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# OD-characterization of Almost Simple Groups Related to $D_4(4)$

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ABSTRACT. Let G be a finite group and  $\pi_e(G)$  be the set of orders of all elements in G. The set  $\pi_e(G)$  determines the prime graph (or Grunberg-Kegel graph)  $\Gamma(G)$  whose vertex set is  $\pi(G)$ . The set of primes dividing the order of G, and two vertices p and q are adjacent if and only if  $pq \in \pi_e(G)$ . The degree deg(p) of a vertex  $p \in \pi(G)$ , is the number of edges incident on p. Let  $\pi(G) = \{p_1, p_2, ..., p_k\}$  with  $p_1 < p_2 < ... < p_k$ . We define  $D(G) := (deg(p_1), deg(p_2), ..., deg(p_k))$ , which is called the degree pattern of G. The group G is called k-fold OD-characterizable if there exist exactly k non-isomorphic groups M satisfying conditions |G| = |M| and D(G) = D(M). Usually a 1-fold OD-characterizable group is simply called OD-characterizable. In this paper, we classify all finite groups with the same order and degree pattern as an almost simple groups related to  $D_4(4)$ .

**Keywords:** Degree pattern, k-fold OD-characterizable, Almost simple group.

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24 1

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#### 1. Introduction

Let G be a finite group,  $\pi(G)$  the set of all prime divisors of |G| and  $\pi_e(G)$  be the set of orders of elements in G. The prime graph (or Grunberg-Kegel graph)  $\Gamma(G)$  of G is a simple graph with vertex set  $\pi(G)$  in which two vertices p and q are joined by an edge ( and we write  $p \sim q$ ) if and only if G contains an element of order pq (i.e.  $pq \in \pi_e(G)$ ).

The degree  $\deg(p)$  of a vertex  $p \in \pi(G)$  is the number of edges incident on p. If  $\pi(G) = \{p_1, p_2, ..., p_k\}$  with  $p_1 < p_2 < ... < p_k$ , then we define  $D(G) := (\deg(p_1), \deg(p_2), ..., \deg(p_k))$ , which is called the degree pattern of G, and leads a following definition.

**Definition 1.1.** The finite group G is called k-fold OD-characterizable if there exist exactly k non-isomorphic groups H satisfying conditions |G| = |H| and D(G) = D(H). In particular, a 1-fold OD-characterizable group is simply called OD-characterizable.

The interest in characterizing finite groups by their degree patterns started in [7] by M. R. Darafsheh and et. all, in which the authors proved that the following simple groups are uniquely determined by their order and degree patterns: All sporadic simple groups, the alternating groups  $A_p$  with p and p-2 primes and some simple groups of Lie type. Also in a series of articles (see [4, 6, 8, 9, 14, 17]), it was shown that many finite simple groups are OD-characterizable.

Let A and B be two groups then a split extension is denoted by A:B. If L is a finite simple group and  $Aut(L) \cong L:A$ , then if B is a cyclic subgroup of A of order n we will write L:n for the split extension L:B. Moreover if there are more than one subgroup of orders n in A, then we will denote them by  $L:n_1, L:n_2$ , etc.

**Definition 1.2.** A group G is said to be an almost simple group related to S if and only if  $S \leq G \lesssim \operatorname{Aut}(S)$ , for some non-abelian simple group S.

In many papers (see [2, 3, 10, 13, 15, 16]), it has been proved, up to now, that many finite almost simple groups are OD-characterizable or k-fold OD-characterizable for certain k > 2.

We denote the socle of G by Soc(G), which is the subgroup generated by the set of all minimal normal subgroups of G. For  $p \in \pi(G)$ , we denote by  $G_p$  and  $Syl_p(G)$  a Sylow p-subgroup of G and the set of all Sylow p-subgroups of G respectively, all further unexplained notation are standard and can be found in [11].

In this article our main aim is to show the recognizability of the almost simple groups related to  $L := D_4(4)$  by degree pattern in the prime graph and

order of the group. In fact, we will prove the following Theorem.

**Main Theorem** Let M be an almost simple group related to  $L := D_4(4)$ . If G is a finite group such that D(G) = D(M) and |G| = |M|, then the following assertions hold:

- (a) If M = L, then  $G \cong L$ .
- (b) If  $M = L : 2_1$ , then  $G \cong L : 2_1$  or  $L : 2_3$ .
- (c) If  $M = L : 2_2$ , then  $G \cong L : 2_2$  or  $\mathbb{Z}_2 \times L$ .
- (d) If  $M = L : 2_3$ , then  $G \cong L : 2_3$  or  $L : 2_1$ .
- (e) If M = L : 3, then  $G \cong L : 3$  or  $\mathbb{Z}_3 \times L$ .
- (f) If  $M = L : 2^2$ , then  $G \cong L : 2^2$ ,  $\mathbb{Z}_2 \times (L : 2_1)$ ,  $\mathbb{Z}_2 \times (L : 2_2)$ ,  $\mathbb{Z}_2 \times (L : 2_3)$ ,  $\mathbb{Z}_4 \times L$  or  $(\mathbb{Z}_2 \times \mathbb{Z}_2) \times L$ .
- (g) If  $M = L : (D_6)_1$ , then  $G \cong L : (D_6)_1$ , L : 6,  $\mathbb{Z}_3 \times (L : 2_1)$ ,  $\mathbb{Z}_3 \times (L : 2_3)$  or  $(\mathbb{Z}_3 \times L) \cdot \mathbb{Z}_2$ .
- (h) If  $M = L : (D_6)_2$ , then  $G \cong L : (D_6)_2$ ,  $\mathbb{Z}_2 \times (L : 3)$ ,  $\mathbb{Z}_3 \times (L : 2_2)$ ,  $(\mathbb{Z}_3 \times L) \cdot \mathbb{Z}_2$ ,  $\mathbb{Z}_6 \times L$  or  $D_6 \times L$ .
- (i) If M = L : 6, then  $G \cong L : 6$ ,  $L : (D_6)_1$ ,  $\mathbb{Z}_3 \times (L : 2_1)$ ,  $\mathbb{Z}_3 \times (L : 2_3)$  or  $(\mathbb{Z}_3 \times L) \cdot \mathbb{Z}_2$ .
- (j) If  $M = L : D_{12}$ , then  $G \cong L : D_{12}$ ,  $\mathbb{Z}_2 \times (L : (D_6)_1)$ ,  $\mathbb{Z}_2 \times (L : (D_6)_2)$ ,  $\mathbb{Z}_2 \times (L : 6)$ ,  $\mathbb{Z}_3 \times (L : 2^2)$ ,  $(\mathbb{Z}_3 \times (L : 2_1)) \cdot \mathbb{Z}_2$ ,  $(\mathbb{Z}_3 \times (L : 2_2)) \cdot \mathbb{Z}_2$ ,  $(\mathbb{Z}_3 \times (L : 2_3)) \cdot \mathbb{Z}_2$ ,  $\mathbb{Z}_4 \times (L : 3)$ ,  $(\mathbb{Z}_2 \times \mathbb{Z}_2) \times (L : 3)$ ,  $(\mathbb{Z}_4 \times L) \cdot \mathbb{Z}_3$ ,  $((\mathbb{Z}_2 \times \mathbb{Z}_2) \times L) \cdot \mathbb{Z}_3$ ,  $\mathbb{Z}_6 \times (L : 2_1)$ ,  $\mathbb{Z}_6 \times (L : 2_2)$ ,  $\mathbb{Z}_6 \times (L : 2_3)$ ,  $(\mathbb{Z}_6 \times L) \cdot \mathbb{Z}_2$ ,  $D_6 \times (L : 2_1)$ ,  $D_6 \times (L : 2_2)$ ,  $D_6 \times (L : 2_3)$ ,  $\mathbb{Z}_{12} \times L$ ,  $(\mathbb{Z}_2 \times \mathbb{Z}_6) \times L$ ,  $(\mathbb{Z}_2 \times L) \cdot D_6$ ,  $\mathbb{A}_4 \times L$ ,  $L \cdot \mathbb{A}_4$ ,  $D_{12} \times L$  or  $T \times L$ .

### 2. Preliminary Results

It is well-known that  $\operatorname{Aut}(D_4(4)) \cong D_4(4): D_{12}$  where  $D_{12}$  denotes the dihedral group of order 12. We remark that  $D_{12}$  has the following non-trivial proper subgroups up to conjugacy: three subgroups of order 2, one cyclic subgroup each of order 3 and 6, two subgroups isomorphic to  $D_6 \cong \mathbb{S}_3$  and one subgroup of order 4 isomorphic to the Klein's four group denoted by  $2^2$ . The field and the duality automorphisms of  $D_4(4)$  are denoted by  $2_1$  and  $2_2$  respectively, and we set  $2_3 = 2_1.2_2$  (field\*duality which is called the diagonal automorphism). Therefore up to conjugacy we have the following almost simple groups related to  $D_4(4)$ .

**Lemma 2.1.** If G is an almost simple group related to  $L:=D_4(4)$ , then G is isomorphic to one of the following groups:  $L, L: 2_1, L: 2_2, L: 2_3, L: 3, L: 2^2, L: (D_6)_1, L: (D_6)_2, L: 6, L: D_{12}$ .

**Lemma 2.2** ([5]). Let G be a Frobenius group with kernel K and complement H. Then:

- (a) K is a nilpotent group.
- (b)  $|K| \equiv 1 \pmod{|H|}$ .

Let  $p \geq 5$  be a prime. We denote by  $\mathfrak{S}_p$  the set of all simple groups with prime divisors at most p. Clearly, if  $q \leq p$ , then  $\mathfrak{S}_q \subseteq \mathfrak{S}_p$ . We list all the simple groups in class  $\mathfrak{S}_{17}$  with their order and the order of their outer automorphisms in TABLE 1, taken from [12].

TABLE 1: Simple groups in  $\mathfrak{S}_p$ ,  $p \leq 17$ .

S	S	$ \mathrm{Out}(S) $	S	S	$ \mathrm{Out}(S) $
$A_5$	$2^2 \cdot 3 \cdot 5$	2	$G_2(3)$	$2^6 \cdot 3^6 \cdot 7 \cdot 13$	2
$A_6$	$2^3 \cdot 3^2 \cdot 5$	4	$^{3}D_{4}(2)$	$2^{12} \cdot 3^4 \cdot 7^2 \cdot 13$	3
$S_4(3)$	$2^6 \cdot 3^4 \cdot 5$	2	$L_2(64)$	$2^6 \cdot 3^2 \cdot 5 \cdot 7 \cdot 13$	6
$L_2(7)$	$2^3 \cdot 3 \cdot 7$	2	$U_4(5)$	$2^7 \cdot 3^4 \cdot 5^6 \cdot 7 \cdot 13$	4
$L_2(8)$	$2^3 \cdot 3^2 \cdot 7$	3	$L_{3}(9)$	$2^7 \cdot 3^6 \cdot 5 \cdot 7 \cdot 13$	4
$U_3(3)$	$2^5 \cdot 3^3 \cdot 7$	2	$S_6(3)$	$2^9 \cdot 3^9 \cdot 5 \cdot 7 \cdot 13$	2
$A_7$	$2^3 \cdot 3^2 \cdot 5 \cdot 7$	2	$O_7(3)$	$2^9 \cdot 3^9 \cdot 5 \cdot 7 \cdot 13$	2
$L_2(49)$	$2^4 \cdot 3 \cdot 5^2 \cdot 7^2$	4	$G_2(4)$	$2^{12}\cdot 3^3\cdot 5^2\cdot 7\cdot 13$	2
$U_3(5)$	$2^4 \cdot 3^2 \cdot 5^3 \cdot 7$	6	$S_4(8)$	$2^{12}\cdot 3^4\cdot 5\cdot 7^2\cdot 13$	6
$L_3(4)$	$2^6 \cdot 3^2 \cdot 5 \cdot 7$	12	$O_8^+(3)$	$2^{12} \cdot 3^{12} \cdot 5^2 \cdot 7 \cdot 13$	24
A <sub>8</sub>	$2^6 \cdot 3^2 \cdot 5 \cdot 7$	2	$L_5(3)$	$2^9 \cdot 3^{10} \cdot 5 \cdot 11^2 \cdot 13$	2
$A_9$	$2^6 \cdot 3^4 \cdot 5 \cdot 7$	2	$A_{13}$	$2^9 \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$	2
$J_2$	$2^7 \cdot 3^3 \cdot 5^2 \cdot 7$	2	$A_{14}$	$2^{10} \cdot 3^5 \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 13$	2
$A_{10}$	$2^7 \cdot 3^4 \cdot 5^2 \cdot 7$	2	$A_{15}$	$2^{10} \cdot 3^6 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13$	2
$U_4(3)$	$2^7 \cdot 3^6 \cdot 5 \cdot 7$	8	$L_6(3)$	$2^{11} \cdot 3^{15} \cdot 5 \cdot 7 \cdot 11^2 \cdot 13^2$	4
$S_4(7)$	$2^8 \cdot 3^2 \cdot 5^2 \cdot 7^4$	2	Suz	$2^{13} \cdot 3^7 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$	2
$S_6(2)$	$2^9 \cdot 3^4 \cdot 5 \cdot 7$	1	$A_{16}$	$2^{14} \cdot 3^6 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13$	2
$O_8^+(2)$	$2^{12}\cdot 3^5\cdot 5^2\cdot 7$	6	$Fi_{22}$	$2^{17} \cdot 3^9 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$	2
$L_2(11)$	$2^2 \cdot 3 \cdot 5 \cdot 11$	2	$L_2(17)$	$2^4 \cdot 3^2 \cdot 17$	2
$M_{11}$	$2^4 \cdot 3^2 \cdot 5 \cdot 11$	1	$L_2(16)$	$2^4 \cdot 3 \cdot 5 \cdot 17$	4
$M_{12}$	$2^6 \cdot 3^3 \cdot 5 \cdot 11$	2	$S_4(4)$	$2^8 \cdot 3^2 \cdot 5^2 \cdot 17$	4
$U_5(2)$	$2^{10}\cdot 3^5\cdot 5\cdot 11$	2	He	$2^{10} \cdot 3^3 \cdot 5^2 \cdot 7^3 \cdot 17$	2
$M_{22}$	$2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$	2	$O_8^-(2)$	$2^{12} \cdot 3^4 \cdot 5 \cdot 7 \cdot 17$	2
$A_{11}$	$2^7 \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 11$	2	$L_4(4)$	$2^{12} \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 17$	4
$M^cL$	$2^7 \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11$	2	$S_8(2)$	$2^{16} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 17$	1
HS	$2^9 \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 11$	2	$U_4(4)$	$2^{12} \cdot 3^2 \cdot 5^3 \cdot 13 \cdot 17$	4
$A_{12}$	$2^9 \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 11$	2	$U_3(17)$	$2^6 \cdot 3^4 \cdot 7 \cdot 13 \cdot 17^3$	6
$U_6(2)$	$2^{15} \cdot 3^6 \cdot 5 \cdot 7 \cdot 11$	6	$O_{10}^{-}(2)$	$2^{20} \cdot 3^6 \cdot 5^2 \cdot 7 \cdot 11 \cdot 17$	2
$L_3(3)$	$2^4 \cdot 3^3 \cdot 13$	2	$L_2(13^2)$	$2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 13^2 \cdot 17$	4
$L_2(25)$	$2^3 \cdot 3 \cdot 5^2 \cdot 13$	4	$S_4(13)$	$2^6 \cdot 3^2 \cdot 5 \cdot 7^2 \cdot 13^4 \cdot 17$	2
$U_3(4)$	$2^6 \cdot 3 \cdot 5^2 \cdot 13$	4	$L_3(16)$	$2^{12} \cdot 3^2 \cdot 5^2 \cdot 7 \cdot 13 \cdot 17$	24
$S_4(5)$	$2^6 \cdot 3^2 \cdot 5^4 \cdot 13$	2	$S_6(4)$	$2^{18} \cdot 3^4 \cdot 5^3 \cdot 7 \cdot 13 \cdot 17$	2
$L_4(3)$	$2^7 \cdot 3^6 \cdot 5 \cdot 13$	4	$O_8^+(4)$	$2^{24} \cdot 3^5 \cdot 5^4 \cdot 7 \cdot 13 \cdot 17^2$	12
$^{2}F_{4}(2)'$	$2^{11}\cdot 3^3\cdot 5^2\cdot 13$	2	$F_4(2)$	$2^{24} \cdot 3^6 \cdot 5^2 \cdot 7^2 \cdot 13 \cdot 17$	2
$L_2(13)$	$2^2 \cdot 3 \cdot 7 \cdot 13$	2	$A_{17}$	$2^{14} \cdot 3^6 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17$	2
$L_2(27)$	$2^2 \cdot 3^3 \cdot 7 \cdot 13$	6	$A_{18}$	$2^{15} \cdot 3^8 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17$	2

**Definition 2.3.** A completely reducible group will be called a CR-group. The center of a CR-group is a direct product of the abelian factor in the decomposition. Hence, a CR-group is centerless, that is, has trivial center, if and only if it is a direct product of non-abelian simple groups. The following Lemma determines the structure of the automorphism group of a centerless CR-group.

**Lemma 2.3** ([11]). Let R be a finite centerless CR-group and write  $R = R_1 \times R_2 \times ... \times R_k$ , where  $R_i$  is a direct product of  $n_i$  isomorphic copies of a simple group  $H_i$ , and  $H_i$  and  $H_j$  are not isomorphic if  $i \neq j$ . Then  $\operatorname{Aut}(R) = \operatorname{Aut}(R_1) \times \operatorname{Aut}(R_2) \times ... \times \operatorname{Aut}(R_k)$  and  $\operatorname{Aut}(R_i) \cong \operatorname{Aut}(H_i) \wr \mathbb{S}_{n_i}$ , where in this wreath product  $\operatorname{Aut}(H_i)$  appears in its right regular representation and the symmetric group  $\mathbb{S}_{n_i}$  in its natural permutation representation. Moreover, these isomorphisms induce isomorphisms  $\operatorname{Out}(R) \cong \operatorname{Out}(R_1) \times \operatorname{Out}(R_2) \times ... \times \operatorname{Out}(R_k)$  and  $\operatorname{Out}(R_i) \cong \operatorname{Out}(H_i) \wr \mathbb{S}_{n_i}$ .

## 3. OD-Characterization of Almost Simple Groups Related to $D_4(4)$

In this section, we study the problem of characterizing almost simple groups by order and degree pattern. Especially we will focus our attention on almost simple groups related to  $L = D_4(4)$ , namely, we will prove the Main Theorem of Sec. 1. We break the proof into a number of separate propositions. By assumption, we depict all possibilities for the prime graph associated with

By assumption, we depict all possibilities for the prime graph associated with G by use of the variables for some vertices in each proposition. Also, we need to know the structure of  $\Gamma(M)$  to determine the possibilities for G in some proposition, therefore we depict the prime graph of all extension of L in pages 18 to 20. Note that the set of order elements in each of the following propositions is calculated using Magma.

**Proposition 3.1.** If M = L, then  $G \cong L$ .

*Proof.* By TABLE 1  $|L| = 2^{24} \cdot 3^5 \cdot 5^4 \cdot 7 \cdot 13 \cdot 17^2$ .  $\pi_e(L) = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 15, 17, 20, 21, 30, 34, 51, 63, 65, 85, 255\}$ , so D(L) = (3, 4, 4, 1, 1, 3). Since |G| = |L| and D(G) = D(L), we conclude that the prime graph of G has following form:

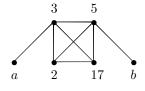


Figure 3.1

where  $\{a, b\} = \{7, 13\}.$ 

We will show that G is isomorphic to  $L = D_4(4)$ . We break up the proof into a several steps.

**Step1.** Let K be the maximal normal solvable subgroup of G. Then K is a  $\{2,3,5\}$ -group. In particular, G is non-solvable.

First we show that K is a 17'-group. Assume the contrary and let  $17 \in \pi(K)$ . Then 13 dose not divide the order of K. Otherwise, we may suppose that T is a Hall  $\{13,17\}$ -subgroup of K. It is seen that T is a nilpotent subgroup of order  $13.17^i$  for i=1 or 2. Thus,  $13.17 \in \pi_e(K) \subseteq \pi_e(G)$ , a contradiction. Thus  $\{17\} \subseteq \pi(K) \subseteq \pi(G) - \{13\}$ . Let  $K_{17} \in \operatorname{Syl}_{17}(K)$ . By Frattini argument,  $G = KN_G(K_{17})$ . Therefore,  $N_G(K_{17})$  contains an element x of order 13. Since G has no element of order 13.17,  $\langle x \rangle$  should act fixed point freely on  $K_{17}$ , that is implying  $\langle x \rangle K_{17}$  is a Frobenius group. By Lemma 2.2(b),  $|\langle x \rangle| |(|K_{17}| - 1)$ . It follows that  $13|17^i - 1$  for i=1 or 2, which is a contradiction.

Next, we show that K is a p'-group for  $p \in \{a,b\}$ . Let p||K| and  $K_p \in \operatorname{Syl}_p(K)$ . Now by Frattini argument,  $G = KN_G(K_p)$ , so 17 must divide the order of  $N_G(K_p)$ . Therefore, the normalizer  $N_G(K_p)$  contains an element of order 17, say x. So  $\langle x \rangle K_p$  is a cyclic subgroup of G of order 17.p, and so  $p \sim 17$  in  $\Gamma(G)$ , which is a contradiction. Therefore K is a  $\{2,3,5\}$ -group. In addition, since K is a proper subgroup of G, it follows that G is non-solvable.

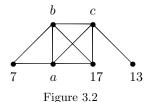
**Step 2.** The quotient G/K is an almost simple group. In fact,  $S \leq G/K \lesssim \operatorname{Aut}(S)$ , where S is a finite non-abelian simple group isomorphic to  $L := D_4(4)$ . Let  $\overline{G} = G/K$ . Then  $S := \operatorname{Soc}(\overline{G}) = P_1 \times P_2 \times ... \times P_m$ , where  $P_i$ 's are finite non-abelian simple groups and  $S \leq \frac{G}{K} \lesssim \operatorname{Aut}(S)$ . If we show that m = 1, the proof of Step 2 will be completed.

Suppose that  $m \geq 2$ . In this case, we claim that 13 does not divide |S|. Assume the contrary and let  $13 \mid |S|$ , on the other hand,  $\{2,3\} \subset \pi(P_i)$  for every i (by TABLE 1), hence  $2 \sim 13$  and  $3 \sim 13$ , which is a contradiction. Now, by step 1, we observe that  $13 \in \pi(\overline{G}) \subseteq \pi(\operatorname{Aut}(S))$ . But  $\operatorname{Aut}(S) = \operatorname{Aut}(S_1) \times \operatorname{Aut}(S_2) \times ... \times \operatorname{Aut}(S_r)$ , where the groups  $S_j$  are direct products of isomorphic  $P_i$ 's such that  $S = S_1 \times S_2 \times ... \times S_r$ . Therefore, for some j, 13 divides the order of an automorphism group of a direct product  $S_j$  of t isomorphic simple groups  $P_i$ . Since  $P_i \in \mathfrak{S}_{17}$ , it follows that  $|\operatorname{Out}(P_i)|$  is not divisible by 13 (see TABLE 1). Now, by Lemma 2.3, we obtain  $|\operatorname{Aut}(S_j)| = |\operatorname{Aut}(P_i)|^{t!} \cdot t!$ . Therefore,  $t \geq 13$  and so  $2^{26}$  must divide the order of G, which is a contradiction. Therefore m = 1 and  $S = P_1$ .

By TABLE 1 and Step 1, it is evident that  $|S| = 2^{\alpha}.3^{\beta}.5^{\gamma}.7.13.17^2$ , where  $2 \leq \alpha \leq 24$ ,  $1 \leq \beta \leq 5$  and  $0 \leq \gamma \leq 4$ . Now, using collected results contained in TABLE 1, we deduce that  $S \cong D_4(4)$  and by Step 2,  $L \subseteq G/K \lesssim \operatorname{Aut}(L)$  is completed. As |G| = |L|, we deduce K = 1, so  $G \cong L$  and the proof is completed.

**Proposition 3.2.** If  $M = L : 2_1$ , then  $G \cong L : 2_1$  or  $L : 2_3$ .

*Proof.* As  $|L:2_1| = 2^{25}.3^5.5^4.7.13.17^2$  and  $\pi_e(L:2_1) = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 14, 15, 16, 17, 18, 20, 21, 24, 30, 34, 51, 63, 65, 85, 255\}$ , then  $D(L:2_1) = (4, 4, 2, 1, 3)$ . Since  $|G| = |L:2_1|$  and  $D(G) = D(L:2_1)$ , we conclude that there exist several possibilities for  $\Gamma(G)$ :



where  $\{a, b, c\} = \{2, 3, 5\}.$ 

**Step1.** Let K be the maximal normal solvable subgroup of G. Then K is a  $\{2,3,5\}$ -group. In particular, G is non-solvable.

By a similar argument to that in Proposition 3.1, we can obtain this assertion.

**Step 2.** The quotient  $\frac{G}{K}$  is an almost simple group. In fact,  $S \leq \frac{G}{K} \lesssim \operatorname{Aut}(S)$ , where S is a finite non-abelian simple group.

The proof is similar to Step 2 of Proposition 3.1.

By TABLE 1 and Step 1, it is evident that  $|S|=2^{\alpha}.3^{\beta}.5^{\gamma}.7.13.17^2$ , where  $2\leq \alpha\leq 25,\ 1\leq \beta\leq 5$  and  $0\leq \gamma\leq 4$ . Now, using collected results contained in TABLE 1, we conclude that  $S\cong D_4(4)$  and by Step 2,  $L\unlhd \frac{G}{K}\lesssim \operatorname{Aut}(L)$ . As  $|G|=|L:2_1|=2|L|$ , we deduce |K|=1 or 2.

If |K| = 1, then  $G \cong L : 2_1, L : 2_2$  or  $L : 2_3$ . Obviously,  $G \cong L : 2_1$  or  $L : 2_3$  because deg(2) = 5 in  $\Gamma(L : 2_2)$  (see page 16).

If |K| = 2, then  $K \leq Z(G)$  and so deg(2) = 5, which is a contradiction.  $\square$ 

**Proposition 3.3.** If  $M = L : 2_2$ , then  $G \cong L : 2_2$  or  $\mathbb{Z}_2 \times L$ .

*Proof.* As  $|L:2_2|=2^{25}.3^5.5^4.7.13.17^2$  and  $\pi_e(L:2_2)=\{1,2,3,4,5,6,7,8,9,10,12,13,14,15,17,18,20,21,24,26,30,34,40,42,51,60,63,65,68,85,102,126,130,170,255\}$ , then  $D(L:2_2)=(5,4,4,2,2,3)$ . By assumption  $|G|=|L:2_2|$  and  $D(G)=D(L:2_2)$ , so the prime graph of G has following form:

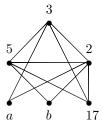


Figure 3.3

where  $\{a, b\} = \{7, 13\}.$ 

**Step1.** Let K be the maximal normal solvable subgroup of G. Then K is a  $\{2,3,5\}$ -group. In particular, G is non-solvable.

By similar arguments as in the proof of Step 1 in Proposition 3.1, we conclude that K is a  $\{2,3,5\}$ -group and G is non-solvable.

**Step 2.** The quotient  $\frac{G}{K}$  is an almost simple group. In fact,  $S \leq \frac{G}{K} \lesssim \operatorname{Aut}(S)$ , where S is a finite non-abelian simple group.

Let  $\overline{G} = \frac{G}{K}$ . Then  $S := \operatorname{Soc}(\overline{G}), S = P_1 \times P_2 \times ... \times P_m$ , where  $P_i$ 's are finite non-abelian simple groups and  $S \leq \frac{G}{K} \lesssim \operatorname{Aut}(S)$ . We are going to prove that m=1 and  $S=P_1$ . Suppose that  $m\geq 2$ . We claim a does not divide |S|. Assume the contrary and let  $a \mid |S|$ , we conclude that a just divide the order of one of the simple groups  $P_i$ 's. Without loss of generality, we assume that  $a||P_1|$ . Then the rest of the  $P_i$ 's must be  $\{2,3\}$ -group (because only 2 and 3 are adjacent to a in  $\Gamma(G)$ ), this is a contradiction because  $P_i$ 's are finite non-abelian simple groups. Now, by Step 1, we observe that  $a \in \pi(\overline{G}) \subseteq \pi(\operatorname{Aut}(S))$ . But  $\operatorname{Aut}(S) = \operatorname{Aut}(S_1) \times \operatorname{Aut}(S_2) \times ... \times \operatorname{Aut}(S_r)$ , where the groups  $S_i$  are direct products of isomorphic  $P_i$ 's such that  $S = S_1 \times S_2 \times ... \times S_r$ . Therefore, for some j, a divides the order of an automorphism group of a direct product  $S_i$  of t isomorphic simple groups  $P_i$ . Since  $P_i \in \mathfrak{S}_{17}$ , it follows that  $|\operatorname{Out}(P_i)|$  is not divisible by a (see TABLE 1), so a does not divide the order of  $Aut(P_i)$ . Now, by Lemma 2.3, we obtain  $|\operatorname{Aut}(S_i)| = |\operatorname{Aut}(P_i)|^{t!} \cdot t!$ . Therefore,  $t \geq a$  and so  $3^a$  must divide the order of G, which is a contradiction. Therefore m=1 and  $S=P_1$ .

By TABLE 1 and Step 1, it is evident that  $|S| = 2^{\alpha}.3^{\beta}.5^{\gamma}.7.13.17^2$ , where  $2 \le \alpha \le 25$ ,  $1 \le \beta \le 5$  and  $0 \le \gamma \le 4$ . Now, using collected results contained in TABLE 1, we conclude that  $S \cong D_4(4)$  and by Step 2,  $L \le \frac{G}{K} \lesssim \operatorname{Aut}(L)$ . As  $|G| = |L: 2_2| = 2|L|$ , we deduce |K| = 1 or 2.

If |K| = 1, then  $G \cong L : 2_1, L : 2_2$  or  $L : 2_3$  because |G| = 2|L|. It is obvious that  $G \cong L : 2_2$ , because deg(13) = 1 in  $\Gamma(L : 2_1)$  and  $\Gamma(L : 2_3)$  (see page 17).

If |K| = 2, then  $G/K \cong L$  and  $K \leq Z(G)$ . It follows that G is a central extension of K by L. If G is a non-split extension of K by L, then |K| must divide the Schur multiplier of L, which is 1. But this is a contradiction, so we obtain that G split over |K|. Hence  $G \cong \mathbb{Z}_2 \times L$ .

**Proposition 3.4.** If  $M = L : 2_3$ , then  $G \cong L : 2_3$  or  $L : 2_1$ .

*Proof.* As  $|L:2_3|=2^{25}.3^5.5^4.7.13.17^2$  and  $\pi_e(L:2_3)=\{1,2,3,4,5,6,7,8,9,10,12,13,14,15,16,17,18,20,21,24,30,34,51,63,65,85,255\}$ , then  $D(L:2_3)=(4,4,4,2,1,3)$ . Since  $|G|=|L:2_3|$  and  $D(G)=D(L:2_3)$ , we conclude that  $\Gamma(G)$  has the following form similarly to Proposition 3.2:

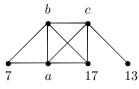


Figure 3.4

where  $\{a, b, c\} = \{2, 3, 5\}.$ 

**Step1.** Let K be the maximal normal solvable subgroup of G. Then K is a  $\{2,3,5\}$ -group. In particular, G is non-solvable.

We can prove this by the similar way to that in Proposition 3.2.

**Step 2.** The quotient  $\frac{G}{K}$  is an almost simple group. In fact,  $S \leq \frac{G}{K} \lesssim \operatorname{Aut}(S)$ , where S is a finite non-abelian simple group.

By using a similar argument, as in the proof of Proposition 3.2, we can verify that  $\frac{G}{K}$  is an almost simple group.

By TABLE 1 and Step 1, it is evident that  $|S|=2^{\alpha}.3^{\beta}.5^{\gamma}.7.13.17^2$ , where  $2 \leq \alpha \leq 25, \ 1 \leq \beta \leq 5$  and  $0 \leq \gamma \leq 4$ . Now, using collected results contained in TABLE 1, we conclude that  $S \cong D_4(4)$  and by Step 2,  $L \unlhd \frac{G}{K} \lesssim \operatorname{Aut}(L)$ . As  $|G|=|L:2_3|=2|L|$ , we deduce |K|=1 or 2.

If |K| = 1, then  $G \cong L : 2_1, L : 2_2$  or  $L : 2_3$  because |G| = 2|L|. Obviously,  $G \cong L : 2_3$  or  $L : 2_1$ , because deg(2) = 5 in  $\Gamma(L : 2_2)$  (see page 16).

If |K| = 2, then  $K \leq Z(G)$  and so deg(2) = 5, which is a contradiction.  $\square$ 

**Proposition 3.5.** If M = L : 3, then  $G \cong L : 3$  or  $\mathbb{Z}_3 \times L$ .

*Proof.* As  $|L:3| = 2^{24}.3^6.5^4.7.13.17^2$  and  $\pi_e(L:3) = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 15, 17, 18, 20, 21, 24, 30, 34, 39, 45, 51, 63, 65, 85, 255\}$ , then D(L:3) = (3, 5, 4, 1, 2, 3). since |G| = |L:3| and D(G) = D(L:3), we conclude that  $\Gamma(G)$  has the following form (like  $\Gamma(L:3)$ ):

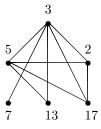


Figure 3.5

**Step1.** Let K be the maximal normal solvable subgroup of G. Then K is a  $\{2,3\}$ -group. In particular, G is non-solvable.

First, we show that K is a p'-group for p = 7, 13 and 17. Since the proof is quite similar to the proof of Step 1 in Proposition 3.1, so we avoid here full explanation of all details.

Next we consider K is a 5'-group. Assume the contrary,  $5 \in \pi_e(K)$ . Let  $K_5 \in \operatorname{Syl}_5(K)$ . By Frattini argument,  $G = KN_G(K_5)$ . Therefore,  $N_G(K_5)$  has an element x of order 7. Since G has no element of order 5.7,  $\langle x \rangle$  should act fixed point freely on  $K_5$ , implying  $\langle x \rangle K_5$  is a Frobenius group. By Lemma 2.2(b),  $|\langle x \rangle||(|K_5|-1)$ , which is impossible. Therefore K is a  $\{2,3\}$ -group. In addition since K is a proper subgroup of G, then G is non-solvable and the proof of this step is completed.

**Step 2.** The quotient  $\frac{G}{K}$  is an almost simple group. In fact,  $S \leq \frac{G}{K} \lesssim \operatorname{Aut}(S)$ , where S is a finite non-abelian simple group. In a similar way as in the proof of Step 2 in Proposition 3.1, we conclude that  $\frac{G}{K}$  is an almost simple group.

By TABLE 1 and Step 1, it is evident that  $|S| = 2^{\alpha}.3^{\beta}.5^{4}.7.13.17^{2}$ , where  $2 \leq \alpha \leq 24$  and  $1 \leq \beta \leq 6$ . Now, using collected results contained in TABLE 1, we conclude that  $S \cong D_{4}(4)$  and by Step 2,  $L \subseteq \frac{G}{K} \lesssim \operatorname{Aut}(L)$ . As |G| = |L|: |G| = |G| = |G| and |G| = |G| = |G| and |G| = |G| = |G| = |G|.

If |K| = 1, then  $G \cong L : 3$ .

If |K|=3, then  $G/K\cong L$ . In this case we have  $G/C_G(K)\lesssim \operatorname{Aut}(K)\cong \mathbb{Z}_2$ . Thus  $|G/C_G(K)|=1$  or 2. If  $|G/C_G(K)|=1$ , then  $K\leq Z(G)$ , that is, G is a central extension of K by L. If G is a non-split extension of K by L, then |K| must divide the Schur multiplier of L, which is 1. But this is a contradiction, so we obtain that G split over K. Hence  $G\cong \mathbb{Z}_3\times L$ . If  $|G/C_G(K)|=2$ , then  $K< C_G(K)$  and  $1\neq C_G(K)/K \subseteq G/K\cong L$ , which is a contradiction since L is simple.

**Proposition 3.6.** If  $M = L : 2^2$ , then  $G \cong L : 2^2$ ,  $\mathbb{Z}_2 \times (L : 2_1)$ ,  $\mathbb{Z}_2 \times (L : 2_2)$ ,  $\mathbb{Z}_2 \times (L : 2_3)$ ,  $\mathbb{Z}_4 \times L$  or  $(\mathbb{Z}_2 \times \mathbb{Z}_2) \times L$ .

*Proof.* As  $|L:2^2|=2^{26}.3^5.5^4.7.13.17^2$  and  $\pi_e(L:2^2)=\{1,2,3,4,5,6,7,8,9,10,12,13,14,15,16,17,18,20,21,24,26,30,34,42,51,60,63,65,68,85,102,126,130,170,255\}$ , then  $D(L:2^2)=(5,4,4,2,2,3)$ . Since  $|G|=|L:2^2|$  and  $D(G)=D(L:2^2)$ , so the prime graph of G has following form similarly to Proposition 3.3:

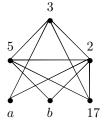


Figure 3.6

where  $\{a, b\} = \{7, 13\}.$ 

**Step1.** Let K be the maximal normal solvable subgroup of G. Then K is a  $\{2,3,5\}$ -group. In particular, G is non-solvable.

According to Step 1 in Proposition 3.3, we have K is a  $\{2,3,5\}$ -group and G is non-solvable.

**Step 2.** The quotient  $\frac{G}{K}$  is an almost simple group. In fact,  $S \leq \frac{G}{K} \lesssim \operatorname{Aut}(S)$ , where S is a finite non-abelian simple group.

We can prove this by the similar argument in Step 2 in Proposition 3.3.

By TABLE 1 and Step 1, it is evident that  $|S| = 2^{\alpha}.3^{\beta}.5^{\gamma}.7.13.17^2$ , where  $2 \le \alpha \le 26$ ,  $1 \le \beta \le 5$  and  $0 \le \gamma \le 4$ . Now, using collected results contained in TABLE 1, we conclude that  $S \cong D_4(4)$  and by Step 2,  $L \le \frac{G}{K} \lesssim \operatorname{Aut}(L)$ . As  $|G| = |L:2^2| = 4|L|$ , we deduce |K| = 1, 2 or 4.

If |K| = 1, then  $G \cong L : 2^2$ .

If |K| = 2, then  $K \leq Z(G)$ . In this case G is a central extension of  $\mathbb{Z}_2$  by  $L: 2_1, L: 2_2$  or  $L: 2_3$ . If G splits over K then  $G \cong \mathbb{Z}_2 \times (L: 2_1), \mathbb{Z}_2 \times (L: 2_2)$  or  $\mathbb{Z}_2 \times (L: 2_3)$ , otherwise we get a contradiction because |K| must divide the Schure multiplier of  $L: 2_1, L: 2_2$  and  $L: 2_3$ , which is impossible.

If |K|=4, then  $G/K\cong L$ . In this case we have  $G/C_G(K)\lesssim \operatorname{Aut}(K)\cong \mathbb{Z}_2$  or  $S_3$ . Thus  $|G/C_G(K)|=1,2,3$  or 6. If  $|G/C_G(K)|=1$ , then  $K\leq Z(G)$ , that is, G is a central extension of K by L. If G is a non-split extension of K by L, then |K| must divide the Schur multiplier of L, which is 1, but this is a contradiction. Therefore G splits over K. Hence  $G\cong K\times L$ . So we have  $G\cong \mathbb{Z}_4\times L$  or  $(\mathbb{Z}_2\times\mathbb{Z}_2)\times L$  because  $K\cong \mathbb{Z}_4$  or  $\mathbb{Z}_2\times\mathbb{Z}_2$ . If  $|G/C_G(K)|=2,3$  or 6, then  $K< C_G(K)$  and  $1\neq C_G(K)/K \leq G/K\cong L$ . Which is a contradiction, since L is simple.

**Proposition 3.7.** If  $M = L : (D_6)_1$ , then  $G \cong L : (D_6)_1$ , L : 6,  $\mathbb{Z}_3 \times (L : 2_1)$ ,  $\mathbb{Z}_3 \times (L : 2_3)$  or  $(\mathbb{Z}_3 \times L) \cdot \mathbb{Z}_2$ .

*Proof.* As  $|L:(D_6)_1| = 2^{25}.3^6.5^4.7.13.17^2$  and  $\pi_e(L:(D_6)_1) = \{1,2,3,4,5,6,7,8,9,10,12,13,14,15,16,17,18,20,21,24,30,34,39,42,45,51,60,63,65,85,255\}$ , then  $D(L:(D_6)_1) = (4,5,4,2,2,3)$ . Since  $|G| = |L:(D_6)_1|$  and  $D(G) = D(L:(D_6)_1)$ , we conclude that there exist several possibilities for  $\Gamma(G)$ :

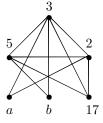


Figure 3.7

where  $\{a, b\} = \{7, 13\}.$ 

**Step1.** Let K be the maximal normal solvable subgroup of G. Then K is a  $\{2,3,5\}$ -group. In particular, G is non-solvable.

By the similar argument to that in Step 1 in Proposition 3.1, we can obtain this assertion.

**Step 2.** The quotient  $\frac{G}{K}$  is an almost simple group. In fact,  $S \leq \frac{G}{K} \lesssim \text{Aut(S)}$ , where S is a finite non-abelian simple group.

The proof is similar to Step 2 in Proposition 3.3.

By TABLE 1 and Step 1, it is evident that  $|S|=2^{\alpha}.3^{\beta}.5^{\gamma}.7.13.17^2$ , where  $2\leq \alpha\leq 25,\ 1\leq \beta\leq 6$  and  $0\leq \gamma\leq 4$ . Now, using collected results contained in TABLE 1, we conclude that  $S\cong D_4(4)$  and by Step 2,  $L\unlhd \frac{G}{K}\lesssim \operatorname{Aut}(L)$ . As  $|G|=|L:D_6)_1|=6|L|$ , we deduce |K|=1,2,3 or 6.

If |K| = 1, then  $G \cong L : (D_6)_1, L : (D_6)_2$  or L : 6 because |G| = 6|L|. Obviously,  $G \cong L : (D_6)_1$  or L : 6 because deg(2) = 5 in  $\Gamma(L : (D_6)_2)$ .

If |K| = 2, then  $K \leq Z(G)$  and so deg(2) = 5, which is a contradiction (see page 18).

If |K|=3, then  $G/K\cong L:2_1, L:2_2$  or  $L:2_3$ . But  $G/C_G(K)\lesssim \operatorname{Aut}(K)\cong \mathbb{Z}_2$ . Thus  $|G/C_G(K)|=1$  or 2. If  $|G/C_G(K)|=1$ , then  $K\leq Z(G)$ , that is, G is a central extension of K by  $L:2_1, L:2_2$  or  $L:2_3$ . If G splits over K, then  $G\cong \mathbb{Z}_3\times (L:2_1)$  or  $\mathbb{Z}_3\times (L:2_3)$  because in  $\Gamma(\mathbb{Z}_3\times (L:2_2))$  the degree of 2 is 5. Otherwise we get a contradiction because |K| must divide the Schure multiplier of  $L:2_1, L:2_2$  and  $L:2_3$ , which is impossible. If  $|G/C_G(K)|=2$ , then  $K< C_G(K)$  and  $1\neq C_G(K)/K \leq G/K\cong L:2_1, L:2_2$  or  $L:2_3$ , we obtain  $C_G(K)/K\cong L$ . Since  $K\leq Z(C_G(K))$ ,  $C_G(K)$  is a central extension of K by L. If  $C_G(K)$  splits over K, then  $C_G(K)\cong \mathbb{Z}_3\times L$ , otherwise we get a contradiction because |K| must divide the Schure multiplier of L, which is impossible. Therefore,  $G\cong (\mathbb{Z}_3\times L).\mathbb{Z}_2$ .

If |K| = 6, then  $G/K \cong L$  and  $K \cong \mathbb{Z}_6$  or  $D_6$ .

If  $K \cong \mathbb{Z}_6$ , then  $G/C_G(K) \lesssim \mathbb{Z}_2$  and so  $|G/C_G(K)| = 1$  or 2. If  $|G/C_G(K)| = 1$ , then  $K \leq Z(G)$ . It follows that deg(2) = 5, a contradiction. If  $|G/C_G(K)| = 2$ , then  $K < C_G(K)$  and  $1 \neq C_G(K)/K \leq G/K \cong L$ , which is a contradiction because L is simple.

If  $K \cong D_6$ , then  $K \cap C_G(K) = 1$  and  $G/C_G(K) \lesssim D_6$ . Thus  $C_G(K) \neq 1$ . Hence,  $1 \neq C_G(K) \cong C_G(K)K/K \trianglelefteq G/K \cong L$ . It follows that  $L \cong G/K \cong C_G(K)$  because L is simple. Therefore,  $G \cong D_6 \times L$ , which implies that deg(2) = 5, a contradiction.

**Proposition 3.8.** If  $M = L : (D_6)_2$ , then  $G \cong L : (D_6)_2$ ,  $\mathbb{Z}_2 \times (L : 3)$ ,  $\mathbb{Z}_3 \times (L : 2_2)$ ,  $(\mathbb{Z}_3 \times L).\mathbb{Z}_2$ ,  $\mathbb{Z}_6 \times L$  or  $S_3 \times L$ .

Proof. As  $|L:(D_6)_2|=2^{25}.3^6.5^4.7.13.17^2$  and  $\pi_e(L:(D_6)_2)=\{1,2,3,4,5,6,7,8,9,10,12,13,14,15,17,18,20,21,24,26,30,34,39,40,42,45,51,60,63,65,68,85,102,126,130,170,255\}$ , then  $D(L:(D_6)_2)=(5,5,4,2,3,3)$ . Since  $|G|=|L:(D_6)_2|$  and  $D(G)=D(L:(D_6)_2)$ , we conclude that Γ(G) has the following form (like Γ(L:(D\_6)\_2)):

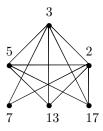


Figure 3.8

**Step1.** Let K be the maximal normal solvable subgroup of G. Then K is a  $\{2,3\}$ -group. In particular, G is non-solvable.

The proof is similar to Step 1 in Proposition 3.5.

**Step 2.** The quotient  $\frac{G}{K}$  is an almost simple group. In fact,  $S \leq \frac{G}{K} \lesssim \operatorname{Aut}(S)$ , where S is a finite non-abelian simple group.

Let  $\overline{G} = \frac{G}{K}$ . Then  $S := \operatorname{Soc}(\overline{G})$ ,  $S = P_1 \times P_2 \times ... \times P_m$ , where  $P_i$ 's are finite non-abelian simple groups and  $S \leq \frac{G}{K} \lesssim \operatorname{Aut}(S)$ . We are going to prove that m = 1 and  $S = P_1$ . Suppose that  $m \geq 2$ . By the same argument in Step 2 of Proposition 3.3 and considering 7 instead of a, we get a contradiction. Therefore m = 1 and  $S = P_1$ .

By TABLE 1 and Step 1, it is evident that  $|S| = 2^{\alpha}.3^{\beta}.5^{4}.7.13.17^{2}$ , where  $2 \leq \alpha \leq 25$  and  $1 \leq \beta \leq 6$ . Now, using collected results contained in TABLE 1, we conclude that  $S \cong D_{4}(4)$  and by Step 2,  $L \trianglelefteq \frac{G}{K} \lesssim \operatorname{Aut}(L)$ . As |G| = |L|:  $|D_{6}|_{2}| = 6|L|$ , we deduce |K| = 1, 2, 3 or 6.

If |K| = 1, then  $G \cong L : (D_6)_1, L : (D_6)_2$  or L : 6 because |G| = 6|L|. Obviously  $G \cong L : (D_6)_2$  because in  $\Gamma(L : (D_6)_1)$  and  $\Gamma(L : 6)$ , we have deg(13) = 2(see page 17).

If |K|=2, then  $K \leq Z(G)$  and  $G/K \cong L:3$ . Hence G is a central extension of K by L:3. If G splits over K, then  $G \cong \mathbb{Z}_2 \times (L:3)$ . Otherwise we get a contradiction because |K| must divide the Schure multiplier of L:3, which is impossible.

If |K|=3, then  $G/K\cong L:2_1,L:2_2$  or  $L:2_3$ . But  $G/C_G(K)\lesssim \operatorname{Aut}(K)\cong \mathbb{Z}_2$ . Thus  $|G/C_G(K)|=1$  or 2. If  $|G/C_G(K)|=1$ , then  $K\leq Z(G)$ , that is, G is a central extension of K by  $L:2_1,L:2_2$  or  $L:2_3$ . If G splits over K, then only  $G\cong \mathbb{Z}_3\times (L:2_2)$  because  $2\nsim 13$  in  $\Gamma(\mathbb{Z}_3\times (L:2_1))$  and  $\Gamma(\mathbb{Z}_3\times (L:2_3))$ . Otherwise we get a contradiction because |K| must divide the Schure multiplier of  $L:2_1,L:2_2$  and  $L:2_3$ , which is impossible. If

 $|G/C_G(K)| = 2$ , then  $K < C_G(K)$  and  $1 \neq C_G(K)/K \leq G/K \cong L : 2_1, L : 2_2$  or  $L : 2_3$ , we obtain  $C_G(K)/K \cong L$ . Since  $K \leq Z(C_G(K)), C_G(K)$  is a central extension of K by L. If  $C_G(K)$  splits over K, then  $C_G(K) \cong \mathbb{Z}_3 \times L$ , otherwise we get a contradiction because |K| must divide the Schure multiplier of L, which is impossible. Therefore,  $G \cong (\mathbb{Z}_3 \times L).\mathbb{Z}_2$ .

If |K|=6, then  $G/K\cong L$  and  $K\cong \mathbb{Z}_6$  or  $D_6$ . If  $K\cong \mathbb{Z}_6$ , then  $G/C_G(K)\lesssim \mathbb{Z}_2$  and so  $|G/C_G(K)|=1$  or 2. If  $|G/C_G(K)|=1$ , then  $K\leq Z(G)$  and  $G/K\cong L$ . Therefore G is a central extension of K by L. If G is a non-split extension of K by L, then |K| must divide the Schure multiplier of L, which is 1. But this is a contradiction. So we obtain that G splits over K. Hence  $G\cong \mathbb{Z}_6\times L$ . If  $|G/C_G(K)|=2$ , then  $K< C_G(K)$  and  $1\neq C_G(K)/K\trianglelefteq G/K\cong L$ , which is a contradiction because L is simple. If  $K\cong D_6$ , then  $K\cap C_G(K)=1$  and  $G/C_G(K)\lesssim D_6$ . Thus  $C_G(K)\neq 1$ . Hence,  $1\neq C_G(K)\cong C_G(K)K/K\trianglelefteq G/K\cong L$ . It follows that  $L\cong G/K\cong C_G(K)$  because L is simple. Therefore  $G\cong D_6\times L$ .

**Proposition 3.9.** If M = L : 6, then  $G \cong L : 6$ ,  $L : (D_6)_1$ ,  $\mathbb{Z}_3 \times (L : 2_1)$ ,  $\mathbb{Z}_3 \times (L : 2_3)$  or  $(\mathbb{Z}_3 \times L).\mathbb{Z}_2$ .

*Proof.* As  $|L:6| = 2^{25}.3^6.5^4.7.13.17^2$  and  $\pi_e(L:6) = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 14, 15, 16, 17, 18, 20, 21, 24, 30, 34, 36, 39, 42, 45, 48, 51, 63, 65, 85, 255\}, then <math>D(L:6) = (4, 5, 4, 2, 2, 3)$ . Since |G| = |L:6| and D(G) = D(L:6), there exist several possibilities for  $\Gamma(G)$  similarly to Proposition 3.7:

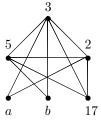


Figure 3.9

where  $\{a, b\} = \{7, 13\}.$ 

**Step1.** Let K be the maximal normal solvable subgroup of G. Then K is a  $\{2,3,5\}$ -group. In particular, G is non-solvable.

The proof is similar to that in Proposition 3.3.

**Step 2.** The quotient  $\frac{G}{K}$  is an almost simple group. In fact,  $S \leq \frac{G}{K} \lesssim \operatorname{Aut}(S)$ , where S is a finite non-abelian simple group.

Again we refer to Step 2 of proposition 3.3 to get the proof.

By TABLE 1 and Step 1, it is evident that  $|S| = 2^{\alpha}.3^{\beta}.5^{\gamma}.7.13.17^{2}$ , where  $2 \le \alpha \le 25$ ,  $1 \le \beta \le 6$  and  $0 \le \gamma \le 4$ . Now, using collected results contained

in TABLE 1, we conclude that  $S \cong D_4(4)$  and by Step 2,  $L \subseteq \frac{G}{K} \lesssim \operatorname{Aut}(L)$ . As |G| = |L:6| = 6|L|, we deduce |K| = 1, 2, 3 or 6.

If |K| = 1, then  $G \cong L : 6$ ,  $L : (D_6)_1$  or  $L : (D_6)_2$  because |G| = 6|L|. Obviously,  $G \cong L : 6$  or  $L : (D_6)_1$  because deg(2) = 5 in  $\Gamma(L : (D_6)_2)$  (see page 18).

If |K| = 2, then  $K \leq Z(G)$  and so deg(2) = 5, which is a contradiction.

If |K|=3, then  $G/K\cong L:2_1, L:2_2$  or  $L:2_3$ . But  $G/C_G(K)\lesssim \operatorname{Aut}(K)\cong \mathbb{Z}_2$ . Thus  $|G/C_G(K)|=1$  or 2. If  $|G/C_G(K)|=1$ , then  $K\leq Z(G)$ , that is, G is a central extension of K by  $L:2_1, L:2_2$  or  $L:2_3$ . If G splits over K, then  $G\cong \mathbb{Z}_3\times (L:2_1)$  or  $\mathbb{Z}_3\times (L:2_3)$  because in  $\Gamma(\mathbb{Z}_3\times (L:2_2))$  the degree of 2 is 5. Otherwise we get a contradiction because |K| must divide the Schure multiplier of  $L:2_1, L:2_2$  and  $L:2_3$ , which is impossible. If  $|G/C_G(K)|=2$ , then  $K< C_G(K)$  and  $1\neq C_G(K)/K \trianglelefteq G/K\cong L:2_1, L:2_2$  or  $L:2_3$ , we obtain  $C_G(K)/K\cong L$ . Since  $K\leq Z(C_G(K))$ ,  $C_G(K)$  is a central extension of K by L. If  $C_G(K)$  splits over K, then  $C_G(K)\cong \mathbb{Z}_3\times L$ , otherwise we get a contradiction because |K| must divide the Schure multiplier of L, which is impossible. Therefore,  $G\cong (\mathbb{Z}_3\times L).\mathbb{Z}_2$ .

If |K| = 6, then  $G/K \cong L$  and  $K \cong \mathbb{Z}_6$  or  $D_6$ . If  $K \cong \mathbb{Z}_6$ , then  $G/C_G(K) \lesssim \mathbb{Z}_2$  and so  $|G/C_G(K)| = 1$  or 2. If  $|G/C_G(K)| = 1$ , then  $K \leq Z(G)$ . It follows that deg(2) = 5, a contradiction. If  $|G/C_G(K)| = 2$ , then  $K < C_G(K)$  and  $1 \neq C_G(K)/K \leq G/K \cong L$ , which is a contradiction because L is simple. If  $K \cong D_6$ , then  $K \cap C_G(K) = 1$  and  $G/C_G(K) \lesssim D_6$ . Thus  $C_G(K) \neq 1$ . Hence,  $1 \neq C_G(K) \cong C_G(K)K/K \leq G/K \cong L$ . It follows that  $L \cong G/K \cong C_G(K)$  because L is simple. Therefore,  $G \cong D_6 \times L$ , which implies that deg(2) = 5, a contradiction.

**Proposition 3.10.** If  $M = L : D_{12}$ , then  $G \cong L : D_{12}, \mathbb{Z}_2 \times (L : (D_6)_1)$ ,  $\mathbb{Z}_2 \times (L : (D_6)_2)$ ,  $\mathbb{Z}_2 \times (L : 6)$ ,  $\mathbb{Z}_3 \times (L : 2^2)$ ,  $(\mathbb{Z}_3 \times (L : 2_1)).\mathbb{Z}_2$ ,  $(\mathbb{Z}_3 \times (L : 2_2)).\mathbb{Z}_2$ ,  $(\mathbb{Z}_3 \times (L : 2_3)).\mathbb{Z}_2$ ,  $\mathbb{Z}_4 \times (L : 3)$ ,  $(\mathbb{Z}_2 \times \mathbb{Z}_2) \times (L : 3)$ ,  $(\mathbb{Z}_4 \times L).\mathbb{Z}_3$ ,  $((\mathbb{Z}_2 \times \mathbb{Z}_2) \times L).\mathbb{Z}_3$ ,  $\mathbb{Z}_6 \times (L : 2_1)$ ,  $\mathbb{Z}_6 \times (L : 2_2)$ ,  $\mathbb{Z}_6 \times (L : 2_3)$ ,  $(\mathbb{Z}_6 \times L).\mathbb{Z}_2$ ,  $\mathbb{Z}_3 \times (L : 2_1)$ ,  $\mathbb{Z}_3 \times (L : 2_2)$ ,  $\mathbb{Z}_3 \times (L : 2_3)$ ,  $\mathbb{Z}_{12} \times L$ ,  $(\mathbb{Z}_2 \times \mathbb{Z}_6) \times L$ ,  $D_{12} \times L$ ,  $(\mathbb{Z}_2 \times L).D_6$ ,  $\mathbb{Z}_4 \times L$ ,  $L.\mathbb{Z}_4$  or  $T \times L$ .

*Proof.* As  $|L:D_{12}| = 2^{26}.3^6.5^4.7.13.17^2$  and  $\pi_e(L:(D_{12})) = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 14, 15, 16, 17, 18, 20, 21, 24, 26, 30, 34, 39, 40, 42, 45, 48, 51, 60, 63, 65, 68, 85, 102, 126, 130, 170, 255\}, then <math>D(L:D_{12}) = (5, 5, 4, 2, 3, 3)$ . Since  $|G| = |L:D_{12}|$  and  $D(G) = D(L:D_{12})$ , we conclude that  $\Gamma(G)$  has the following form (like  $\Gamma(L:D_{12})$ ):

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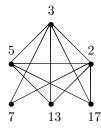


Figure 3.10

**Step1.** Let K be the maximal normal solvable subgroup of G. Then K is a  $\{2,3\}$ -group. In particular, G is non-solvable.

The proof is similar to Step 1 in Proposition 3.5.

**Step 2.** The quotient  $\frac{G}{K}$  is an almost simple group. In fact,  $S \leq \frac{G}{K} \lesssim \operatorname{Aut}(S)$ , where S is a finite non-abelian simple group.

To get the proof, follow the way in the proof of Step 2 in proposition 3.5.

By TABLE 1 and Step 1, it is evident that  $|S| = 2^{\alpha}.3^{\beta}.5^{4}.7.13.17^{2}$ , where  $2 \leq \alpha \leq 26$  and  $1 \leq \beta \leq 6$ . Now, using collected results contained in TABLE 1, we conclude that  $S \cong D_{4}(4)$  and by Step 2,  $L \trianglelefteq \frac{G}{K} \lesssim \operatorname{Aut}(L)$ . As |G| = |L|:  $D_{12}|=12|L|$ , we deduce |K|=1,2,3,4,6 or 12.

If |K| = 1, then  $G \cong L : D_{12}$ .

If |K| = 2, then  $G/K \cong L : (D_6)_1, L : (D_6)_2$  or L : 6 and  $K \leq Z(G)$ . It follows that G is a central extension of K by  $L : (D_6)_1, L : (D_6)_2$  or L : 6. If G splits over K, then  $G \cong \mathbb{Z}_2 \times (L : (D_6)_1), \mathbb{Z}_2 \times (L : (D_6)_2)$  or  $\mathbb{Z}_2 \times (L : 6)$ . Otherwise  $G \cong \mathbb{Z}_2.(L : (D_6)_1)$  or  $\mathbb{Z}_2.(L : (D_6)_2)$ .

If |K|=3, then  $G/K\cong L:2^2$ . But  $G/C_G(K)\lesssim \operatorname{Aut}(K)\cong \mathbb{Z}_2$ . Thus  $|G/C_G(K)|=1$  or 2. If  $|G/C_G(K)|=1$ , then  $K\leq Z(G)$ , that is, G is a central extension of K by  $L:2^2$ . If G splits over K, then  $G\cong \mathbb{Z}_3\times (L:2^2)$ , Otherwise we get a contradiction because |K| must divide the Schure multiplier of  $L:2^2$ , which is impossible. If  $|G/C_G(K)|=2$ , then  $K< C_G(K)$  and  $1\neq C_G(K)/K \subseteq G/K\cong L:2^2$ , and we obtain  $C_G(K)/K\cong L:2_1,L:2_2$  or  $L:2_3$ . Since  $K\leq Z(C_G(K))$ ,  $C_G(K)$  is a central extension of K by  $L:2_1,L:2_2$  or  $L:2_3$ . Thus  $C_G(K)\cong \mathbb{Z}_3\times (L:2_1)$ ,  $\mathbb{Z}_3\times (L:2_2)$  or  $\mathbb{Z}_3\times (L:2_3)$ , otherwise we get a contradiction because 3 must divide the Schure multiplier of  $L:2_1,L:2_2$  or  $L:2_3$ , which is impossible. Therefore,  $G\cong (\mathbb{Z}_3\times (L:2_1)).\mathbb{Z}_2, (\mathbb{Z}_3\times (L:2_2)).\mathbb{Z}_2$  or  $(\mathbb{Z}_3\times (L:2_3)).\mathbb{Z}_2$ .

If |K|=4, then  $G/K\cong L:3$  and  $K\cong \mathbb{Z}_4$  or  $\mathbb{Z}_2\times \mathbb{Z}_2$ . In this case we have  $G/C_G(K)\lesssim \operatorname{Aut}(K)\cong \mathbb{Z}_2$  or  $S_3$ . Thus  $|G/C_G(K)|=1,2,3$  or 6. If  $|G/C_G(K)|=1$ , then  $K\leq Z(G)$ , that is, G is a central extension of K by L:3. If G split over K by L:3, then  $G\cong \mathbb{Z}_4\times (L:3)$  or  $(\mathbb{Z}_2\times \mathbb{Z}_2)\times (L:3)$ . Otherwise we get a contradiction because |K| must divide the Schure multiplier of L:3, which is impossible. If  $|G/C_G(K)|\neq 1$ , since  $|G/C_G(K)|=2,3$  or 6, it follows that  $K< C_G(K)$ . As L is simple, we conclude that  $1\neq C_G(K)/K$  must

be an extension of L. Hence  $|G/C_G(K)| = 3$  and therefore  $C_G(K)/K \cong L$ . Now, since  $K \leq Z(C_G(K))$ , we conclude that  $C_G(K)$  is a central extension of K by L. Thus  $C_G(K) \cong \mathbb{Z}_4 \times L$ , or  $(\mathbb{Z}_2 \times \mathbb{Z}_2) \times L$ , otherwise |K| must divide the Schure multiplier of L, which is 1 and it is impossible. Therefore,  $G \cong (\mathbb{Z}_4 \times L).\mathbb{Z}_3$  or  $((\mathbb{Z}_2 \times \mathbb{Z}_2) \times L).\mathbb{Z}_3$ .

If |K| = 6, then  $G/K \cong L : 2_1, L : 2_2$  or  $L : 2_3$  and  $K \cong \mathbb{Z}_6$  or  $D_6$ . If  $K \cong \mathbb{Z}_6$ , then  $G/C_G(K) \lesssim \mathbb{Z}_2$  and so  $|G/C_G(K)| = 1$  or 2. If  $|G/C_G(K)| = 1$ , then  $K \leq Z(G)$ , that is G is a central extension of  $\mathbb{Z}_6$  by  $L: 2_1, L: 2_2$ or  $L: 2_3$ . If G splits over K, we obtain  $G \cong \mathbb{Z}_6 \times (L: 2_1), \mathbb{Z}_6 \times (L: 2_2)$ or  $\mathbb{Z}_6 \times (L:2_3)$ , otherwise we get a contradiction because |K| must divide the Schure multiplier of  $L: 2_1, L: 2_2$  or  $L: 2_3$ , which is impossible. If  $|G/C_G(K)| = 2$ , then  $K < C_G(K)$  and  $1 \neq C_G(K)/K \leq G/K \cong L : 2_1$ ,  $L: 2_2 \text{ or } L: 2_3, \text{ and we obtain } C_G(K)/K \cong L. \text{ Since } K \leq Z(C_G(K)),$  $C_G(K)$  is a central extension of K by L. Thus  $C_G(K) \cong \mathbb{Z}_6 \times L$ , otherwise we get a contradiction because |K| must divide the Schure multiplier of L. Therefore  $G \cong (\mathbb{Z}_6 \times L).\mathbb{Z}_2$ . If  $K \cong D_6$ , then  $G/C_G(K) \lesssim D_6$  and so  $|G/C_G(K)| = 1, 2, 3$  or 6. If  $|G/C_G(K)| = 1$ , then  $K \leq Z(G)$ , that is a contradiction. If  $|G/C_G(K)| = 2$ , then we have  $|KC_G(K)| = 6 \cdot |G|/2 = 3|G|$ because  $K \cap C_G(K) = 1$ , which is a contradiction. If  $|G/C_G(K)| = 3$ , then we have  $|KC_G(K)| = 6.|G|/3 = 2|G|$  because  $K \cap C_G(K) = 1$ , which is a contradiction. If  $|G/C_G(K)| = 6$ , then  $G/C_G(K) \cong D_6$  and  $C_G(K) \neq 1$ . Hence,  $1 \neq C_G(K) \cong C_G(K)K/K \leq G/K \cong L : 2_1, L : 2_2 \text{ or } L : 2_3$ . It follows that  $C_G(K) \cong L: 2_1, L: 2_2$  or  $L: 2_3$  because L is simple. Therefore,  $G \cong D_6 \times (L:2_1), D_6 \times (L:2_2) \text{ or } D_6 \times (L:2_3).$ 

Before processing the last case, we recall the following facts.

There exist five non-isomorphic groups of order 12. Two of them are abelian and three are non-abelian. The non-abelian groups are: alternating group  $A_4$ , dihedral group  $D_{12}$  and the dicyclic group T with generators a and b, subject to the relations  $a^6 = 1$ ,  $a^3 = b^2$  and  $b^{-1}ab = a^{-1}$ .

If |K| = 12, then  $G/K \cong L$  and  $K \cong \mathbb{Z}_{12}$ ,  $\mathbb{Z}_2 \times \mathbb{Z}_6$ ,  $D_{12}$ ,  $\mathbb{A}_4$  or T. But  $C_G(K)K/K \subseteq G/K \cong L$ . If  $C_G(K)K/K = 1$ , then  $C_G(K) \subseteq K$  and hence  $|L| = |G/K|||G/C_G(K)|||\operatorname{Aut}(K)||$ . Thus  $|L|||\operatorname{Aut}(K)||$ , a contradiction. Therefore,  $C_G(K)K/K \neq 1$  and since L is simple group, we conclude that  $G = C_G(K)K$  and hence,  $G/C_G(K) \cong K/Z(K)$ . Now, we should consider the following cases:

If  $K \cong \mathbb{Z}_{12}$  or  $\mathbb{Z}_2 \times \mathbb{Z}_6$ , then  $G/C_G(K) = 1$ . Therefore  $K \leq Z(G)$ , that is G is a central extension of  $\mathbb{Z}_{12}$  or  $\mathbb{Z}_2 \times \mathbb{Z}_6$  by L. If G splits over K, we obtain  $G \cong \mathbb{Z}_{12} \times L$  or  $(\mathbb{Z}_2 \times \mathbb{Z}_6) \times L$ , otherwise we get a contradiction because |K| must divide the Schure multiplier of L, which is 1 and it is impossible.

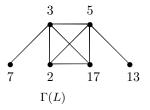
[ DOI: 10.7508/ijmsi.2015.01.003 ]

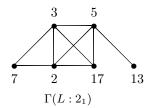
If  $K \cong D_{12}$ , then G = K.L and  $G/C_G(K) \cong D_6$ . Since  $C_G(K)/Z(K) \cong G/K \cong L$  and  $Z(K) \leq Z(C_G(K))$ , we conclude that  $C_G(K)$  is a central extension of  $Z(K) \cong \mathbb{Z}_2$  by L. If  $C_G(K)$  is a non-split extension, then 2 must divide the Schure multiplier of L, which is 1 and it is impossible. Thus  $C_G(K) \cong \mathbb{Z}_2 \times L$  and hence, G is a split extension of K by L. Now, since  $Hom(L, Aut(D_{12}))$  is trivial, we have  $G \cong D_{12} \times L$ .

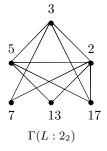
If  $K \cong \mathbb{A}_4$ , then  $G/C_G(K) \cong \mathbb{A}_4$ . As  $G = C_G(K)K$ , It follows that  $C_G(K) \cong L$ . Therefore  $G \cong L \times \mathbb{A}_4$  or  $L.\mathbb{A}_4$ .

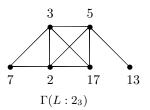
If  $K \cong T$ , then By the similar way in case  $K \cong D_{12}$ , we can conclude that G is a split extension of K by L. Also, since  $\operatorname{Hom}(L,\operatorname{Aut}(T))$  is trivial, we have  $G \cong T \times L$ .

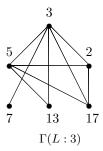
According to what we said before the proof, here we depict  $\Gamma(M)$  by |M| and  $\pi_e(M)$ , where M is an almost simple group related to  $L = D_4(4)$ .

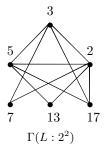


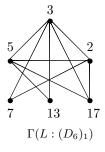




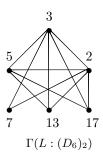


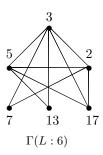


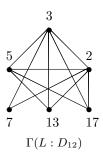




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## 4. Acknowledgments

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