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# The $SL_\Phi$-integral in Locally Convex Topological Vector Spaces

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Abstract. In this paper, we use the Minkowski functional to introduce an $SL$-type property or condition and then define a $SL$-type integral for a function taking values in a locally convex topological vector space (LCTVS). We show that this integral is equivalent to the $SH_1$ integral, a version of the Henstock-Kurzweil integral in a LCTVS.

**Keywords:** Strong Lusin condition, Strong Lusin integral, Strongly Henstock integral.

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## 1. Introduction

Jaroslav Kurzweil and Ralph Henstock independently introduced an integral that bears their name – the Henstock-Kurzweil integral or simply the $HK$-integral. This integral is a generalization of the Lebesgue integral but possesses the structural definition of the Riemann integral. Various studies of the integral in more abstract spaces have been done in the past decades. However, when extended to Banach spaces, the Henstock-Kurzweil integral no longer satisfies the well-known Henstock Lemma [2]. This shortcoming had led to the definition

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of a stronger version of the $HK$-integral (called $HL$-integral) for Banach-valued functions.

To be able to consider or define an integral that would be equivalent to the $HK$-integral, P.Y. Lee [12] introduced the concept of strong Lusin condition. This concept lies between the Lusin condition $N$ [7, 12] and the absolute continuity property. This property had been used by Lee and Vyborny [11] to introduce the $SL$-integral. The authors showed that this integral is indeed equivalent to the $HK$-integral. Recently, Maza and Canoy [5] also introduced an $SL$-type integral for locally convex topological vector space (LCTVS)-valued functions and showed that such an integral is equivalent to a version of the Henstock integral (called the $SH$-integral) in a LCTVS. In [6], the authors also defined and studied another integral for LCTVS-valued functions.

In this paper, we make use of the Minkowski's functional $\Phi$ to introduce an $SL_\Phi$ condition and define an $SL$-type integral. Specifically, we define the $SL_\Phi$-integral of a function taking values in a locally convex topological vector space. It will be shown, as one of our main results, that this integral is equivalent to the $SH_1$-integral, a version of the $HK$ integral in the LCTVS setting.

Recall that a **topological vector space** $X$ is real vector space together with a topology defined on it such that scalar multiplication and vector addition are continuous with respect to the topology and that every point of $X$ is closed [10]. Equivalently, $(X, \tau)$ is a topological vector space if $X$ is a real vector space and $\tau$ is Hausdorff topology on $X$ such that scalar multiplication and vector addition are continuous with respect to $\tau$. Continuity would then imply that for every open set $U$, there are open sets $V_1$ and $V_2$ such that $V_1 + V_2 \subseteq U$. More generally, for every $\theta$-nbd $U$ (an open set containing the zero vector $\theta$ of $X$) and $n \in \mathbb{N}$ there are $\theta$-nbds $V_1, V_2, \ldots, V_n$ such that $V_1 + V_2 + \cdots + V_n \subseteq U$ (see [4] and [10]). Note that $X$ being a Hausdorff space implies that only the zero vector $\theta$ is contained in all of the $\theta$-nbds.

A set $A \subseteq X$, where $X$ is a topological vector space, is **absorbing** if for every $x \in X$ there is $t > 0$ such that $x \in tA$; it is **convex** if for every $x, y \in A$ and $0 \leq t \leq 1$, $tx+(1-t)y \in A$; and it is **balanced** if $\alpha A \subseteq A$ for every $|\alpha| \leq 1$. The convex property can be extended as follows: for every $x_1, x_2, \ldots, x_n \in A$ and positive real numbers $\alpha_1, \alpha_2, \ldots, \alpha_n$ such that $\alpha_1 + \alpha_2 + \cdots + \alpha_n \leq 1$ we have $\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n \in A$. A topological vector space $X$ is said to be **locally convex** if there is a local base consisting of convex sets in $X$. It is known that every locally convex topological vector space has a local base at $\theta$ consisting of absorbing, balanced, and convex sets.

For a given set $A \subseteq X$, the **Minkowski functional** of $A$ on $X$ is defined by $\Phi_A(x) = \inf\{\lambda > 0 : x \in \lambda A\}$ for every $x \in X$. If $U \subseteq X$ is a balanced, absorbing and convex set, then $(i)$ $U = \{x \in X : \Phi_U(x) < 1\}$, and $(ii)$ $\Phi_U$ is a semi-norm on $X$, that is, $\Phi_U(u + v) \leq \Phi_U(u) + \Phi_U(v)$ for all $u, v \in X$ (sub-additivity), and $\Phi_U(ku) = |k|\Phi_U(u)$ for any real number $k$. Also, for any

$V \subseteq X$, $\Phi_{rV}(x) = \frac{1}{r}\Phi_V(x)$ for all positive real number $r$ and $x \in X$. For any given absorbing sets $A \subseteq B \subseteq X$, $\Phi_B(t) \leq \Phi_A(t)$ for all $t \in X$. One may refer to [10, 1] for the definition, the earlier mentioned results, and a detailed discussion of the Minkowski functional.

A function $\delta : [a, b] \to R$ is called a **tight gauge** if it takes on non-negative values and a **gauge** if it takes on positive values [12]. A finite collection of ordered pairs $\{(I_i, t_i)\}_{i=1}^n$ of non-overlapping closed intervals of $[a, b]$ and real numbers is called a **partial partition** (resp. **partition**) of $[a, b]$ if $\bigcup_{i=1}^n I_i \subseteq [a, b]$ (resp. $\bigcup_{i=1}^n I_i = [a, b]$). A collection $\{(I_i, t_i)\}_{i=1}^n$ is called a **$\delta$-fine partial partition** (**$\delta$-fine partition**) of $[a, b]$ if $\{(I_i, t_i)\}_{i=1}^n$ is a partial partition (resp. partition) of $[a, b]$ and $t_i \in I_i \subseteq (t_i - \delta(t_i), t_i + \delta(t_i))$ for each $i \in \{1, 2, \ldots, n\}$. We will occasionally denote a given $\delta$-fine partial partition (or partition) $D = \{([u_i, v_i], t_i) : 1, \leq i \leq n\}$ by $D = \{([u, v], t)\}$.

A unitary sequence is a sequence of positive numbers $\{r_i\}_{i=1}^n$ such that $\sum_{i=1}^n r_i = 1$. A function $f : [a, b] \to X$, where $X$ is a LCTVS, is $SH_1$-integrable (see [8]) if there exists a function $F : [a, b] \to X$ such that for any $\theta$-nbd $V$, there exists a gauge $\delta$ on $[a, b]$ such that for every $\delta$-fine partition $D = \{([x_{i-1}, x_i], t_i) : 1 \leq i \leq n\}$ of $[a, b]$, there exists a unitary sequence $\{r_i\}_{i=1}^n$ such that

$$F(x_i) - F(x_{i-1}) - f(t_i)(x_i - x_{i-1}) \in r_i V$$

for each $1 \leq i \leq n$. The difference $F(b) - F(a)$ is the $SH_1$ integral of $f$ on $[a, b]$ and write $(SH_1) \int_a^b f = F(b) - F(a)$.

A function $F : [a, b] \to X$ is said to satisfy the $SL_\Phi$ **condition** if given a subset $E$ of $[a, b]$ of measure zero, a $\theta$-nbd $U$, and $\varepsilon > 0$, there exists a gauge $\delta$ such that for any $\delta$-fine partial partition $D = \{([u, v], t)\}$ with $t_i \in E$, we have

$$(D) \sum \Phi_U(F(u, v)) < \varepsilon$$

where $F(u, v) = F(v) - F(u)$. A function satisfying the $SL_\Phi$ condition is called an $SL_\Phi$ function.

A function $f : [a, b] \to X$ is said to be **$SL_\Phi$-integrable** on $[a, b]$ if there exists a $SL_\Phi$ function $F$ with the property that for every $\theta$-nbd $U$ and $\varepsilon > 0$, there is a tight gauge $\delta$ such that for every $\delta$-fine partial partition $D = \{([u, v], t)\}$ of $[a, b]$, $(D) \sum \Phi_U(F(u, v) - f(t)(v - u)) < \varepsilon$. The vector $F(a, b)$ is the $SL_\Phi$ integral of $f$ and is denoted by $(SL_\Phi) \int_a^b f$. We call the function $F$ an **$SL_\Phi$-primitive** of $f$.

## 2. MAIN RESULTS

Throughout this section, $X$ is a locally convex topological vector space.

**Theorem 2.1.** *Let $c \in X$. Then the constant function $F : [a,b] \to X$ given by $F(t) = c$ is an $SL_\Phi$ function.*

*Proof.* Let $E \subseteq [a,b]$ be of measure zero, $U$ a $\theta$-nbd, and $\varepsilon > 0$. Choose any gauge $\delta$ on $[a,b]$. Then for any $\delta$-fine partial partition $D = \{([u,v],t)\}$ of $[a,b]$ with $t \in E$,

$$(D)\sum \Phi_U(F(u,v)) = \sum_{i=1}^{n} \Phi_U(c-c) = \sum_{i=1}^{n} \Phi_U(\theta) = 0 < \varepsilon.$$

□

**Theorem 2.2.** *Let $F, G : [a,b] \to X$ be $SL_\Phi$ functions and $c \in R$. Then $cF$ and $F+G$ are $SL_\Phi$ functions.*

*Proof.* By Theorem 2.1, $cF$ is a $SL_\Phi$ function if $c = 0$. Suppose $c \neq 0$ and let $E \subset [a,b]$ be a set of measure zero, $V$ a $\theta$-nbd, and $\varepsilon > 0$. Then there is an absorbing, balanced and convex $\theta$-nbd $U$ with $U \subseteq V$. By assumption, there exists a gauge $\delta_1$ such that for any $\delta_1$-fine partial partition $D = \{([u,v],t)\}$ with $t_i \in E$, we have

$$(D)\sum \Phi_U(F(u,v)) < \frac{1}{|c|}\varepsilon.$$

Hence,

$$\begin{aligned}(D)\sum \Phi_V(cF(v) - cF(u)) &\leq (D)\sum \Phi_U(cF(v) - cF(u))\\ &= (D)\sum \Phi_U\left(c(F(u,v))\right)\\ &= (D)\sum |c|\Phi_U\left(F(u,v)\right) < \varepsilon.\end{aligned}$$

This proves that $cF$ is $SL_\Phi$.

For the second part, again let $E \subset [a,b]$ be a given set of measure zero, $V$ a $\theta$-nbd, and $\varepsilon > 0$. Let $U \subseteq V$ be an absorbing, balanced and convex $\theta$-nbd. Since $F$ and $G$ are $SL_\Phi$ functions, there is a common gauge $\delta$ on $[a,b]$ such that for any $\delta$-fine partial partition $D = \{([u,v],t)\}$ of $[a,b]$ with $t_i \in E$,

$$(D)\sum \Phi_U(F(u,v)) < \frac{1}{2}\varepsilon \quad \text{and} \quad \sum_{i=1}^{n} \Phi_U(G(u,v)) < \frac{1}{2}\varepsilon$$

It follows that

$$\begin{aligned}(D)\sum \Phi_V((F+G)(u,v)) &\leq (D)\sum \Phi_U((F+G)(u,v))\\ &\leq (D)\sum \Phi_U(F(u,v)) + (D)\sum \Phi_U(G(u,v)) \leq \varepsilon.\end{aligned}$$

Therefore, $F+G$ satisfies the $SL_\Phi$ condition. □

**Theorem 2.3.** *Let $F : [a,b] \to X$ be an $SL_\Phi$ function. Then the restriction $F|_{[c,d]}$ of $F$ to $[c,d] \subseteq [a,b]$ is also a $SL_\Phi$ function. function.*

*Proof.* Let $U$ be a $\theta$-nbd, $\varepsilon > 0$, and $E \subseteq [c,d]$ be of measure zero. Since $F$ is a $SL_\Phi$ function on $[a,b]$, there exists a gauge $\delta$ on $[a,b]$ such that for any $\delta$-fine partial partition $D = \{([u,v],t)\}$ of $[a,b]$ with $t_i \in E$, we have

$$(D)\sum \Phi_U(F(u,v)) < \varepsilon.$$

Let $\delta_0$ be the restriction of $\delta$ to $[c,d]$. Suppose $D' = \{([u',v'],t')\}$ with $t'_i \in E$ is a $\delta_0$-fine partial partition of $[c,d]$. Then $D'$ is a $\delta$-fine partial partition of $[a,b]$. Hence,

$$(D')\sum \Phi_U(F(u',v')) < \varepsilon.$$

Therefore, $F|_{[c,d]} : [c,d] \to X$ is an $SL_\Phi$ function. □

**Theorem 2.4.** *Let $F : [a,b] \to X$ be a function and let $c \in (a,b)$. Suppose the restrictions of $F$ to $[a,c]$ and $[c,b]$ are $SL_\Phi$ functions on $[a,c]$ and $[c,b]$, respectively. Then $F$ is an $SL_\Phi$ function on $[a,b]$.*

*Proof.* Let $E \subset [a,b]$ be of measure zero, $W$ a $\theta$-nbd, and $\varepsilon > 0$. Let $U \subseteq W$ be an absorbing, balanced and convex $\theta$-nbd. We may assume that $a < c < b$. Then $E \cap [a,c]$ and $E \cap [c,b]$ are both of measure zero. Since $F$ is a $SL_\Phi$ function on $[a,c]$ and $[c,b]$ there are gauges $\delta_1$ on $[a,c]$ and $\delta_2$ on $[c,b]$ that satisfies the $SL_\Phi$-condition with respect to $\varepsilon$ and $U$. Define a function $\delta : [a,b] \to R^+$ by

$$\delta(t) = \begin{cases} \min\{\delta_1(t), c-t\} & t \in [a,c) \\ \min\{\delta_2(t), t-c\} & t \in (c,b] \\ \min\{\delta_1(c), \delta_2(c)\} & t = c. \end{cases}$$

Let $D = \{([u_i,v_i],t_i) : 1 \le i \le n\}$ be a $\delta$-fine partial partition of $[a,b]$.
**Case 1.** Suppose either $c = u_k$ or $c = v_k$ for some $k \in \{1,2,\ldots,n\}$ or $c \notin [u,v]$ for all $([u,v],t) \in D$. We write $D = D_1 \cup D_2$ where $D_1$ contains subintervals of $[a,c]$ and $D_2$ contains subintervals of $[c,d]$. Then $D_1$ is $\delta_1$-fine partial partition of $[a,c]$ and $D_2$ is $\delta_2$-fine partial partition of $[c,b]$. Hence,

$$(D_1)\sum \Phi_U(F(u,v)) < \frac{\varepsilon}{2} \quad \text{and} \quad (D_2)\sum \Phi_U(F(u,v)) < \frac{\varepsilon}{2}$$

It follows that

$$(D)\sum \Phi_W(F(u,v) \le (D)\sum_{i=1}^{n} \Phi_U(F(u,v)) < \varepsilon.$$

**Case 2.** Suppose $c \in (u_k, v_k)$. By the definition of $\delta$, $c = t_k$. A refinement of $D'$ of $D$ can be expressed as $D' = D_1 \cup D_2$ where $D_1$ a $\delta_1$-fine partial partition of $[a,c]$ and $D_2$ a $\delta_2$-fine partial partition of $[c,b]$. Hence,

$$(D_1)\sum \Phi_U(F(v)-F(u)) < \frac{\varepsilon}{2} \text{ and } (D_2)\sum \Phi_U(F(v)-F(u)) < \frac{\varepsilon}{2}.$$

Therefore,

$$(D)\sum \Phi_W(F(u,v)) \le (D)\sum \Phi_U(F(u,v)) \le (D_1)\sum \Phi_U(F(u,v)) + (D_2)\sum \Phi_U(F(u,v)) < \varepsilon.$$

Accordingly, $F$ satisfies the $SL_\Phi$ condition on $[a,b]$. □

### 2.1. **The Φ-Strong Lusin integral.**

**Theorem 2.5.** *Let $f : [a,b] \to X$ be $SL_\Phi$-integrable with $SL_\Phi$-primitive $F$. For any $[c,d] \subseteq [a,b]$, the restriction $f|_{[c,d]} : [c,d] \to X$ is $SL_\Phi$-integrable with $SL_\Phi$-primitive $F|_{[c,d]} : [c,d] \to X$.*

*Proof.* Suppose $f$ is $SL_\Phi$-integrable with $SL_\Phi$-primitive $F$ on $[a,b]$. Let $U$ be a $\theta$-nbd and $\varepsilon > 0$. Then there is a tight gauge $\delta_0$ on $[a,b]$ associated to $f$, $U$, and $\epsilon$ in the definition of the $SL_\Phi$ integral. The restriction of $F$ on $[c,d]$ satisfies the $SL_\Phi$ condition by Theorem 2.3. Let $\delta$ be the restriction of $\delta_0$ on $[c,d]$ and let $D$ be $\delta$-fine partial partition of $[c,d]$. Then $D$ is a $\delta_0$-fine partial partition of $[a,b]$. Hence,

$$(D)\sum \Phi_U(F(u,v) - f(t)(v-u)) < \varepsilon,$$

showing that $f|_{[c,d]}$ is $SL_\Phi$-integrable on $[c,d]$ with $SL_\Phi$-primitive $F|_{[c,d]}$ : $[c,d] \to X$. □

**Theorem 2.6.** *Let $f : [a,b] \to X$ be $SL_\Phi$-integrable on $[a,b]$. Suppose $F$ and $G$ are $SL_\Phi$-primitives of $f$. Then $F(c,d) = G(c,d)$ for every $[c,d] \subseteq [a,b]$. In particular, the $SL_\Phi$-integral of $f$ is unique.*

*Proof.* Let $[c,d] \subseteq [a,b]$, $V$ a $\theta$-nbd, and $\varepsilon > 0$. Let $U$ be an absorbing, balanced and convex $\theta$-nbd with $U \subseteq V$. Since $f$ is $SL_\Phi$-integrable on $[c,d]$, there exists a tight gauge $\delta_0$ such that for a $\delta_0$-fine partial partition $D'$ of $[c,d]$, we have

$$(D')\sum \Phi_U(F_1(u,v) - f(t)(v-u)) < \frac{\varepsilon}{3} \quad \text{and}$$
$$(D')\sum \Phi_U(f(t)(v-u) - G_1(u,v)) < \frac{\varepsilon}{3}.$$

Let $N_{\delta_0} = \{t \in [a,b] : \delta_0(t) = 0\}$. Then $m^*(N_{\delta_0}) = 0$. Now, $F_1 = F|_{[c,d]}$ and $G_1 = G|_{[c,d]}$ are $SL_\Phi$-primitives of $f|_{[c,d]}$ by Theorem 2.5. Thus, $H = F_1 - G_1$ is a $SL_\Phi$-function on $[c,d]$ by Theorem 2.2. Hence, there exists a gauge $\delta_1$ on $[a,b]$ such that for any $\delta_1$-fine partial partition $D'' = \{([u,v],t)\}$ with $t \in N_{\delta_0}$, $(D'')\sum \Phi_U(H(u,v)) < \frac{\varepsilon}{3}$. Define $\delta(t) = \delta_0(t)$ if $t \notin N_{\delta_0}$ and $\delta(t) = \delta_1(t)$ if $t \in N_{\delta_0}$. Then $\delta$ is a gauge on $[a,b]$. Let $D$ be a $\delta$-fine partition of $[a,b]$. Let $D_0 = \{([u,v],t) \in D : t \notin N_{\delta_0}\}$ and $D_1 = D \setminus D_0$. Then $D_0$ is a $\delta_0$-fine partial partition of $[a,b]$ and $D_1$ is a $\delta_1$-fine partial partition of $[a,b]$. Consequently,

$$\begin{aligned}\Phi_V(F_1(c,d)-G_1(c,d)) &\le \Phi_U(F_1(c,d)-G_1(c,d))\\ &= \Phi_U((D)\sum(F_1(u,v)-G_1(u,v)))\\ &\le \Phi_U g[(D_0)\sum(F_1(u,v)-f(t)(v-u))\\ &\qquad + (D_0)\sum(f(t)(v-u)-G_1(u,v))\\ &\qquad\quad + (D_1)\sum(F_1(u,v)-G_1(u,v))g]\\ &< \frac{\varepsilon}{3}+\frac{\varepsilon}{3}+(D_1)\sum(H(u,v)) \le \varepsilon.\end{aligned}$$

Since $V$ and $\varepsilon$ were arbitrarily chosen, $F(c,d)=G(c,d)$. □

**Theorem 2.7.** *Let $f,g:[a,b]\to X$ be $SL_\Phi$-integrable functions and $c\in R$. Then each of the following holds:*

*$(i)$ $cf$ is $SL_\Phi$-integrable on $[a,b]$ and*

$$(SL_\Phi)\int_a^b c\cdot f = c\cdot(SL_\Phi)\int_a^b f \text{ and,}$$

*$(ii)$ $f+g$ is $SL_\Phi$-integrable on $[a,b]$ and*

$$(SL_\Phi)\int_a^b (f+g) = (SL_\Phi)\int_a^b f + (SL_\Phi)\int_a^b g.$$

*Proof.* $(i)$ Let $F$ be a $SL_\Phi$-primitive of $f$. We may assume that $c\neq 0$. Let $V$ be a given $\theta$-nbd and $\varepsilon>0$. Then there is an absorbing, balanced and convex $\theta$-nbd $U\subseteq V$. Thus, there is a tight gauge $\delta>0$ such that for any $\delta$-fine partial partition $D=\{([u,v],t)\}$, we have

$$(D)\sum\Phi_U(F(u,v)-f(t)(u,v)) < \frac{\varepsilon}{|c|}.$$

Thus,

$$(D)\sum\Phi_V((cF)(u,v)-cf(t)(v-u)) \le \sum_{i=1}^{n}\Phi_U(c(F(u,v)-f(t)(v-u))) < \varepsilon.$$

This and Theorem 2.2 would imply that $cf$ is $SL_\Phi$-integrable with $SL_\Phi$-primitive $cF$ on $[a,b]$. Moreover,

$$(SL_\Phi)\int_a^b c\cdot f = (cF)(a,b) = c(F(a,b)) = c\cdot(SL_\Phi)\int_a^b f.$$

$(ii)$ Let $F$ and $G$ be $SL_\Phi$-primitives for the functions $f$ and $g$, respectively. Let $V$ be a $\theta$-nbd and $\varepsilon>0$. Let $U$ be an absorbing, balanced and convex $\theta$-nbd with $U\subseteq V$. Let $\delta_1$ and $\delta_2$ be tight gauges on $[a,b]$ associated $f$, $U$ and $\frac{\varepsilon}{2}$, and $g$, $U$ and $\frac{\varepsilon}{2}$, respectively, in the definition of the $SL_\Phi$ integral. Define $\delta(t)=\min\{\delta_1(t),\delta_2(t)\}$. Then $\delta$ is a tight gauge $[a,b]$. Let $D=\{([u,v],t)\}$ be a $\delta$-fine partial partition of $[a, b]$. Note that $D$ is both a $\delta_1$-fine and a $\delta_2$-fine partial partition of $[a, b]$. Hence,

$$(D)\sum \Phi_U(F(u,v) - f(t)(v-u)) < \frac{\varepsilon}{2}$$
$$(D)\sum \Phi_U(G(u,v) - g(t)(v-u)) < \frac{\varepsilon}{2}.$$

Thus,

$$\begin{aligned}(D)\sum \Phi_V((F+G)(u,v) &- (f+g)(t)(u,v)) \\ &\leq (D)\sum \Phi_V((F+G)(u,v) - (f+g)(t)(u,v)) \\ &\leq (D)\sum \Phi_U(F(u,v) - f(t)(u,v)) \\ &\quad + (D)\sum \Phi_U(G(u,v) - g(t)(u,v)) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

Thus, $f+g$ is $SL_\Phi$-integrable with $SL_\Phi$-primitive $F+G$ and

$$\begin{aligned}(\Phi\text{-}SL)\int_a^b (f+g) &= (F+G)(a,b) \\ &= F(a,b) + G(a,b) \\ &= (\Phi\text{-}SL)\int_a^b f + (\Phi\text{-}SL)\int_a^b g.\end{aligned}$$

□

**Theorem 2.8.** *Let $f : [a, b] \to X$ be a function and $c \in (a, b)$. Suppose the restrictions of $f$ on $[a, c]$ and $[c, b]$ are $SL_\Phi$-integrable on $[a, c]$ and $[c, b]$, respectively. Then $f$ is $SL_\Phi$-integrable on $[a, b]$ and*

$$(SL_\Phi)\int_a^b f = (SL_\Phi)\int_a^c f + (SL_\Phi)\int_c^b f.$$

*Proof.* Let $F_1 : [a, c] \to X$ and $F_2 : [c, b] \to X$ be $SL_\Phi$-primitives of the restrictions of $f$ on $[a, c]$ and $[c, b]$, respectively. Define $F : [a, b] \to X$ by

$$F(t) = \begin{cases} F_1(t) & \text{for } t \in [a,c] \\ F_2(c,t) + F_1(c) & \text{for } t \in [c,b] \end{cases}$$

The restrictions of $F(t)$ to $[a, c]$ and $[c, b]$ are $F_1(t)$ and $G(t) = F_2(c, t) + F_1(c)$, respectively. By Theorem 2.2 and Theorem 2.3, $G$ is a $SL_\Phi$ function on $[c, b]$. Thus, $F$ is a $SL_\Phi$ function on $[a, b]$ by Theorem 2.4.

It now remains to show that $F$ is a $SL_\Phi$-primitive of $f$ so that $f$ is $SL_\Phi$-integrable on $[a, b]$. To this end, let $V$ be a $\theta$-nbd and $\varepsilon > 0$. Choose any absorbing, balanced, and convex $\theta$-nbd $U \subseteq V$. Let $\delta_1$ and $\delta_2$ be tight gauges associated with the restrictions $f|_{[a,c]}$ and $f|_{[c,b]}$, $U$, and $\frac{\varepsilon}{2}$ in the definition of $SL_\Phi$ integral. Define $\delta$ as follows:

$$\delta(t) = \begin{cases} \min\{\delta_1(t), c-t\} & \text{for } t \in [a,c) \\ \min\{\delta_1(c), \delta_2(c)\} & \text{for } t = c \\ \min\{\delta_2(t), t-c\} & \text{for } t \in (c,b] \end{cases}$$

Consider any $\delta$-fine partial partition $D = \{([u,v],t)\}$ of $[a,b]$.

**Case 1.** Suppose $c = u_k$ or $c = v_k$ for some $k \in \{1,2,\dots,n\}$ or $c \notin [u_i, v_i]$ for all $i \in \{1,2,\dots,n\}$. Then $D$ is a disjoint union of $D_1$ and $D_2$ where the elements in $D_1$ are tagged intervals contained in $[a,c]$ and the elements in $D_2$ are tagged intervals contained in $[c,b]$. Then $D_1$ is $\delta_1$-fine partial partition of $[a,c]$ and $D_2$ is $\delta_2$-fine partial partition of $[c,b]$. Hence,

$$(D_1)\sum \Phi_U(F_1(u,v) - f(t)(v-u)) < \frac{\varepsilon}{2}$$
$$(D_2)\sum \Phi_U(F_2(u,v) - f(t)(v-u)) < \frac{\varepsilon}{2}.$$

It follows that

$$\begin{aligned}(D)\sum \Phi_U(F(u,v) - f(t)(v-u)) \\ &= (D_1)\sum \Phi_U(F_1(u,v) - f(t)(v-u)) \\ &\quad + (D_2)\sum \Phi_U(G(u,v) - f(t)(v-u)) \\ &= (D_1)\sum \Phi_U(F_1(u,v) - f(t)(v-u)) \\ &\quad + (D_2)\sum \Phi_U(F_2(u,v) - f(t)(v-u)) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

**Case 2.** Suppose $c \in (u_k, v_k)$. Then $t_k = c$. Let $D' = D_1 \cup D_2$ be a refinement of $D$ where $D_1 = \{([u_i,v_i],t_i) : 1 \le i \le k-1\} \cup \{([u_k,c],c)\}$ and $D_2 = \{([u_i,v_i],t_i) : k+1 \le i \le n\} \cup \{([c,v_k],c)\}$. Then $D_1$ is a $\delta_1$-fine partial partition of $[a,c]$ and $D_2$ is a $\delta_2$-fine partial partition of $[c,b]$. Hence,

$$\Phi_U(F_1(u_k,c) - f(c)(c-u_k)) + (D_1)\sum \Phi_U(F_1(u,v) - f(t)(v-u)) < \frac{\varepsilon}{2}$$

and

$$\Phi_U(F_2(c,v_k) - f(c)(v_k-c)) + (D_2)\sum \Phi_U(F_2(u,v) - f(t)(v-u)) < \frac{\varepsilon}{2}.$$

Since

$$F(u_k,v_k) - f(t_k)(v_k-u_k) = F_2(u_k,v_k) - f(c)(v_k-c) + F_1(u_k,v_k) - f(c)(c-u_k),$$

it follows that

$$\begin{aligned}&(D)\sum\Phi_U(F(u,v)-f(t)(v-u))\\&\le(D_1)\sum\Phi_U(F(u,v)-f(t)(v-u))+(D_2)\sum\Phi_U(F(u,v)-f(t)(v-u))\\&\qquad+\Phi_U(F(u_k,v_k)-f(t_k)(v_k-u_k))\\&<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.\end{aligned}$$

In either case, we have

$$(D)\sum\Phi_V(F(u,v)-f(t)(v-u))\le(D)\sum\Phi_U(F(u,v)-f(t)(v-u))<\varepsilon.$$

Hence, $F$ is a $SL_\Phi$-primitive of $f$ on $[a,b]$ and

$$(SL_\Phi)\int_a^b f=F(a,b)=F(a,c)+F(c,b)=(SL_\Phi)\int_a^c f+(SL_\Phi)\int_c^b f.$$

□

**Corollary 2.9.** *Let $f:[a,b]\to X$ be $SL_\Phi$-integrable on $[a,b]$ with $SL_\Phi$-primitive $F$. For any $c\in[a,b]$, we have*

$$(SL_\Phi)\int_a^b f=(SL_\Phi)\int_a^c f+(SL_\Phi)\int_c^b f.$$

In what follows, $\Delta_\Phi(U,F,f)$ is given by

$$\begin{aligned}\Delta_\Phi(U,F,f)=\{t\in[a,b]:\forall\delta>0,\exists[u,v]\subseteq[a,b]\text{ with }t\in[u,v]\text{ and}\\|v-u|<\delta,\ \Phi_U\left(F(v)-F(u)-f(t)(v-u)\right)\ge v-u\}.\end{aligned}$$

**Theorem 2.10.** *Let $f,F:[a,b]\to X$ be functions where $F$ is a $SL_\Phi$-function. If $\Delta_\Phi(U,F,f)$ is of measure zero for each $\theta$-nbd $U$, then $f$ is $SL_\Phi$-integrable with $SL_\Phi$-primitive $F$.*

*Proof.* Let $V$ be a given $\theta$-nbd and $\varepsilon>0$. Then there is an absorbing, balanced and convex $\theta$-nbd $U$ such that $U\subseteq V$. If $t\notin\Delta_\Phi(\frac{\varepsilon}{b-a}U,F,f)$, then there is a real number $\delta_0(t)>0$ such that $\Phi_{\frac{\varepsilon}{b-a}U}(F(u,v)-f(t)(v-u))<1$ whenever $|v-u|<\delta_0(t)$ and $t\in[u,v]$ or equivalently, $\Phi_U(F(u,v)-f(t)(v-u))<\frac{\varepsilon(v-u)}{b-a}$ whenever $|v-u|<\delta_0(t)$ and $t\in[u,v]$. Define

$$\delta(t)=\begin{cases}0 & \text{if } t\in\Delta_\Phi(\frac{\varepsilon}{b-a}U,F,f)\\ \frac{\delta_0(t)}{2} & \text{if } t\notin\Delta_\Phi(\frac{\varepsilon}{b-a}U,F,f)\end{cases}$$

Then $\delta(t)$ is a tight gauge because $\Delta_\Phi(\frac{\varepsilon}{b-a}U,F,f)$ has measure zero. Consider a given $\delta$-fine partial partition $D=\{([u,v],t)\}$ of $[a,b]$. Suppose there exists $([u,v],t)\in D$ such that $\delta(t)=0$. Then $t\in[u,v]\subseteq(t-\delta(t),t+\delta(t))=\varnothing$, a contradiction. Hence, $\delta(t)\neq 0$ for all $([u,v],t)\in D$. This implies that for each $([u,v],t)\in D$,

$$\Phi_U(F(u,v)-f(t)(v-u))<\frac{\varepsilon(v-u)}{b-a}.$$

Thus, we have

$$\begin{aligned}(D)\sum \Phi_V(F(u,v)-f(t)(v-u)) &\leq (D)\sum \Phi_U(F(u,v)-f(t)(v-u)) \\ &< (D)\sum \left(\frac{\varepsilon(v-u)}{b-a}\right) \\ &\leq \varepsilon\frac{b-a}{b-a}=\varepsilon.\end{aligned}$$

Therefore, $f$ is $SL_\Phi$-integrable with $SL_\Phi$-primitive $F$. ☐

Let $F:[a,b]\to X$ be a function and $t\in[a,b]$. Then $F$ is **differentiable** at $t$ ($F'(t)$ is the derivative $F$ at $t$) if for every $\theta$-nbd $U$, there is a $\delta>0$ for which $F(v)-F(u)-F'(t)(v-u)\in(v-u)U$ whenever $t\in[u,v]\subseteq[a,b]$ and $|v-u|<\delta$ (see [9]).

**Theorem 2.11.** *Let $f,F:[a,b]\to X$ be functions and $\mathcal{D}(F,f)$ be the set of all $t\in[a,b]$ such that $F'(t)$ does not exist or $F'(t)\neq f(t)\}$. Then*

$$\mathcal{D}(F,f)=\bigcup_{\theta\text{-}nbd\ U}\Delta_\Phi(U,F,f).$$

*Proof.* If $t\in\mathcal{D}(F,f)$, then there is a $\theta$-nbd $U$ such that for each $\delta>0$, there is $[u,v]\subseteq(t-\delta,t+\delta)$ containing $t$ for which $\Phi_U(F(u,v)-f(t)(v-u))>u-v$. Equivalently, $t\in\Delta_\Phi(U,F,f)$. Thus,

$$\mathcal{D}(F,f)\subseteq\bigcup_{\theta\text{-nbd}\ U}\Delta_\Phi(U,F,f).$$

Next, let $t\in\bigcup_{\theta\text{-nbd}\ U}\Delta_\Phi(U,F,f)$. Then there exists a $\theta$-nbd $U$ such that $t\in\Delta_\Phi(U,F,f)$. Let $V$ be an absorbing, balanced, and convex set such that $V\subseteq U$. Then for all $\delta>0$, there exists $[u,v]\subseteq[a,b]$ with $t\in[u,v]$ and $(v-u)<\delta$ such that

$$\Phi_V(F(u,v)-f(t)(v-u))\geq\Phi_U(F(u,v)-f(t)(v-u))\geq(v-u).$$

Since $V$ is absorbing, balanced and convex, it follows that $F(u,v)-f(t)(v-u)\notin(v-u)V$. Thus, $t\in\mathcal{D}(F,f)$, showing that

$$\bigcup_{\theta\text{-nbd}\ U}\Delta_\Phi(U,F,f)\subseteq\mathcal{D}(F,f).$$

This proves the desired equality. ☐

Example 2.12. The zero function $f:[a,b]\to X$ is $SL_\Phi$-integrable with $SL_\Phi$-primitive $F:[a,b]\to X$ given by the constant function $F=\alpha$ where $\alpha$ is any vector in $X$. In fact, for any $\theta$-nbd $U$, $\Delta_\Phi(U,F,f)=\emptyset$. Indeed, if there is a $t\in\Delta(U,F,f)$, then for all $\delta>0$, there exists $[u,v]\subseteq[a,b]$ containing $t$ with $(v-u)<\delta$ and $F(v)-F(u)-f(t)(v-u)\notin(v-u)U$. However, we see that

$F(v) - F(u) - f(t)(v-u) = \alpha - \alpha - \theta(v-u) \in (v-u)U$, a contradiction. By Theorem 2.10, $f$ is $SL_\Phi$-integrable with $SL_\Phi$-primitive $F = \alpha$ and

$$(SL_\Phi)\int_a^b f = F(b) - F(a) = \theta.$$

**Theorem 2.13.** *Let $F : [a,b] \to X$ be a $SL_\Phi$-function. Suppose that $F'(x) = \theta$ almost everywhere on $[a,b]$. Then $F$ is a constant function.*

*Proof.* Let $f : [a,b] \to X$ be the zero function. Since $F'(x) = \theta = f(x)$ almost everywhere on $[a,b]$, $\mathcal{D}(F,f)$ has measure zero. Thus, $\Delta_\Phi(U,F,f)$ is of measure zero for each $\theta$-nbd $U$. By Theorem 2.10 and by Example 2.12,

$$\theta = (SL_\Phi)\int_a^b f = F(b) - F(a).$$

Hence, $F(b) = F(a)$. We know that $F$ is a $SL_\Phi$ function on $[a,x]$ by Theorem 2.3 and that $F'(x) = \theta$ almost everywhere on $[a,x]$ for all $x \in (a,b]$. By replacing $b$ with $x \in (a,b]$ in the proof above, we have $F(x) = F(a)$. Therefore, $F$ is a constant function. □

**Theorem 2.14.** *A function $f : [a,b] \to X$ is $SH_1$ integrable on $[a,b]$ if and only if there exists a function $F : [a,b] \to X$ with the property that for every $\theta$-nbd $U$ and $\varepsilon > 0$, there is a gauge $\delta$ on $[a,b]$ such that for every $\delta$-fine partition $D = \{([u,v],t)\}$ we have*

$$(D)\sum \Phi_U(F(u,v) - f(t)(v-u)) < \varepsilon.$$

*In this case,*

$$(SH_1)\int_a^b f = F(a,b).$$

*Proof.* Let $f : [a,b] \to X$ be $SH_1$-integrable on $[a,b]$ and let $F$ be its $SH_1$-primitive. Let $U$ be a $\theta$-nbd and $\varepsilon > 0$. Then there is an absorbing, balanced and convex $\theta$-nbd $V \subseteq U$. By the definition of $SH_1$-integral applied to $\varepsilon V$, there is a gauge $\delta$ on $[a,b]$ such that for every $\delta$-fine partition $D = \{([u_i,v_i],t_i) : 1 \le i \le n\}$ of $[a,b]$, there is a unitary sequence $\langle r_i\rangle_{i=1}^n$ such that

$$F(u_i,v_i) - f(t_i)(v_i - u_i) \in r_i \varepsilon V$$

for each $i \in \{1,2,\cdots,n\}$. Since $V$ is balanced, convex and absorbing,

$$\Phi_V(F(u_i,v_i) - f(t_i)(v_i-u_i)) \le \Phi_V(F(u_i,v_i) - f(t_i)(v_i-u_i)) < r_i\varepsilon$$

for each $i \in \{1,2,\cdots,n\}$. Hence,

$$(D)\sum \Phi_U(F(u,v) - f(t)(v-u)) < \sum_{i=1}^n r_i\varepsilon = \varepsilon.$$

Conversely, suppose $f$ and $F$ satisfy the condition. Let $U$ be a $\theta$-nbd. Then by assumption there is a gauge $\delta$ on $[a,b]$ such that for every $\delta$-fine partition $D = \{([u,v],t)\}$ we have

$$(D)\sum \Phi_U(F(u,v) - f(t)(v-u)) < 1.$$

Choose $\alpha > 0$ so that $\alpha + (D)\sum \Phi_U(F(v) - F(u) - f(t)(v-u)) = 1$ and let $r_i = \frac{\alpha}{n} + \Phi_U(F(v_i) - F(u_i) - f(t_i)(v_i - u_i))$ for each $i \in \{1, 2, \cdots, n\}$. Then $\sum_{i=1}^n r_i = 1$ and

$$F(u_i, v_i) - f(t_i)(v_i - u_i) \in r_i U \text{ for each } i \in \{1, 2, \cdots, n\}.$$

Therefore, $f$ is $SH_1$ integrable with $SH_1$ primitive $F$. □

**Theorem 2.15.** *Let $f : [a,b] \to X$. If $\Phi_U \circ f = 0$ almost everywhere on $[a,b]$ for each $\theta$-nbd $U$, then $f$ is $SH_1$ integrable and $(SH_1)\int_a^b f = \theta$. In particular, if $f(t) = \theta$ almost everywhere on $[a,b]$, then $f$ is $SH_1$ integrable and $(SH_1)\int_a^b f = \theta$.*

*Proof.* Let $F : [a,b] \to X$ with $F(t) = \theta$ for all $t \in [a,b]$. Let $V$ be a $\theta$-nbd and let $\varepsilon > 0$. Choose any balanced, convex and absorbing $\theta$-nbd $U \subseteq V$. Let $S = \{t \in [a,b] : \Phi_U(f(t)) \neq 0\}$ and $E_k = \{t \in S : k-1 < \Phi_U(f(t)) \leq k\}$ for each positive integer $k$. Then $m(S) = 0$ and $m(E_k) = 0$ for each $k > 0$. Thus, there exists an open set $G_k$ such that $E_k \subseteq G_k$ and $m(G_k) < \frac{\varepsilon}{k2^k}$. Also, $\{E_i\}_{i=1}^\infty$ is a pairwise disjoint collection whose union is $S$. Define $\delta(t) = 1$ if $t \in [a,b] \setminus S$ and $\delta(t) > 0$ be a real number such that $t \in (t - \delta(t), t + \delta(t)) \subseteq G_k$ if $t \in E_k$. Let $D = \{([u,v],t)\}$ be a $\delta$-fine partition of $[a,b]$. Let $D_0 = \{([u,v],t) \in D : t \in [a,b] \setminus S\}$ and let $D_k = \{([u,v],t) \in D : t \in E_k\}$ for each $k > 0$. Since $U$ is convex, $\Phi_U(f(t)) < j$ for each $t \in E_j$. Also, since $\bigcup\{[u_i, v_i] : t_i \in E_j\} \subseteq G_j$ for integer $j > 0$, $\sum_{t_i \in E_j}(v_i - u_i) \leq m(G_j) < \frac{\varepsilon}{j2^j}$. Hence,

$$\begin{aligned}
&(D)\sum \Phi_V(F(u,v) - f(t)(v-u)) \leq (D)\sum \Phi_U(F(u,v) - f(t)(v-u)) \\
&= (D_0)\sum \Phi_U(F(u,v) - f(t)(v-u)) + (D \setminus D_0)\sum \Phi_U(F(u,v) - f(t)(v-u)).
\end{aligned}$$

Since $f(t) = \theta$ for all $t \notin S$ and $F(t) = \theta$ for all $t \in [a,b]$, we have

$$\begin{aligned}
(D)\sum \Phi_V(F(u,v) - f(t)(v-u)) &\leq 0 + \sum_{j=1}^\infty \sum_{t \in E_j} (v_i - u_i)\Phi_U(-f(t_i)) \\
&\leq \sum_{j=1}^\infty \sum_{t_i \in E_j} (v_i - u_i) j < \sum_{j=1}^\infty \frac{\varepsilon}{j2^j} j = \varepsilon.
\end{aligned}$$

By Theorem 2.14, $f$ is $SH_1$ integrable with $SH_1$ primitive $F$ and

$$(SH_1)\int_a^b f = F(b) - F(a) = \theta.$$

The second assertion directly follows. □

**Theorem 2.16.** *Let $f : [a,b] \to X$ be $SL_\Phi$-integrable on $[a,b]$ with $SL_\Phi$-primitive $F$. Then $f$ is $SH_1$-integrable on $[a,b]$ with $SH_1$ primitive $F$ and*

$$(SH_1)\int_a^b f = (SL_\Phi)\int_a^b f.$$

*Proof.* Let $V$ be a $\theta$-nbd and let $\varepsilon > 0$. Let $U$ be an absorbing, balanced, and convex $\theta$-nbd with $U \subseteq V$. Since $f$ is $SL_\Phi$-integrable with $SL_\Phi$-primitive $F$, there is a tight gauge $\delta_1$ such that for any $\delta_1$-fine partial partition $D_1 = \{([u,v],t)\}$ of $[a,b]$,

$$(D_1)\sum \Phi_U(F(u,v) - f(t)(v-u)) < \frac{\varepsilon}{3}.$$

Let $Z = \{t \in [a,b] : \delta_1(t) = 0\}$. Then $Z$ has measure zero. Define $f_0 = f \cdot 1_Z$. Then $f_0$ is zero almost everywhere on $[a,b]$ and hence, $f_0$ is $SH_1$-integrable with primitive $F_0 : [a,b] \to X$ given by $F_0 = \theta$ by Theorem 2.15. Thus by Theorem 2.14, there exists a gauge $\delta_2$ such that for any $\delta_2$-fine partition $D_2 = \{([u,v],t)\}$ of $[a,b]$,

$$(D_2)\sum \Phi_U(-f_0(t)(v-u)) < \frac{\varepsilon}{3}.$$

In particular, if $D_2 = \{([u,v],t)\}$ is a $\delta_2$-fine partition of $[a,b]$ with $t \in Z$ for each $([u,v],t) \in D_2$, then

$$(D_2)\sum \Phi_U(-f(t)(v-u)) < \frac{\varepsilon}{3}.$$

Since $F$ is a $SL_\Phi$ function, there is a gauge $\gamma$ such that for every $\gamma$-fine partial partition $D_3 = \{([u,v],t)\}$ of $[a,b]$ with $t \in Z$,

$$(D_3)\sum \Phi_U(F(u,v)) < \frac{\varepsilon}{3}.$$

Define

$$\lambda(t) = \begin{cases} \delta_1(t) & \text{if } t \in [a,b] \setminus Z \\ \min\{\delta_2(t), \gamma(t)\} & \text{otherwise.} \end{cases}$$

Then $\lambda$ is a gauge on $[a,b]$. Let $D = \{([u,v],t)\}$ be a $\lambda$-fine partition of $[a,b]$. Write $D$ as a disjoint union of $D_1'$ and $D_2'$, where $D_1' = \{([u,v],t) \in D : t \in [a,b] \setminus Z\}$ and $D_2' = D \setminus D_1'$. Then $D_1'$ is $\delta_1$-fine partial partition of $[a,b]$ and $D_2'$ is both a $\delta_2$-fine and a $\gamma$-fine partial partition of $[a,b]$. Thus,

$$\begin{aligned}(D)\sum \Phi_U(F(u,v) - f(t)(v-u)) &\le (D_1')\sum \Phi_U(F(u,v) - f(t)(v-u)) \\ &+ (D_2')\sum \Phi_U(-f(t)(v-u)) + (D_2')\sum \Phi_U(F(u,v)) \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.\end{aligned}$$

Consequently,

$$(D)\sum \Phi_V(F(v)-F(u)-f(t)(v-u)) \le (D)\sum \Phi_U(F(v)-F(u)-f(t)(v-u)) < \varepsilon.$$

Therefore, $f$ is $SH_1$-integrable with primitive $F$ and

$$(SH_1)\int_a^b f = F(b) - F(a) = (SL_\Phi)\int_a^b f.$$

This proves the assertion. □

**Theorem 2.17.** *Let $f : [a,b] \to X$ be $SH_1$-integrable with $SH_1$ primitive $F$. Then $f$ is $SL_\Phi$-integrable with $SL_\Phi$-primitive $F$ and*

$$(SL_\Phi)\int_a^b f = (SH_1)\int_a^b f.$$

*Proof.* Let $S \subseteq [a,b]$ be of measure zero, $V$ a $\theta$-nbd, and $\varepsilon > 0$. Let $U$ be balanced, convex, and absorbing $\theta$-nbd with $U \subseteq V$. Let $f_0 = f \cdot 1_S$. Since $f_0 = \theta$ almost everywhere, $f_0$ is $SH_1$-integrable with primitive $F_0 = \theta$ by Theorem 2.15. Hence by Theorem 2.14, there exists a gauge $\delta_1$ such that for any $\delta_1$-fine partition $D_1 = \{([u,v],t)\}$ on $[a,b]$,

$$(D_1)\sum_{i=1}^{n}\Phi_U(-f_0(t_i)(v_i-u_i)) = (D_1^*)\sum_{i=1}^{n}\Phi_U(-f(t_i)(v_i-u_i)) < \frac{\varepsilon}{2},$$

where $D_1^* = \{([u,v],t) \in D_1 : t \in S\}$. Since $f$ is $SH_1$-integrable, there exists a gauge $\delta_2$ such that for every $\delta_2$-fine partition $D_2 = \{([u,v],t)\}$ of $[a,b]$,

$$(D_2)\sum \Phi_U(F(u,v) - f(t)(v-u)) < \frac{\varepsilon}{2}.$$

Let $\delta(t) = \min\{\delta_1(t), \delta_2(t)\}$. Then $\delta$ is a gauge. If $D = \{([u,v],t) : 1 \le i \le n\}$ is a partial partition of $[a,b]$ for which the tags are in $S$, then $D$ is both a $\delta_1$-fine and a $\delta_2$-fine partial partition of $[a,b]$. Thus,

$$(D)\sum \Phi_U(-f(t)(v-u)) < \frac{\varepsilon}{2}$$

and

$$(D)\sum \Phi_U(F(u,v) - f(t)(v-u)) < \frac{\varepsilon}{2}.$$

This implies that

$$(D)\sum \Phi_V(F(u,v)) \le (D)\sum \Phi_U(F(u,v))$$
$$\le (D)\sum \Phi_U(f(t)(v-u)) + (D)\sum \Phi_U(F(u,v) - f(t)(v-u)) < \varepsilon.$$

This shows that $F$ is a $SL_\Phi$ function. Finally, let $V$ be a $\theta$-nbd and $\varepsilon > 0$. Let $U$ be a balanced, convex and absorbing $\theta$-nbd with $U \subseteq V$. Since $f$ is $SH_1$ integrable on $[a,b]$, there is a gauge $\delta$ on $[a,b]$ such that for every $\delta$-fine partition $D = \{([u,v],t)\}$ of $[a,b]$, we have

$$(D)\sum \Phi_U(F(u,v) - f(t)(v-u) < \varepsilon.$$

Clearly, $\delta$ is a tight gauge and for each $\delta$-fine partial partition $D' = \{([u,v],t)\}$ of $[a,b]$,

$$(D')\sum \Phi_V(F(u,v) - f(t)(v-u) \le (D')\sum \Phi_U(F(u,v) - f(t)(v-u) < \varepsilon.$$

Accordingly, $f$ is $SL_\Phi$-integrable with $SL_\Phi$-primitive $F$ on $[a,b]$ and

$$(SL_\Phi)\int_a^b f = F(b)-F(a) = (SH_1)\int_a^b f.$$

This completes the proof of the theorem. □

**Theorem 2.18.** *Let $F, f : [a,b] \to X$ be functions. Then the following statements are equivalent:*

*$(i)$ $f$ is $SH_1$-integrable with $SH_1$ primitive $F$.*

*$(ii)$ $f$ is $SL_\Phi$-integrable with $SL_\Phi$-primitive $F$.*

*$(iii)$ $F$ is a $SL_\Phi$ and $\Delta_\Phi(U,F,f)$ is of measure zero for each $\theta$-nbd $U$.*

*In this case,*

$$\big(SL_\Phi\big)\int_a^b f = \big(SH_1\big)\int_a^b f.$$

*Proof.* The equivalence of $(i)$ and $(ii)$ follows immediately from Theorems 2.16 and 2.17. By Theorem 2.10, $(iii)$ implies $(ii)$. It remains only to show that $(ii)$ implies $(iii)$ and we only need to prove that $\Delta_\Phi(U,F,f)$ is of measure zero for all $\theta$-nbd $U$. Assume $(ii)$ and let $U$ be a $\theta$-nbd. Then $f$ is $SH_1$-integrable with $SH_1$ primitive $F$.

Let $\varepsilon > 0$. By Theorem 2.14, there is a gauge $\delta$ on $[a,b]$ such that for every $\delta$-fine partition $D = \{([u_i,v_i],t_i) : 1 \le i \le n\}$ we have

$$\sum_{i=1}^n \Phi_U\big(F(v_i)-F(u_i)-f(t_i)(v_i-u_i)\big) < \frac{\varepsilon}{2}.$$

Let $\mathcal{F} = \{[u,v] \subseteq [a,b] : \exists t \in [a,b], \Phi_U\big(F(v)-F(u)-f(t)(v-u)\big) \ge (v-u)$ and $|v-u| < \delta(t)\}$. Let $t \in \Delta_\Phi(U,F,f)$ and $\eta > 0$. Then by the defintion of $\Delta_\Phi(U,F,f)$, there is $[u,v] \subseteq [a,b]$ with $t \in [u,v]$, $m(I) = v-u < \eta$ and $\Phi_U\big(F(v)-F(u)-f(t)(v-u)\big) \ge (v-u)$. Hence, $\mathcal{F}$ is a Vitali cover for $\Delta_\Phi(U,F,f)$. By Vitali's theorem, there is a finite collection of non-degenerate close intervals $\{I_i\}_{i=1}^n \subseteq \mathcal{F}$ such that $m^*\big(\Delta_\Phi(U,F,f) \setminus \bigcup_{i=1}^n I_i\big) < \frac{\varepsilon}{2}$. Let $t_i \in I_i$ so that $I_i \in \mathcal{F}$ and $I_i \subseteq \Big(t_i - \delta(t_i), t_i + \delta(t_i)\Big)$ for each $i \in \{1,2,\dots,n\}$. Let $D = \{([x_i,y_i],s_i) : 1 \le i \le n\}$ be a $\delta$-fine partition so that for each $i \in \{1,2,\dots,n\}$, $(I_i,t_i) \in D$. Thus,

$$\begin{aligned} m^*\big(\Delta_\Phi(U,F,f)\big) &< m^*\Big(\bigcup_{i=1}^n I_i\Big) + m^*\Big(\Delta_\Phi(U,F,f) \setminus \bigcup_{i=1}^n I_i\Big) \\ &\le \sum_{i=1}^n m(I_i) + \frac{\varepsilon}{2} \\ &\le \sum_{i=1}^n \Phi_U\big(F(v_i)-F(u_i)-f(t_i)(v_i-u_i)\big) + \frac{\varepsilon}{2} \\ &\le \varepsilon. \end{aligned}$$

Since $\varepsilon$ was arbitrary, it follows that $\Delta_\Phi(U, F, f)$ is of measure zero. □

**Corollary 2.19.** *Let $X$ be a first countable LCTVS and let $f : [a, b] \to X$ be a function. Then $f$ is $SL_\Phi$-integrable on $[a, b]$ if and only if there is a $SL_\Phi$ function $F : [a, b] \to X$ such that $F' = f$ almost everywhere on $[a, b]$.*

*Proof.* Suppose $f$ is $SL_\Phi$-integrable with $SL_\Phi$-primitive $F : [a, b] \to X$. By Theorem 2.18, $\Delta_\Phi(U, F, f)$ is of measure zero for each $\theta$-nbd $U$. Since $X$ is first countable, there is a countable local basis $\mathcal{B}$ at $\theta$ for the topology associated with $X$. By Theorem 2.11,

$$\mathcal{D}(F, f) \subseteq \bigcup_{U \in \mathcal{B}} \Delta_\Phi(U, F, f),$$

implying that $\mathcal{D}(F, f)$ is of measure zero. Thus, $F'(x) = f(x)$ almost everywhere on $[a, b]$.

Conversely, if $F$ is a $SL_\Phi$ function such that $F'(x) = f(x)$ almost everywhere on $[a, b]$, then $\mathcal{D}(F, f)$ is of measure zero. By Theorem 2.11,

$$\mathcal{D}(F, f) = \bigcup_{\theta\text{-nbd } U} \Delta_\Phi(U, F, f).$$

Therefore, $\Delta_\Phi(U, F, f)$ is of measure zero for each $\theta$-nbd $U$. □

Example 2.20. The collection $\mathbb{R}^{[a,b]}$ of all functions from $[a, b]$ to $\mathbb{R}$ is a real vector space with respect to the usual addition and scalar multiplication of functions. For each $\alpha \in [a, b]$, define the evaluation map $\rho_\alpha : \mathbb{R}^{[a,b]} \to \mathbb{R}$ by $\rho_\alpha(f) = |f(\alpha)|$. Note that for every $f, g \in \mathbb{R}^{[a,b]}$ and $c \in \mathbb{R}$,

$$\rho_\alpha(f + g) = |f(\alpha) + g(\alpha)| \leq |f(\alpha)| + |g(\alpha)| = \rho_\alpha(f) + \rho_\alpha(g)$$
$$\rho_\alpha(cf) = |cf(\alpha)| = |c||f(\alpha)| = |c|\rho_\alpha(f).$$

This shows that each evaluation map $\rho_\alpha$ is a semi-norm on $\mathbb{R}^{[a,b]}$ making the real vector space $\mathbb{R}^{[a,b]}$ into a locally convex space with absorbing, balance, and convex $\theta$-nbds $V(\rho_\alpha) = \{f \in \mathbb{R}^{[a,b]} : \rho_\alpha(f) < 1\}$.

Now, from the family of functions $\{e_\alpha : \alpha \in [a, b]\} \subseteq \mathbb{R}^{[a,b]}$, where

$$e_\alpha(x) = \begin{cases} 1 & \text{if} \quad x = \alpha \\ 0 & \text{otherwise,} \end{cases}$$

define the function $\Theta : [a, b] \to \mathbb{R}^{[a,b]}$ by $\Theta(t) = e_t$ for each $t \in [a, b]$. Then $\rho_\alpha\big(\Theta(t)\big) = 0$ for all $\alpha \neq t \in [a, b]$. Here we find that $\Theta$ satisfies the assumption of Theorem 2.15. Hence, the zero function $F(t) = \theta$ on $[a, b]$ is a $SH_1$-primitive of $\Theta$. Notice that $F$ has a derivative equal to itself everywhere on $[a, b]$ and $F$ is not equal to $\Theta$ at every point on $[a, b]$. Thus, the second condition in $(iii)$ of Theorem 2.18 is not equivalent to $F'(t) = \Theta(t)$ almost everywhere on $[a, b]$ as stated in Corollary 2.19.

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