magma.

Proposition 3.1. If M = L, then $G \cong L$.

Proof. By TABLE 1 $|L| = 2^{24} \cdot 3^5 \cdot 5^4 \cdot 7 \cdot 13 \cdot 17^2$. $\pi_e(L) = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 15, 17, 20, 21, 30, 34, 51, 63, 65, 85, 255\}$, so D(L) = (3, 4, 4, 1, 1, 3). Since |G| = |L| and D(G) = D(L), we conclude that the prime graph of G has following form:



Figure 3.1

where $\{a, b\} = \{7, 13\}.$

We will show that G is isomorphic to $L = D_4(4)$. We break up the proof into a several steps.

Step1. Let K be the maximal normal solvable subgroup of G. Then K is a $\{2,3,5\}$ -group. In particular, G is non-solvable.

First we show that K is a 17'-group. Assume the contrary and let $17 \in \pi(K)$. Then 13 dose not divide the order of K. Otherwise, we may suppose that T is a Hall {13,17}-subgroup of K. It is seen that T is a nilpotent subgroup of order 13.17ⁱ for i = 1 or 2. Thus, $13.17 \in \pi_e(K) \subseteq \pi_e(G)$, a contradiction. Thus $\{17\} \subseteq \pi(K) \subseteq \pi(G) - \{13\}$. Let $K_{17} \in \text{Syl}_{17}(K)$. By Frattini argument, $G = KN_G(K_{17})$. Therefore, $N_G(K_{17})$ contains an element x of order 13. Since G has no element of order 13.17, $\langle x \rangle$ should act fixed point freely on K_{17} , that is implying $\langle x \rangle K_{17}$ is a Frobenius group. By Lemma 2.2(b), $|\langle x \rangle || (|K_{17}| - 1)$. It follows that $13|17^i - 1$ for i = 1 or 2, which is a contradiction.

Next, we show that K is a p'-group for $p \in \{a, b\}$. Let p||K| and $K_p \in \operatorname{Syl}_p(K)$. Now by Frattini argument, $G = KN_G(K_p)$, so 17 must divide the order of $N_G(K_p)$. Therefore, the normalizer $N_G(K_p)$ contains an element of order 17, say x. So $\langle x \rangle K_p$ is a cyclic subgroup of G of order 17.p, and so $p \sim 17$ in $\Gamma(G)$, which is a contradiction. Therefore K is a $\{2, 3, 5\}$ -group. In addition, since K is a proper subgroup of G, it follows that G is non-solvable.

Step 2. The quotient G/K is an almost simple group. In fact, $S \leq G/K \lesssim$ Aut(S), where S is a finite non-abelian simple group isomorphic to $L := D_4(4)$.

Let $\overline{G} = G/K$. Then $S := \operatorname{Soc}(\overline{G}) = P_1 \times P_2 \times \ldots \times P_m$, where P_i 's are finite non-abelian simple groups and $S \leq \frac{G}{K} \leq \operatorname{Aut}(S)$. If we show that m = 1, the proof of Step 2 will be completed.

Suppose that $m \geq 2$. In this case, we claim that 13 does not divide |S|. Assume the contrary and let 13 |S|, on the other hand, $\{2,3\} \subset \pi(P_i)$ for every *i* (by TABLE 1), hence $2 \sim 13$ and $3 \sim 13$, which is a contradiction. Now, by step 1, we observe that $13 \in \pi(\overline{G}) \subseteq \pi(\operatorname{Aut}(S))$. But $\operatorname{Aut}(S) =$ $\operatorname{Aut}(S_1) \times \operatorname{Aut}(S_2) \times \ldots \times \operatorname{Aut}(S_r)$, where the groups S_j are direct products of isomorphic P_i 's such that $S = S_1 \times S_2 \times \ldots \times S_r$. Therefore, for some *j*, 13 divides the order of an automorphism group of a direct product S_j of *t* isomorphic simple groups P_i . Since $P_i \in \mathfrak{S}_{17}$, it follows that $|\operatorname{Out}(P_i)|$ is not divisible by 13 (see TABLE 1). Now, by Lemma 2.3, we obtain $|\operatorname{Aut}(S_j)| =$ $|\operatorname{Aut}(P_i)|^{t!} t!$. Therefore, $t \geq 13$ and so 2^{26} must divide the order of *G*, which is a contradiction. Therefore m = 1 and $S = P_1$.

By TABLE 1 and Step 1, it is evident that $|S| = 2^{\alpha}.3^{\beta}.5^{\gamma}.7.13.17^2$, where $2 \leq \alpha \leq 24, 1 \leq \beta \leq 5$ and $0 \leq \gamma \leq 4$. Now, using collected results contained in TABLE 1, we deduce that $S \cong D_4(4)$ and by Step 2, $L \trianglelefteq G/K \lesssim \operatorname{Aut}(L)$ is completed. As |G| = |L|, we deduce K = 1, so $G \cong L$ and the proof is completed.

Proposition 3.2. If $M = L : 2_1$, then $G \cong L : 2_1$ or $L : 2_3$.

Proof. As $|L:2_1| = 2^{25} \cdot 3^5 \cdot 5^4 \cdot 7 \cdot 13 \cdot 17^2$ and $\pi_e(L:2_1) = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 14, 15, 16, 17, 18, 20, 21, 24, 30, 34, 51, 63, 65, 85, 255\}$, then $D(L:2_1) = (4, 4, 4, 2, 1, 3)$. Since $|G| = |L:2_1|$ and $D(G) = D(L:2_1)$, we conclude that there exist several possibilities for $\Gamma(G)$:



where $\{a, b, c\} = \{2, 3, 5\}.$

Step1. Let K be the maximal normal solvable subgroup of G. Then K is a $\{2,3,5\}$ -group. In particular, G is non-solvable.

By a similar argument to that in Proposition 3.1, we can obtain this assertion. **Step 2.** The quotient $\frac{G}{K}$ is an almost simple group. In fact, $S \leq \frac{G}{K} \lesssim$ Aut(S), where S is a finite non-abelian simple group.

The proof is similar to Step 2 of Proposition 3.1.

By TABLE 1 and Step 1, it is evident that $|S| = 2^{\alpha}.3^{\beta}.5^{\gamma}.7.13.17^2$, where $2 \leq \alpha \leq 25, 1 \leq \beta \leq 5$ and $0 \leq \gamma \leq 4$. Now, using collected results contained in TABLE 1, we conclude that $S \cong D_4(4)$ and by Step 2, $L \leq \frac{G}{K} \lesssim \operatorname{Aut}(L)$. As $|G| = |L:2_1| = 2|L|$, we deduce |K| = 1 or 2.

If |K| = 1, then $G \cong L : 2_1, L : 2_2$ or $L : 2_3$. Obviously, $G \cong L : 2_1$ or $L : 2_3$ because deg(2) = 5 in $\Gamma(L : 2_2)$ (see page 16).

If |K| = 2, then $K \leq Z(G)$ and so deg(2) = 5, which is a contradiction. \Box

Proposition 3.3. If $M = L : 2_2$, then $G \cong L : 2_2$ or $\mathbb{Z}_2 \times L$.

Proof. As $|L:2_2| = 2^{25} \cdot 3^5 \cdot 5^4 \cdot 7 \cdot 13 \cdot 17^2$ and $\pi_e(L:2_2) = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 14, 15, 17, 18, 20, 21, 24, 26, 30, 34, 40, 42, 51, 60, 63, 65, 68, 85, 102, 126, 130, 170, 255\}$, then $D(L:2_2) = (5, 4, 4, 2, 2, 3)$. By assumption $|G| = |L:2_2|$ and $D(G) = D(L:2_2)$, so the prime graph of G has following form:



where $\{a, b\} = \{7, 13\}.$

Step1. Let K be the maximal normal solvable subgroup of G. Then K is a $\{2, 3, 5\}$ -group. In particular, G is non-solvable.

By similar arguments as in the proof of Step 1 in Proposition 3.1, we conclude that K is a $\{2, 3, 5\}$ -group and G is non-solvable.

Step 2. The quotient $\frac{G}{K}$ is an almost simple group. In fact, $S \leq \frac{G}{K} \lesssim$ Aut(S), where S is a finite non-abelian simple group.

Let $\overline{G} = \frac{G}{K}$. Then $S := \operatorname{Soc}(\overline{G}), S = P_1 \times P_2 \times \ldots \times P_m$, where P_i 's are finite non-abelian simple groups and $S \leq \frac{G}{K} \lesssim \operatorname{Aut}(S)$. We are going to prove that m = 1 and $S = P_1$. Suppose that $m \ge 2$. We claim a does not divide |S|. Assume the contrary and let $a \mid |S|$, we conclude that a just divide the order of one of the simple groups P_i 's. Without loss of generality, we assume that $a||P_1|$. Then the rest of the P_i 's must be $\{2,3\}$ -group (because only 2 and 3 are adjacent to a in $\Gamma(G)$), this is a contradiction because P_i 's are finite non-abelian simple groups. Now, by Step 1, we observe that $a \in \pi(\overline{G}) \subseteq \pi(\operatorname{Aut}(S))$. But $\operatorname{Aut}(S) = \operatorname{Aut}(S_1) \times \operatorname{Aut}(S_2) \times \ldots \times \operatorname{Aut}(S_r)$, where the groups S_i are direct products of isomorphic P_i 's such that $S = S_1 \times S_2 \times ... \times S_r$. Therefore, for some j, a divides the order of an automorphism group of a direct product S_i of t isomorphic simple groups P_i . Since $P_i \in \mathfrak{S}_{17}$, it follows that $|\operatorname{Out}(P_i)|$ is not divisible by a (see TABLE 1), so a does not divide the order of $\operatorname{Aut}(P_i)$. Now, by Lemma 2.3, we obtain $|\operatorname{Aut}(S_i)| = |\operatorname{Aut}(P_i)|^{t!} t!$. Therefore, $t \ge a$ and so 3^a must divide the order of G, which is a contradiction. Therefore m = 1 and $S = P_1.$

By TABLE 1 and Step 1, it is evident that $|S| = 2^{\alpha}.3^{\beta}.5^{\gamma}.7.13.17^2$, where $2 \leq \alpha \leq 25, 1 \leq \beta \leq 5$ and $0 \leq \gamma \leq 4$. Now, using collected results contained in TABLE 1, we conclude that $S \cong D_4(4)$ and by Step 2, $L \trianglelefteq \frac{G}{K} \lesssim \operatorname{Aut}(L)$. As $|G| = |L:2_2| = 2|L|$, we deduce |K| = 1 or 2.

If |K| = 1, then $G \cong L : 2_1, L : 2_2$ or $L : 2_3$ because |G| = 2|L|. It is obvious that $G \cong L : 2_2$, because deg(13) = 1 in $\Gamma(L : 2_1)$ and $\Gamma(L : 2_3)$ (see page 17).

If |K| = 2, then $G/K \cong L$ and $K \leq Z(G)$. It follows that G is a central extension of K by L. If G is a non-split extension of K by L, then |K| must divide the Schur multiplier of L, which is 1. But this is a contradiction, so we obtain that G split over |K|. Hence $G \cong \mathbb{Z}_2 \times L$.

Proposition 3.4. If $M = L : 2_3$, then $G \cong L : 2_3$ or $L : 2_1$.

Proof. As $|L:2_3| = 2^{25} \cdot 3^5 \cdot 5^4 \cdot 7 \cdot 13 \cdot 17^2$ and $\pi_e(L:2_3) = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 14, 15, 16, 17, 18, 20, 21, 24, 30, 34, 51, 63, 65, 85, 255\}$, then $D(L:2_3) = (4, 4, 4, 2, 1, 3)$. Since $|G| = |L:2_3|$ and $D(G) = D(L:2_3)$, we conclude that $\Gamma(G)$ has the following form similarly to Proposition 3.2:



where $\{a, b, c\} = \{2, 3, 5\}.$

Step1. Let K be the maximal normal solvable subgroup of G. Then K is a $\{2,3,5\}$ -group. In particular, G is non-solvable.

We can prove this by the similar way to that in Proposition 3.2.

Step 2. The quotient $\frac{G}{K}$ is an almost simple group. In fact, $S \leq \frac{G}{K} \lesssim$ Aut(S), where S is a finite non-abelian simple group.

By using a similar argument, as in the proof of Proposition 3.2, we can verify that $\frac{G}{K}$ is an almost simple group.

By TABLE 1 and Step 1, it is evident that $|S| = 2^{\alpha}.3^{\beta}.5^{\gamma}.7.13.17^2$, where $2 \leq \alpha \leq 25, 1 \leq \beta \leq 5$ and $0 \leq \gamma \leq 4$. Now, using collected results contained in TABLE 1, we conclude that $S \cong D_4(4)$ and by Step 2, $L \trianglelefteq \frac{G}{K} \lesssim \operatorname{Aut}(L)$. As $|G| = |L:2_3| = 2|L|$, we deduce |K| = 1 or 2.

If |K| = 1, then $G \cong L : 2_1, L : 2_2$ or $L : 2_3$ because |G| = 2|L|. Obviously, $G \cong L : 2_3$ or $L : 2_1$, because deg(2) = 5 in $\Gamma(L : 2_2)$ (see page 16).

If |K| = 2, then $K \leq Z(G)$ and so deg(2) = 5, which is a contradiction. \Box

Proposition 3.5. If M = L : 3, then $G \cong L : 3$ or $\mathbb{Z}_3 \times L$.

Proof. As $|L:3| = 2^{24} \cdot 3^6 \cdot 5^4 \cdot 7 \cdot 13 \cdot 17^2$ and $\pi_e(L:3) = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 15, 17, 18, 20, 21, 24, 30, 34, 39, 45, 51, 63, 65, 85, 255\}$, then D(L:3) = (3, 5, 4, 1, 2, 3). since |G| = |L:3| and D(G) = D(L:3), we conclude that $\Gamma(G)$ has the following form (like $\Gamma(L:3)$):



Step1. Let K be the maximal normal solvable subgroup of G. Then K is a $\{2,3\}$ -group. In particular, G is non-solvable.

First, we show that K is a p'-group for p = 7,13 and 17. Since the proof is quite similar to the proof of Step 1 in Proposition 3.1, so we avoid here full explanation of all details.

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Next we consider K is a 5'-group. Assume the contrary, $5 \in \pi_e(K)$. Let $K_5 \in \text{Syl}_5(K)$. By Frattini argument, $G = KN_G(K_5)$. Therefore, $N_G(K_5)$ has an element x of order 7. Since G has no element of order 5.7, $\langle x \rangle$ should act fixed point freely on K_5 , implying $\langle x \rangle K_5$ is a Frobenius group. By Lemma 2.2(b), $|\langle x \rangle||(|K_5|-1)$, which is impossible. Therefore K is a $\{2,3\}$ -group. In addition since K is a proper subgroup of C then C is non-solvable and the

In addition since K is a proper subgroup of G, then G is non-solvable and the proof of this step is completed.

Step 2. The quotient $\frac{G}{K}$ is an almost simple group. In fact, $S \leq \frac{G}{K} \lesssim$ Aut(S), where S is a finite non-abelian simple group.

In a similar way as in the proof of Step 2 in Proposition 3.1, we conclude that $\frac{G}{K}$ is an almost simple group.

By TABLE 1 and Step 1, it is evident that $|S| = 2^{\alpha}.3^{\beta}.5^{4}.7.13.17^{2}$, where $2 \leq \alpha \leq 24$ and $1 \leq \beta \leq 6$. Now, using collected results contained in TABLE 1, we conclude that $S \cong D_{4}(4)$ and by Step 2, $L \trianglelefteq \frac{G}{K} \lesssim \operatorname{Aut}(L)$. As |G| = |L: 3| = 3|L|, we deduce |K| = 1 or 3.

If |K| = 1, then $G \cong L : 3$.

If |K| = 3, then $G/K \cong L$. In this case we have $G/C_G(K) \lesssim \operatorname{Aut}(K) \cong \mathbb{Z}_2$. Thus $|G/C_G(K)| = 1$ or 2. If $|G/C_G(K)| = 1$, then $K \leq Z(G)$, that is, G is a central extension of K by L. If G is a non-split extension of K by L, then |K| must divide the Schur multiplier of L, which is 1. But this is a contradiction, so we obtain that G split over K. Hence $G \cong \mathbb{Z}_3 \times L$. If $|G/C_G(K)| = 2$, then $K < C_G(K)$ and $1 \neq C_G(K)/K \leq G/K \cong L$, which is a contradiction since L is simple.

Proposition 3.6. If $M = L : 2^2$, then $G \cong L : 2^2$, $\mathbb{Z}_2 \times (L : 2_1)$, $\mathbb{Z}_2 \times (L : 2_2)$, $\mathbb{Z}_2 \times (L : 2_3)$, $\mathbb{Z}_4 \times L$ or $(\mathbb{Z}_2 \times \mathbb{Z}_2) \times L$.

Proof. As $|L:2^2| = 2^{26}.3^5.5^4.7.13.17^2$ and $\pi_e(L:2^2) = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 14, 15, 16, 17, 18, 20, 21, 24, 26, 30, 34, 42, 51, 60, 63, 65, 68, 85, 102, 126, 130, 170, 255\}$, then $D(L:2^2) = (5, 4, 4, 2, 2, 3)$. Since $|G| = |L:2^2|$ and $D(G) = D(L:2^2)$, so the prime graph of G has following form similarly to Proposition 3.3:



where $\{a, b\} = \{7, 13\}.$

Step1. Let K be the maximal normal solvable subgroup of G. Then K is a $\{2,3,5\}$ -group. In particular, G is non-solvable.

According to Step 1 in Proposition 3.3, we have K is a $\{2, 3, 5\}$ -group and G is non-solvable.

Step 2. The quotient $\frac{G}{K}$ is an almost simple group. In fact, $S \leq \frac{G}{K} \lesssim$ Aut(S), where S is a finite non-abelian simple group.

We can prove this by the similar argument in Step 2 in Proposition 3.3.

By TABLE 1 and Step 1, it is evident that $|S| = 2^{\alpha}.3^{\beta}.5^{\gamma}.7.13.17^2$, where $2 \leq \alpha \leq 26, 1 \leq \beta \leq 5$ and $0 \leq \gamma \leq 4$. Now, using collected results contained in TABLE 1, we conclude that $S \cong D_4(4)$ and by Step 2, $L \leq \frac{G}{K} \leq \operatorname{Aut}(L)$. As $|G| = |L:2^2| = 4|L|$, we deduce |K| = 1, 2 or 4.

If |K| = 1, then $G \cong L : 2^2$.

If |K| = 2, then $K \leq Z(G)$. In this case G is a central extension of \mathbb{Z}_2 by $L: 2_1, L: 2_2$ or $L: 2_3$. If G splits over K then $G \cong \mathbb{Z}_2 \times (L: 2_1), \mathbb{Z}_2 \times (L: 2_2)$ or $\mathbb{Z}_2 \times (L: 2_3)$, otherwise we get a contradiction because |K| must divide the Schure multiplier of $L: 2_1, L: 2_2$ and $L: 2_3$, which is impossible.

If |K| = 4, then $G/K \cong L$. In this case we have $G/C_G(K) \lesssim \operatorname{Aut}(K) \cong \mathbb{Z}_2$ or S_3 . Thus $|G/C_G(K)| = 1, 2, 3$ or 6. If $|G/C_G(K)| = 1$, then $K \leq Z(G)$, that is, G is a central extension of K by L. If G is a non-split extension of Kby L, then |K| must divide the Schur multiplier of L, which is 1, but this is a contradiction. Therefore G splits over K. Hence $G \cong K \times L$. So we have $G \cong \mathbb{Z}_4 \times L$ or $(\mathbb{Z}_2 \times \mathbb{Z}_2) \times L$ because $K \cong \mathbb{Z}_4$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$. If $|G/C_G(K)| = 2, 3$ or 6, then $K < C_G(K)$ and $1 \neq C_G(K)/K \leq G/K \cong L$. Which is a contradiction, since L is simple. \Box

Proposition 3.7. If $M = L : (D_6)_1$, then $G \cong L : (D_6)_1$, L : 6, $\mathbb{Z}_3 \times (L : 2_1)$, $\mathbb{Z}_3 \times (L : 2_3)$ or $(\mathbb{Z}_3 \times L) . \mathbb{Z}_2$.

Proof. As $|L:(D_6)_1| = 2^{25} \cdot 3^6 \cdot 5^4 \cdot 7 \cdot 13 \cdot 17^2$ and $\pi_e(L:(D_6)_1) = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 14, 15, 16, 17, 18, 20, 21, 24, 30, 34, 39, 42, 45, 51, 60, 63, 65, 85, 255\}$, then $D(L:(D_6)_1) = (4, 5, 4, 2, 2, 3)$. Since $|G| = |L:(D_6)_1|$ and $D(G) = D(L:(D_6)_1)$, we conclude that there exist several possibilities for $\Gamma(G)$:



where $\{a, b\} = \{7, 13\}.$

Step1. Let K be the maximal normal solvable subgroup of G. Then K is a $\{2, 3, 5\}$ -group. In particular, G is non-solvable.

By the similar argument to that in Step 1 in Proposition 3.1, we can obtain this assertion.

Step 2. The quotient $\frac{G}{K}$ is an almost simple group. In fact, $S \leq \frac{G}{K} \lesssim \text{Aut}(S)$, where S is a finite non-abelian simple group. The proof is similar to Step 2 in Proposition 3.3.

By TABLE 1 and Step 1, it is evident that $|S| = 2^{\alpha}.3^{\beta}.5^{\gamma}.7.13.17^2$, where $2 \leq \alpha \leq 25, 1 \leq \beta \leq 6$ and $0 \leq \gamma \leq 4$. Now, using collected results contained in TABLE 1, we conclude that $S \cong D_4(4)$ and by Step 2, $L \trianglelefteq \frac{G}{K} \lesssim \operatorname{Aut}(L)$. As $|G| = |L: D_6)_1| = 6|L|$, we deduce |K| = 1, 2, 3 or 6.

If |K| = 1, then $G \cong L : (D_6)_1, L : (D_6)_2$ or L : 6 because |G| = 6|L|. Obviously, $G \cong L : (D_6)_1$ or L : 6 because deg(2) = 5 in $\Gamma(L : (D_6)_2)$.

If |K| = 2, then $K \leq Z(G)$ and so deg(2) = 5, which is a contradiction (see page 18).

If |K| = 3, then $G/K \cong L : 2_1, L : 2_2$ or $L : 2_3$. But $G/C_G(K) \lesssim \operatorname{Aut}(K) \cong \mathbb{Z}_2$. Thus $|G/C_G(K)| = 1$ or 2. If $|G/C_G(K)| = 1$, then $K \leq Z(G)$, that is, G is a central extension of K by $L : 2_1, L : 2_2$ or $L : 2_3$. If G splits over K, then $G \cong \mathbb{Z}_3 \times (L : 2_1)$ or $\mathbb{Z}_3 \times (L : 2_3)$ because in $\Gamma(\mathbb{Z}_3 \times (L : 2_2))$ the degree of 2 is 5. Otherwise we get a contradiction because |K| must divide the Schure multiplier of $L : 2_1, L : 2_2$ and $L : 2_3$, which is impossible. If $|G/C_G(K)| = 2$, then $K < C_G(K)$ and $1 \neq C_G(K)/K \trianglelefteq G/K \cong L : 2_1, L : 2_2$ or $L : 2_3$, we obtain $C_G(K)/K \cong L$. Since $K \leq Z(C_G(K)), C_G(K)$ is a central extension of K by L. If $C_G(K)$ splits over K, then $C_G(K) \cong \mathbb{Z}_3 \times L$, otherwise we get a contradiction because |K| must divide the Schure is impossible. Therefore, $G \cong (\mathbb{Z}_3 \times L).\mathbb{Z}_2$.

If |K| = 6, then $G/K \cong L$ and $K \cong \mathbb{Z}_6$ or D_6 .

If $K \cong \mathbb{Z}_6$, then $G/C_G(K) \leq \mathbb{Z}_2$ and so $|G/C_G(K)| = 1$ or 2. If $|G/C_G(K)| = 1$, then $K \leq Z(G)$. It follows that deg(2) = 5, a contradiction. If $|G/C_G(K)| = 2$, then $K < C_G(K)$ and $1 \neq C_G(K)/K \leq G/K \cong L$, which is a contradiction because L is simple.

If $K \cong D_6$, then $K \cap C_G(K) = 1$ and $G/C_G(K) \leq D_6$. Thus $C_G(K) \neq 1$. Hence, $1 \neq C_G(K) \cong C_G(K)K/K \trianglelefteq G/K \cong L$. It follows that $L \cong G/K \cong C_G(K)$ because L is simple. Therefore, $G \cong D_6 \times L$, which implies that deg(2) = 5, a contradiction.

Proposition 3.8. If $M = L : (D_6)_2$, then $G \cong L : (D_6)_2$, $\mathbb{Z}_2 \times (L : 3)$, $\mathbb{Z}_3 \times (L : 2_2)$, $(\mathbb{Z}_3 \times L) \cdot \mathbb{Z}_2$, $\mathbb{Z}_6 \times L$ or $S_3 \times L$.

Proof. As $|L:(D_6)_2| = 2^{25} \cdot 3^6 \cdot 5^4 \cdot 7 \cdot 13 \cdot 17^2$ and $\pi_e(L:(D_6)_2) = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 14, 15, 17, 18, 20, 21, 24, 26, 30, 34, 39, 40, 42, 45, 51, 60, 63, 65, 68, 85, 102, 126, 130, 170, 255\}$, then $D(L:(D_6)_2) = (5, 5, 4, 2, 3, 3)$. Since $|G| = |L:(D_6)_2|$ and $D(G) = D(L:(D_6)_2)$, we conclude that $\Gamma(G)$ has the following form (like $\Gamma(L:(D_6)_2)$):



Step1. Let K be the maximal normal solvable subgroup of G. Then K is a $\{2,3\}$ -group. In particular, G is non-solvable.

The proof is similar to Step 1 in Proposition 3.5.

Step 2. The quotient $\frac{G}{K}$ is an almost simple group. In fact, $S \leq \frac{G}{K} \lesssim$ Aut(S), where S is a finite non-abelian simple group.

Let $\overline{G} = \frac{G}{K}$. Then $S := \operatorname{Soc}(\overline{G})$, $S = P_1 \times P_2 \times \ldots \times P_m$, where P_i 's are finite non-abelian simple groups and $S \leq \frac{G}{K} \leq \operatorname{Aut}(S)$. We are going to prove that m = 1 and $S = P_1$. Suppose that $m \geq 2$. By the same argument in Step 2 of Proposition 3.3 and considering 7 instead of a, we get a contradiction. Therefore m = 1 and $S = P_1$.

By TABLE 1 and Step 1, it is evident that $|S| = 2^{\alpha}.3^{\beta}.5^{4}.7.13.17^{2}$, where $2 \leq \alpha \leq 25$ and $1 \leq \beta \leq 6$. Now, using collected results contained in TABLE 1, we conclude that $S \cong D_{4}(4)$ and by Step 2, $L \trianglelefteq \frac{G}{K} \lesssim \operatorname{Aut}(L)$. As $|G| = |L : D_{6})_{2}| = 6|L|$, we deduce |K| = 1, 2, 3 or 6.

If |K| = 1, then $G \cong L : (D_6)_1, L : (D_6)_2$ or L : 6 because |G| = 6|L|. Obviously $G \cong L : (D_6)_2$ because in $\Gamma(L : (D_6)_1)$ and $\Gamma(L : 6)$, we have deg(13) = 2 (see page 17).

If |K| = 2, then $K \leq Z(G)$ and $G/K \cong L : 3$. Hence G is a central extension of K by L : 3. If G splits over K, then $G \cong \mathbb{Z}_2 \times (L : 3)$. Otherwise we get a contradiction because |K| must divide the Schure multiplier of L : 3, which is impossible.

If |K| = 3, then $G/K \cong L : 2_1, L : 2_2$ or $L : 2_3$. But $G/C_G(K) \lesssim \operatorname{Aut}(K) \cong \mathbb{Z}_2$. Thus $|G/C_G(K)| = 1$ or 2. If $|G/C_G(K)| = 1$, then $K \leq Z(G)$, that is, G is a central extension of K by $L : 2_1, L : 2_2$ or $L : 2_3$. If G splits over K, then only $G \cong \mathbb{Z}_3 \times (L : 2_2)$ because $2 \not\approx 13$ in $\Gamma(\mathbb{Z}_3 \times (L : 2_1))$ and $\Gamma(\mathbb{Z}_3 \times (L : 2_3))$. Otherwise we get a contradiction because |K| must divide the Schure multiplier of $L : 2_1, L : 2_2$ and $L : 2_3$, which is impossible. If

 $|G/C_G(K)| = 2$, then $K < C_G(K)$ and $1 \neq C_G(K)/K \leq G/K \cong L : 2_1, L : 2_2$ or $L : 2_3$, we obtain $C_G(K)/K \cong L$. Since $K \leq Z(C_G(K)), C_G(K)$ is a central extension of K by L. If $C_G(K)$ splits over K, then $C_G(K) \cong \mathbb{Z}_3 \times L$, otherwise we get a contradiction because |K| must divide the Schure multiplier of L, which is impossible. Therefore, $G \cong (\mathbb{Z}_3 \times L).\mathbb{Z}_2$.

If |K| = 6, then $G/K \cong L$ and $K \cong \mathbb{Z}_6$ or D_6 . If $K \cong \mathbb{Z}_6$, then $G/C_G(K) \lesssim \mathbb{Z}_2$ and so $|G/C_G(K)| = 1$ or 2. If $|G/C_G(K)| = 1$, then $K \leq Z(G)$ and $G/K \cong L$. Therefore G is a central extension of K by L. If G is a non-split extension of K by L, then |K| must divide the Schure multiplier of L, which is 1. But this is a contradiction. So we obtain that G splits over K. Hence $G \cong \mathbb{Z}_6 \times L$. If $|G/C_G(K)| = 2$, then $K < C_G(K)$ and $1 \neq C_G(K)/K \trianglelefteq G/K \cong L$, which is a contradiction because L is simple. If $K \cong D_6$, then $K \cap C_G(K) = 1$ and $G/C_G(K) \lesssim D_6$. Thus $C_G(K) \neq 1$. Hence, $1 \neq C_G(K) \cong C_G(K)K/K \trianglelefteq G/K \cong L$. It follows that $L \cong G/K \cong C_G(K)$ because L is simple. Therefore $G \cong D_6 \times L$.

Proposition 3.9. If M = L : 6, then $G \cong L : 6$, $L : (D_6)_1$, $\mathbb{Z}_3 \times (L : 2_1)$, $\mathbb{Z}_3 \times (L : 2_3)$ or $(\mathbb{Z}_3 \times L) \cdot \mathbb{Z}_2$.

Proof. As $|L:6| = 2^{25} \cdot 3^6 \cdot 5^4 \cdot 7 \cdot 13 \cdot 17^2$ and $\pi_e(L:6) = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 14, 15, 16, 17, 18, 20, 21, 24, 30, 34, 36, 39, 42, 45, 48, 51, 63, 65, 85, 255\}$, then D(L:6) = (4, 5, 4, 2, 2, 3). Since |G| = |L:6| and D(G) = D(L:6), there exist several possibilities for $\Gamma(G)$ similarly to Proposition 3.7:



where $\{a, b\} = \{7, 13\}.$

Step1. Let K be the maximal normal solvable subgroup of G. Then K is a $\{2, 3, 5\}$ -group. In particular, G is non-solvable.

The proof is similar to that in Proposition 3.3.

Step 2. The quotient $\frac{G}{K}$ is an almost simple group. In fact, $S \leq \frac{G}{K} \lesssim \text{Aut}(S)$, where S is a finite non-abelian simple group.

Again we refer to Step 2 of proposition 3.3 to get the proof.

By TABLE 1 and Step 1, it is evident that $|S| = 2^{\alpha}.3^{\beta}.5^{\gamma}.7.13.17^2$, where $2 \le \alpha \le 25, 1 \le \beta \le 6$ and $0 \le \gamma \le 4$. Now, using collected results contained

in TABLE 1, we conclude that $S \cong D_4(4)$ and by Step 2, $L \trianglelefteq \frac{G}{K} \lesssim \operatorname{Aut}(L)$. As |G| = |L:6| = 6|L|, we deduce |K| = 1, 2, 3 or 6.

If |K| = 1, then $G \cong L : 6$, $L : (D_6)_1$ or $L : (D_6)_2$ because |G| = 6|L|. Obviously, $G \cong L : 6$ or $L : (D_6)_1$ because deg(2) = 5 in $\Gamma(L : (D_6)_2)$ (see page 18).

If |K| = 2, then $K \leq Z(G)$ and so deg(2) = 5, which is a contradiction.

If |K| = 3, then $G/K \cong L : 2_1, L : 2_2$ or $L : 2_3$. But $G/C_G(K) \lesssim \operatorname{Aut}(K) \cong \mathbb{Z}_2$. Thus $|G/C_G(K)| = 1$ or 2. If $|G/C_G(K)| = 1$, then $K \leq Z(G)$, that is, G is a central extension of K by $L : 2_1, L : 2_2$ or $L : 2_3$. If G splits over K, then $G \cong \mathbb{Z}_3 \times (L : 2_1)$ or $\mathbb{Z}_3 \times (L : 2_3)$ because in $\Gamma(\mathbb{Z}_3 \times (L : 2_2))$ the degree of 2 is 5. Otherwise we get a contradiction because |K| must divide the Schure multiplier of $L : 2_1, L : 2_2$ and $L : 2_3$, which is impossible. If $|G/C_G(K)| = 2$, then $K < C_G(K)$ and $1 \neq C_G(K)/K \leq G/K \cong L : 2_1, L : 2_2$ or $L : 2_3$, we obtain $C_G(K)/K \cong L$. Since $K \leq Z(C_G(K)), C_G(K)$ is a central extension of K by L. If $C_G(K)$ splits over K, then $C_G(K) \cong \mathbb{Z}_3 \times L$, otherwise we get a contradiction because |K| must divide the Schure multiplier of L, which is impossible. Therefore, $G \cong (\mathbb{Z}_3 \times L).\mathbb{Z}_2$.

If |K| = 6, then $G/K \cong L$ and $K \cong \mathbb{Z}_6$ or D_6 . If $K \cong \mathbb{Z}_6$, then $G/C_G(K) \lesssim \mathbb{Z}_2$ and so $|G/C_G(K)| = 1$ or 2. If $|G/C_G(K)| = 1$, then $K \leq Z(G)$. It follows that deg(2) = 5, a contradiction. If $|G/C_G(K)| = 2$, then $K < C_G(K)$ and $1 \neq C_G(K)/K \leq G/K \cong L$, which is a contradiction because L is simple. If $K \cong D_6$, then $K \cap C_G(K) = 1$ and $G/C_G(K) \lesssim D_6$. Thus $C_G(K) \neq 1$. Hence, $1 \neq C_G(K) \cong C_G(K)K/K \leq G/K \cong L$. It follows that $L \cong G/K \cong C_G(K)$ because L is simple. Therefore, $G \cong D_6 \times L$, which implies that deg(2) = 5, a contradiction.

Proposition 3.10. If $M = L : D_{12}$, then $G \cong L : D_{12}, \mathbb{Z}_2 \times (L : (D_6)_1)$, $\mathbb{Z}_2 \times (L : (D_6)_2), \mathbb{Z}_2 \times (L : 6), \mathbb{Z}_3 \times (L : 2^2), (\mathbb{Z}_3 \times (L : 2_1)).\mathbb{Z}_2, (\mathbb{Z}_3 \times (L : 2_2)).\mathbb{Z}_2, (\mathbb{Z}_3 \times (L : 2_3)).\mathbb{Z}_2, \mathbb{Z}_4 \times (L : 3), (\mathbb{Z}_2 \times \mathbb{Z}_2) \times (L : 3), (\mathbb{Z}_4 \times L).\mathbb{Z}_3, ((\mathbb{Z}_2 \times \mathbb{Z}_2) \times L).\mathbb{Z}_3, \mathbb{Z}_6 \times (L : 2_1), \mathbb{Z}_6 \times (L : 2_2), \mathbb{Z}_6 \times (L : 2_3), (\mathbb{Z}_6 \times L).\mathbb{Z}_2, \mathbb{Z}_3 \times (L : 2_1), S_3 \times (L : 2_2), S_3 \times (L : 2_3), \mathbb{Z}_{12} \times L, (\mathbb{Z}_2 \times \mathbb{Z}_6) \times L, D_{12} \times L, (\mathbb{Z}_2 \times L).D_6, \mathbb{A}_4 \times L, L.\mathbb{A}_4 \text{ or } T \times L.$

Proof. As $|L:D_{12}| = 2^{26} \cdot 3^6 \cdot 5^4 \cdot 7 \cdot 13 \cdot 17^2$ and $\pi_e(L:(D_{12})) = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 14, 15, 16, 17, 18, 20, 21, 24, 26, 30, 34, 39, 40, 42, 45, 48, 51, 60, 63, 65, 68, 85, 102, 126, 130, 170, 255\}$, then $D(L:D_{12}) = (5, 5, 4, 2, 3, 3)$. Since $|G| = |L:D_{12}|$ and $D(G) = D(L:D_{12})$, we conclude that $\Gamma(G)$ has the following form (like $\Gamma(L:D_{12})$):



Step1. Let K be the maximal normal solvable subgroup of G. Then K is a $\{2,3\}$ -group. In particular, G is non-solvable.

The proof is similar to Step 1 in Proposition 3.5.

Step 2. The quotient $\frac{G}{K}$ is an almost simple group. In fact, $S \leq \frac{G}{K} \lesssim \text{Aut}(S)$, where S is a finite non-abelian simple group.

To get the proof, follow the way in the proof of Step 2 in proposition 3.5.

By TABLE 1 and Step 1, it is evident that $|S| = 2^{\alpha}.3^{\beta}.5^{4}.7.13.17^{2}$, where $2 \leq \alpha \leq 26$ and $1 \leq \beta \leq 6$. Now, using collected results contained in TABLE 1, we conclude that $S \cong D_{4}(4)$ and by Step 2, $L \trianglelefteq \frac{G}{K} \lesssim \operatorname{Aut}(L)$. As $|G| = |L : D_{12}| = 12|L|$, we deduce |K| = 1, 2, 3, 4, 6 or 12.

If |K| = 1, then $G \cong L : D_{12}$.

If |K| = 2, then $G/K \cong L : (D_6)_1, L : (D_6)_2$ or L : 6 and $K \leq Z(G)$. It follows that G is a central extension of K by $L : (D_6)_1, L : (D_6)_2$ or L : 6. If G splits over K, then $G \cong \mathbb{Z}_2 \times (L : (D_6)_1), \mathbb{Z}_2 \times (L : (D_6)_2)$ or $\mathbb{Z}_2 \times (L : 6)$. Otherwise $G \cong \mathbb{Z}_2.(L : (D_6)_1)$ or $\mathbb{Z}_2.(L : (D_6)_2)$.

If |K| = 3, then $G/K \cong L : 2^2$. But $G/C_G(K) \lesssim \operatorname{Aut}(K) \cong \mathbb{Z}_2$. Thus $|G/C_G(K)| = 1$ or 2. If $|G/C_G(K)| = 1$, then $K \leq Z(G)$, that is, G is a central extension of K by $L : 2^2$. If G splits over K, then $G \cong \mathbb{Z}_3 \times (L : 2^2)$, Otherwise we get a contradiction because |K| must divide the Schure multiplier of $L : 2^2$, which is impossible. If $|G/C_G(K)| = 2$, then $K < C_G(K)$ and $1 \neq C_G(K)/K \leq G/K \cong L : 2^2$, and we obtain $C_G(K)/K \cong L : 2_1, L : 2_2$ or $L : 2_3$. Since $K \leq Z(C_G(K)), C_G(K)$ is a central extension of K by $L : 2_1, L : 2_2$ or $L : 2_3$. Thus $C_G(K) \cong \mathbb{Z}_3 \times (L : 2_1), \mathbb{Z}_3 \times (L : 2_2)$ or $\mathbb{Z}_3 \times (L : 2_3)$, otherwise we get a contradiction because 3 must divide the Schure multiplier of $L : 2_1, L : 2_2$ or $L : 2_3$, which is impossible. Therefore, $G \cong (\mathbb{Z}_3 \times (L : 2_1)).\mathbb{Z}_2, (\mathbb{Z}_3 \times (L : 2_2)).\mathbb{Z}_2$ or $(\mathbb{Z}_3 \times (L : 2_3)).\mathbb{Z}_2$.

If |K| = 4, then $G/K \cong L$: 3 and $K \cong \mathbb{Z}_4$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$. In this case we have $G/C_G(K) \leq \operatorname{Aut}(K) \cong \mathbb{Z}_2$ or S_3 . Thus $|G/C_G(K)| = 1, 2, 3$ or 6. If $|G/C_G(K)| = 1$, then $K \leq Z(G)$, that is, G is a central extension of K by L: 3. If G split over K by L: 3, then $G \cong \mathbb{Z}_4 \times (L:3)$ or $(\mathbb{Z}_2 \times \mathbb{Z}_2) \times (L:3)$. Otherwise we get a contradiction because |K| must divide the Schure multiplier of L: 3, which is impossible. If $|G/C_G(K)| \neq 1$, since $|G/C_G(K)| = 2, 3$ or 6, it follows that $K < C_G(K)$. As L is simple, we conclude that $1 \neq C_G(K)/K$ must

be an extension of L. Hence $|G/C_G(K)| = 3$ and therefore $C_G(K)/K \cong L$. Now, since $K \leq Z(C_G(K))$, we conclude that $C_G(K)$ is a central extension of K by L. Thus $C_G(K) \cong \mathbb{Z}_4 \times L$, or $(\mathbb{Z}_2 \times \mathbb{Z}_2) \times L$, otherwise |K| must divide the Schure multiplier of L, which is 1 and it is impossible. Therefore, $G \cong (\mathbb{Z}_4 \times L).\mathbb{Z}_3$ or $((\mathbb{Z}_2 \times \mathbb{Z}_2) \times L).\mathbb{Z}_3$.

If |K| = 6, then $G/K \cong L : 2_1, L : 2_2$ or $L : 2_3$ and $K \cong \mathbb{Z}_6$ or D_6 . If $K \cong \mathbb{Z}_6$, then $G/C_G(K) \lesssim \mathbb{Z}_2$ and so $|G/C_G(K)| = 1$ or 2. If $|G/C_G(K)| = 1$, then $K \leq Z(G)$, that is G is a central extension of \mathbb{Z}_6 by $L: 2_1, L: 2_2$ or $L: 2_3$. If G splits over K, we obtain $G \cong \mathbb{Z}_6 \times (L: 2_1), \mathbb{Z}_6 \times (L: 2_2)$ or $\mathbb{Z}_6 \times (L : 2_3)$, otherwise we get a contradiction because |K| must divide the Schure multiplier of $L : 2_1, L : 2_2$ or $L : 2_3$, which is impossible. If $|G/C_G(K)| = 2$, then $K < C_G(K)$ and $1 \neq C_G(K)/K \leq G/K \approx L : 2_1$, $L: 2_2$ or $L: 2_3$, and we obtain $C_G(K)/K \cong L$. Since $K \leq Z(C_G(K))$, $C_G(K)$ is a central extension of K by L. Thus $C_G(K) \cong \mathbb{Z}_6 \times L$, otherwise we get a contradiction because |K| must divide the Schure multiplier of L. Therefore $G \cong (\mathbb{Z}_6 \times L) \mathbb{Z}_2$. If $K \cong D_6$, then $G/C_G(K) \lesssim D_6$ and so $|G/C_G(K)| = 1, 2, 3$ or 6. If $|G/C_G(K)| = 1$, then $K \leq Z(G)$, that is a contradiction. If $|G/C_G(K)| = 2$, then we have $|KC_G(K)| = 6 |G|/2 = 3|G|$ because $K \cap C_G(K) = 1$, which is a contradiction. If $|G/C_G(K)| = 3$, then we have $|KC_G(K)| = 6 |G|/3 = 2|G|$ because $K \cap C_G(K) = 1$, which is a contradiction. If $|G/C_G(K)| = 6$, then $G/C_G(K) \cong D_6$ and $C_G(K) \neq 1$. Hence, $1 \neq C_G(K) \cong C_G(K)K/K \trianglelefteq G/K \cong L : 2_1, L : 2_2$ or $L : 2_3$. It follows that $C_G(K) \cong L : 2_1, L : 2_2$ or $L : 2_3$ because L is simple. Therefore, $G \cong D_6 \times (L:2_1), D_6 \times (L:2_2) \text{ or } D_6 \times (L:2_3).$

Before processing the last case, we recall the following facts.

There exist five non-isomorphic groups of order 12. Two of them are abelian and three are non-abelian. The non-abelian groups are: alternating group A_4 , dihedral group D_{12} and the dicyclic group T with generators a and b, subject to the relations $a^6 = 1$, $a^3 = b^2$ and $b^{-1}ab = a^{-1}$.

If |K| = 12, then $G/K \cong L$ and $K \cong \mathbb{Z}_{12}$, $\mathbb{Z}_2 \times \mathbb{Z}_6$, D_{12} , \mathbb{A}_4 or T. But $C_G(K)K/K \trianglelefteq G/K \cong L$. If $C_G(K)K/K = 1$, then $C_G(K) \le K$ and hence $|L| = |G/K|||G/C_G(K)|||\operatorname{Aut}(K)|$. Thus $|L||\operatorname{Aut}(K)|$, a contradiction. Therefore, $C_G(K)K/K \neq 1$ and since L is simple group, we conclude that $G = C_G(K)K$ and hence, $G/C_G(K) \cong K/Z(K)$. Now, we should consider the following cases:

If $K \cong \mathbb{Z}_{12}$ or $\mathbb{Z}_2 \times \mathbb{Z}_6$, then $G/C_G(K) = 1$. Therefore $K \leq Z(G)$, that is G is a central extension of \mathbb{Z}_{12} or $\mathbb{Z}_2 \times \mathbb{Z}_6$ by L. If G splits over K, we obtain $G \cong \mathbb{Z}_{12} \times L$ or $(\mathbb{Z}_2 \times \mathbb{Z}_6) \times L$, otherwise we get a contradiction because |K| must divide the Schure multiplier of L, which is 1 and it is impossible.

If $K \cong D_{12}$, then G = K.L and $G/C_G(K) \cong D_6$. Since $C_G(K)/Z(K) \cong G/K \cong L$ and $Z(K) \leq Z(C_G(K))$, we conclude that $C_G(K)$ is a central extension of $Z(K) \cong \mathbb{Z}_2$ by L. If $C_G(K)$ is a non-split extension, then 2 must divide the Schure multiplier of L, which is 1 and it is impossible. Thus $C_G(K) \cong \mathbb{Z}_2 \times L$ and hence, G is a split extension of K by L. Now, since $Hom(L, Aut(D_{12}))$ is trivial, we have $G \cong D_{12} \times L$.

If $K \cong \mathbb{A}_4$, then $G/C_G(K) \cong \mathbb{A}_4$. As $G = C_G(K)K$, It follows that $C_G(K) \cong L$. Therefore $G \cong L \times \mathbb{A}_4$ or $L.\mathbb{A}_4$.

If $K \cong T$, then By the similar way in case $K \cong D_{12}$, we can conclude that G is a split extension of K by L. Also, since Hom(L, Aut(T)) is trivial, we have $G \cong T \times L$.

According to what we said before the proof, here we depict $\Gamma(M)$ by |M|and $\pi_e(M)$, where M is an almost simple group related to $L = D_4(4)$.



















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