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Fixed Point Theorems for Multivalued Mappings in Banach Algebras and an Application for Fractional Integral Inclusion

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Abstract. In this paper, we establish some fixed point results for the sum and the product of three multivalued mappings, with weakly sequentially closed graph under weak topology features in a Banach algebra. Satisfying a certain sequential condition $(\mathcal{P})$. As an application, our results are used to prove the existence of solutions for a certain non-linear integral inclusion of fractional order.

Keywords: Measures of weak non-compactness, Multivalued mapping, Weakly condensing, Weakly sequentially closed graph, Fixed point theorems, Integral inclusion.

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1. Introduction

Recently, many authors were concerned in the study of non-linear integral inclusion in a Banach algebra via fixed point techniques. Some of these inclusions can be formulated into non-linear operator inclusion:

$$x \in A(x)B(x) + C(x), \qquad x \in \Omega, \tag{1.1}$$

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where Ω is a non-empty closed convex subset of a Banach algebra E. Leray-Schauder's (resp. Krasnoselskii) fixed point theorems for the sum and the product of three operators is one of the most important techniques that gives the existence of solutions for (1.1). See for example, [14, 15]. In some type of applications one may encounter the case that the operators A, B and C may be not continuous, and the product of two weakly convergent sequences is not necessary weakly convergent. Ben Amar [5] overcame this problem by presenting a new class of Banach algebras satisfying a sequential condition $(\mathcal{P})$, see Definition 2.8. Also, they initiated the study of fixed point theorems for the sum and the product of weakly sequentially continuous operators, with applications to a non-linear integral equations under weak topology settings. This definition plays an important role in many other works and set a major cornerstone in the field. For instance, in [4] there are some non-linear alternatives of Leray-Schauder type involving three operators in Banach algebra satisfying a sequential condition $(\mathcal{P})$.

Recall the function (β) was introduced by De Blasi [13] as a measure of weak non-compactness that can be regarded as the counterpart for the weak topology of the classical Hausdorff measure of norm non-compactness. A. Ben Amar and D. O'regan, [8] proved that some fixed point theorems for the sum and product of three non-linear weakly sequentially continuous operators, in a certain Banach algebra satisfying sequential condition $(\mathcal{P})$, via the measure of weak non-compactness. Moreover, the single-valued mapping $\left(\frac{I-C}{A}\right)$ and its invertibility play a fundamental role in that argument, where the single-valued mapping A is quasi-regular. An extension of these results to establish some non-linear alternatives of Leray-Schauder type in a Banach algebra satisfying the sequential condition $(\mathcal{P})$ is found in [2].

The multivalued mapping with weak topology has a wide interest in many areas of application. On one hand, in [17], there are some fixed point theorems in Banach algebras for the multivalued mapping AB, where A is Lipschitzian and B is compact and upper semi-continuous. Ben Amar, Boumaiza and O'Regan [10] introduce a new class of multivalued mappings of the form $\left(\frac{I-C}{A}\right)$ where A and C are multivalued mappings acting on Banach algebras. They also use the properties of $\mathcal{D}-$Lipschitzian and $\mathcal{D}-$set-Lipschitzian with respect to the De Blasi measure of weak non-compactness for the mapping A and C. These results generalize, extend and improve that the well known results for weakly sequentially single-valued mappings in [5, 7, 8]. On the other hand, O'Regan and Taoudi [20] give some versions of the Krasnoselskii fixed point theorems in the framework of weak topologies for multivalued mapping $(I-B)^{-1}A$, where A is a multivalued mapping with weakly sequentially closed graph, and B is a weakly sequentially continuous single-valued map. In addition, Ben Amar in

[9] gives some multivalued analogues of Krasnoselskii fixed point theorem for mappings of the form $T+S$ on a non-empty closed convex set of a Banach space, where T is weakly completely continuous and S is weakly condensing (resp. 1-set weakly contractive) mapping with weakly sequentially closed graph. The authors in [1] extend these results to obtain new multivalued analogues of Leray-Schauder alternatives (or Krasnoselskii fixed point theorems) for the sum of two mappings $A+B$, where A is weakly compact with weakly sequentially closed graph and B is $\Phi-$condensing (or hemi-weakly compact) with weakly sequentially closed graph.

In this paper, we establish some new fixed point results to obtain new multivalued analogues of Leray-Schauder alternative (or Krasnoselskii) fixed point theorems in a Banach algebras that satisfies a sequential condition $(\mathcal{P})$, for the sum and the product of three multivalued mappings $AB+C$, where A, B and C are weakly sequentially closed graphs. We also use the properties of $\Phi-$ condensing, Φ-non-expansive and hemi-weakly compact. These results complement the recent literature [1, 2, 4, 5, 7, 8, 9, 10, 17, 20]. The main condition in our results is formulated in terms of axiomatic measures of weak non-compactness. Finally, we apply these fixed point results to study the existence of a solution for the following non-linear integral inclusion of fractional order α.

$$x(t) \in F(t,x(t)) + K(t,x(t)) \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} G(s,x(s))ds, \tag{1.2}$$

where F, K and $G : J \times X \longrightarrow P(X)$, $t \in J = [0,T]$ and $\alpha \in (0,1)$.

2. Preliminaries

Throughout this section, we shall introduce some necessary notations and definitions which will be needed to achieve our work. Let E be a Hausdorff linear topological space. Now

$$\begin{aligned}
P(E) &= \Big\{ D \subset E : D \text{ is non-empty} \Big\}, \\
P_{\mathrm{bd}}(E) &= \Big\{ D \subset E : D \text{ is non-empty and bounded } \Big\}, \\
P_{\mathrm{cv}}(E) &= \Big\{ D \subset E : D \text{ is non-empty and convex } \Big\}, \\
P_{\mathrm{cl,bd}}(E) &= \Big\{ D \subset E : D \text{ is non-empty closed and bounded } \Big\}, \\
P_{\mathrm{cl,bd,cv}}(E) &= \Big\{ D \subset E : D \text{ is non-empty closed, bounded and convex } \Big\}.
\end{aligned}$$

Let Z be a non-empty subset of Banach space Y and $F : Z \to P(E)$ be a multivalued mapping. We denote

$$R(F) = \bigcup_{y \in Z} F(y), \ \ and \ \ GrF = \{(z,x) \in Z \times E : x \in F(z)\}$$

the range and the graph of F respectively, moreover, for every subset A of E, we put

$$F^{-1}(A) = \{z \in Z : F(z) \cap A \neq \phi\}.$$

Suppose that E is a Banach space with θ and Z is weakly closed in Y. Now F is said to have weakly sequentially closed graph if for every sequence $\{x_n\} \subset Z$ with $x_n \rightharpoonup x \in Z$ and for every sequence $\{y_n\}$ with $y_n \in F(x_n)$ for all $n \in \mathbb{N}$, $y_n \rightharpoonup y$ in E implies $y \in F(x)$; here $\rightharpoonup$ denotes weak convergence. F is called weakly compact, if $F(A)$ is a relatively weakly compact subset of E for every bounded subset $A \in P_{bd}(Z)$, in addition, F is called weakly upper semi-continuous if and only if $F^{-1}(A)$ is weakly closed for all weakly closed sets $A \subset E$. If F is single-valued mapping, then F is said to be weakly sequentially continuous if for every sequence $\{x_n\} \subset Z$ with $x_n \rightharpoonup x \in Z$, we have $F(x_n) \rightharpoonup F(x)$. Now F is said to be sequentially weakly upper semi-compact in Z, *s.w.u.sco* for short, if for any weakly convergent sequence $\{x_n\}$ in Z and an arbitrary $y_n \in F(x_n)$, the sequence $\{y_n\}$ has a weakly convergent subsequence in E. If F is a single-valued mapping, F is sequentially weakly upper semi-compact if for any weakly convergent sequence $\{x_n\}$ in Z the sequence $\{Fx_n\}$ has a weakly convergent subsequence in E.

Lemma 2.1 ([10])**.** *If $F : Z \to P(X)$ is a s.w.u.sco. multivalued mapping in Z and Z is relatively weakly compact then,*
1- *The set $F(x)$ is relatively weakly compact for each $x \in Z$,*
2- *The set $F(Z)$ is relatively weakly compact.*

Definition 2.2. Let X be a Banach space and C a lattice with a least element, which is denoted by 0. By a measure of weak non-compactness $(MWNC)$ on X we mean a function Φ defined on a set of all bounded subsets of X with values in C, such that for any $\Omega_1, \Omega_2 \in P_{bd}(X)$:
(1) $\Phi(\overline{co}(\Omega_1)) = \Phi(\Omega_1)$, where $\overline{co}$ denotes the closed convex hull of Ω_1,
(2) $\Omega_1 \subseteq \Omega_2$ implies $\Phi(\Omega_1) \leq \Phi(\Omega_2)$,
(3) $\Phi(\Omega_1 \cup \{a\}) = \Phi(\Omega_1)$, for all $a \in X$,
(4) $\Phi(\Omega_1) = 0$ if and only if Ω_1 is relatively weakly compact in X.

If the lattice is a cone of vector space, then the $(MWNC)$ Φ is said to positive homogeneous provided $\Phi(\lambda\Omega) = \lambda\Phi(\Omega)$ for all $\lambda > 0$ and $\Omega \in P_{bd}(X)$, and it is called semi-additive iff $\Phi(\Omega_1 + \Omega_2) \leq \Phi(\Omega_1) + \Phi(\Omega_2)$ for all $\Omega_1, \Omega_2 \in P_{bd}(X)$. These notations is a generalization of the important well known De Blasi measure of weak non-compactness β [13] which was defined on each bounded set Ω

of X by

$$\beta(\Omega) = \inf\{r > 0 : \text{there exists a weakly compact set D such that } \Omega \subseteq D + B_r(0)\},$$

where $B_r(0)$ is the closed ball with radius r and center 0.
It is well known that β enjoys these properties: for any $\Omega_1, \Omega_2 \in P_{bd}(E)$,
(5) $\beta(\Omega_1 \cup \Omega_2) = \max\{\beta(\Omega_1), \beta(\Omega_2)\}$,
(6) $\beta(\lambda\Omega_1) = \lambda\beta(\Omega_1)$ for all $\lambda > 0$,
(7) $\beta(\Omega_1 + \Omega_2) \leq \beta(\Omega_1) + \beta(\Omega_2)$.

Definition 2.3. Let $F : \Omega \to P(X)$, Ω be a non-empty subset of Banach space X and Φ a $(MWNC)$ on X, we can say that

- *A-* F is Φ-*condensing* if F is bonded and $\Phi(F(D)) < \Phi(D)$ for all bounded sets $D \subseteq \Omega$ with $\Phi(D) \neq 0$.
- *B-* F is Φ-*non-expansive* if F is bonded and $\Phi(F(D)) \leq \Phi(D)$ for all bounded sets $D \subseteq \Omega$ with $\Phi(D) \neq 0$.
- *C-* F is *hemi-weakly compact* if for each sequence $\{x_n\}$ has a weakly convergent subsequence whenever there exists $y_n \in F(x_n)$ such that the sequence $\{x_n - y_n\}$ is weakly convergent.

In the sequel, we shall need the following theorems.

Theorem 2.4 ([6])**.** *Let Ω be a non-empty, closed, convex subset of Banach space E. Suppose that $F : \Omega \to P_{cv}(\Omega)$ has a weakly sequentially closed graph and $F(\Omega)$ is relatively weakly compact. Then F has a fixed point.*

Theorem 2.5 ([6])**.** *Let Ω be a non-empty, closed, convex subset of Banach space E. and Φ is $(MWNC)$ on E. Suppose that $F : \Omega \to P_{cv}(\Omega)$ has a weakly sequentially closed graph, is Φ-condensing and $F(\Omega)$ is bounded. Then F has a fixed point.*

Theorem 2.6 ([1])**.** *Let Ω be a non-empty closed convex subset of a Banach space E and U be a weakly open subset of Ω with $\theta \in U$. Assume $F : \overline{U^w} \to P_{cv}(\Omega)$ has weakly sequentially closed graph. In addition, suppose that $F(\overline{U^w})$ is relatively weakly compact. Then, either*

- *A_1- F has a fixed point, or*
- *A_2- there is a point $x \in \partial_\Omega U$ (the weak boundary of U in Ω) and $\lambda \in (0, 1)$ with $x \in \lambda F(x)$.*

Theorem 2.7 ([11])**.** ***Eberlien-Šmulian's Theorem****. In the weak topology on a normed space, compactness and sequential compactness coincide. That is, a subset D of a normed space X is relatively weakly compact (respectively, weakly compact) if and only if every sequence in D has a weakly convergent subsequence in X (respectively, in D).*

An algebra is any vector space X equipped with an associative binary operation of multiplication satisfying the condition $(\alpha x)(\beta y) = (\alpha\beta)(xy)$ for any elements $x, y \in X$ and any scalars α, β. A norm algebra is an algebra which is norm, as a vector space, and in which

$$\|xy\| \leq \|x\|.\|y\|$$

for all x, y. A complete norm algebra is called Banach algebra.

Definition 2.8 ([5])**.** We call that the Banach algebra E satisfies a sequential condition $(\mathcal{P})$, if for any sequences $\{x_n\}$ and $\{y_n\}$ in E such that $x_n \rightharpoonup x$ and $y_n \rightharpoonup y$ implies that $x_n y_n \rightharpoonup xy$.

Note that, every finite dimensional Banach algebra satisfies condition $(\mathcal{P})$. If X satisfies condition $(\mathcal{P})$ and K is a hausdorff compact space then $\mathcal{C}(K, X)$ is also a Banach algebra satisfying condition $(\mathcal{P})$. This consequence from Dobrokov's Theorem, see [[18] Theorem (9)].
A Banach space X is said to have the Dunford-Pettis property if for each Banach space Y, every weakly compact linear operator $F : X \to Y$ takes weakly compact sets in X into norm compact sets of Y. It was proved in [3] that every Banach algebra having the Dunford-Pettis property satisfies condition $(\mathcal{P})$.

Remark 2.9. Let E be a Banach algebra with the condition $(\mathcal{P})$, and suppose that A_1 and A_2 be two arbitrary weakly compact subsets of E. Then the product $A_1 A_2$ is weakly compact.

Lemma 2.10 ([7])**.** *Let E be a Banach algebra with a condition $(\mathcal{P})$. Then for any bounded subset D of E and weakly compact subset K of E, we have $\beta(D.K) \leq \|K\|\beta(D)$, where $\|K\| = \sup\{\|x\|, x \in K\}$.*

Definition 2.11 ([10])**.** Let E be a Banach algebra and $A, C : E \to P(E)$ be multivalued mappings. We say that the mapping $\left(\frac{I-C}{A}\right)$ is well defined on $x \in E$ and we write $y \in \left(\frac{I-C}{A}\right)(x)$ if $x \in yA(x) + C(x)$.

3. Krasnoselskii type fixed point theorems

Throughout this section we present some existence results for the following non-linear operator inclusion $x \in A(x)B(x) + C(x)$.

Theorem 3.1. *Let Ω be a non-empty closed convex subset of a Banach algebra E satisfying condition $(\mathcal{P})$ and Φ a semi-additive (MWNC) on E. Let $A, B, C : \Omega \to P(E)$ be three multivalued mappings satisfying the following conditions:*

(i)- A, B and C have weakly sequentially closed graphs,
(ii)- A and B are weakly compacts, and C is $\Phi-$condensing,
(iii)- For all $x \in \Omega$, $A(x)B(x) + C(x) \in P_{cv}(\Omega)$,
(iv)- $(AB + C)(\Omega)$ is bounded.

Then there exists $x \in \Omega$ with $x \in A(x)B(x) + C(x)$.

Proof. Let $F := AB + C : \Omega \to P_{cv}(\Omega)$. We show that F has weakly sequentially closed graph. Let $x_n \rightharpoonup x$ and $y_n \in F(x_n)$ such that $y_n \rightharpoonup y$. There exists $u_n \in A(x_n)$, $v_n \in B(x_n)$ and $w_n \in C(x_n)$ such that $y_n = u_n.v_n + w_n$. Since A and B are weakly compacts with weakly sequentially closed graphs and $\{x_n\}$ is bounded, it follows that by the Eberlien-Šmulian's Theorem $u_{n_k} \rightharpoonup u \in A(x)$ and $v_{n_k} \rightharpoonup v \in B(x)$. Since E satisfies a sequential condition $(\mathcal{P})$, and C has a weakly sequentially closed graph. Then

$$w_{n_k} = y_{n_k} - u_{n_k}.v_{n_k} \rightharpoonup y - u.v \in C(x).$$

Hence $y \in A(x).B(x) + C(x)$. Consequently, F has weakly sequentially closed graph.
Now, we claim that F is $\Phi-$condensing. Let D be arbitrary bounded subset of Ω with $\Phi(D) \neq 0$. By using Remark 2.9

$$\Phi(F(D)) \leq \Phi(\overline{A(D)}^w.\overline{B(D)^w}) + \Phi(C(D)) < \Phi(D).$$

Apply Theorem 2.5 to deduce that $F := AB + C$ has a fixed point in Ω. □

Theorem 3.2. *Let Ω be a non-empty closed convex subset of a Banach algebra E satisfying condition $(\mathcal{P})$ and Φ a positive homogeneous semi-additive $(MWNC)$ on E. Let $A, B, C : \Omega \to P(E)$ be three multivalued mappings satisfying the following conditions:*

(i)- A, B and C have weakly sequentially closed graphs,
(ii)- A and B are weakly compacts, and C is $\Phi-$non-expansive hemi-weakly compact,
(iii)- There exists a bounded set Ω_0 of E and a sequence $\{\lambda_n\} \subseteq (0,1)$ such that $\lambda_n \to 1$, for all $x \in \Omega$, $(AB + \lambda_n C)(x) \in P_{cv}(\Omega)$ and $(AB + \lambda_n C)(\Omega) \subset \Omega_0$ for all n.

Then there exists $x \in \Omega$ with $x \in A(x)B(x) + C(x)$.

Proof. Define $F_n := AB + \lambda_n C$, for all $n \in \mathbb{N}$. Then by assumption (iii), $F_n : \Omega \to P_{cv}(\Omega)$ is well defined and $F_n(\Omega)$ is bounded. In view of Theorem 3.1, F_n has weakly sequentially closed graph. Let D be an arbitrary bounded subset of Ω with $\Phi(D) \neq 0$. By using Remark 2.9, and C is $\Phi-$non-expansive then,

$$\Phi(F_n(D)) \leq \Phi(\overline{A(D)^w}.\overline{B(D)^w}) + \lambda_n \Phi(C(D)) < \Phi(D).$$

Therefore, F_n is $\Phi-$ condensing. Theorem 2.5 guarantees that F_n has fixed point $x_n \in \Omega$. That is,

$$x_n \in A(x_n).B(x_n) + \lambda_n C(x_n).$$

Then there exists $u_n \in A(x_n)$, $v_n \in B(x_n)$ and $w_n \in C(x_n)$ such that $x_n = u_n.v_n + \lambda_n w_n$. Since A and B are weakly compacts with weakly sequentially closed graphs, E satisfies a condition $(\mathcal{P})$, and $\{x_n\}$ is bounded. Then we can find subsequences $\{u_{n_k}\}$ with $u_{n_k} \rightharpoonup u \in A(x)$ and $\{v_{n_k}\}$ with $v_{n_k} \rightharpoonup v \in B(x)$ such that $u_{n_k}.v_{n_k} \rightharpoonup u.v$. Obviously, the sequence $\{w_n\}$ is bounded and $\lambda_n \to 1$, then we get

$$x_{n_k} - w_{n_k} = u_{n_k}.v_{n_k} + (\lambda_n - 1)w_{n_k} \rightharpoonup u.v.$$

By assumption C is a hemi-weakly compact. This implies $\{x_{n_k}\}$ has a weakly convergent subsequence, say $\{x_{n_{k_j}}\}$. Also, C with weakly sequentially closed graph. Hence $w_{n_{k_j}} \rightharpoonup x - u.v \in C(x)$. Therefore $x \in A(x).B(x) + C(x)$. □

Remark 3.3. Theorem 3.1 and Theorem 3.2 extends theorem (2.1) and Theorem (2.2) in [9] respectively, for the case of the sum and the product of three multivalued mappings.

Theorem 3.4. *Let Ω be a non-empty closed convex subset of a Banach algebra E satisfying condition $(\mathcal{P})$. Let $A, C : E \to P(E)$ and $B : \Omega \to P(E)$ be three multivalued mappings satisfying the following conditions:*

(i)- A, B and C have weakly sequentially closed graphs,
(ii)- A and B are weakly compacts, and C is hemi-weakly compact,
(iii)- $\left(\frac{I-C}{A}\right)^{-1}$ exists on $B(\Omega)$,
(iv)- For each $x \in \Omega$, $\left(\frac{I-C}{A}\right)^{-1} B(x) \in P_{cv}(\Omega)$,
(v)- $\left(\frac{I-C}{A}\right)^{-1} B(\Omega)$ is bounded.

Then there exists $x \in \Omega$ with $x \in A(x)B(x) + C(x)$.

Proof. From the hypotheses (iii), the multivalued mapping $\left(\frac{I-C}{A}\right)^{-1}$ exists on $B(\Omega)$. Let $F = \left(\frac{I-C}{A}\right)^{-1} B$. Then by assumption (iv) $F : \Omega \to P_{cv}(\Omega)$ is well defined. To show that F has a weakly sequentially closed graph, let $x_n \rightharpoonup x \in \Omega$ and $y_n \in F(\Omega)$, $y_n \rightharpoonup y$ with $y_n \in F(x_n)$. Then $\left(\frac{I-C}{A}\right)(y_n) \cap B(x_n) \neq \emptyset$, so there exists $z_n \in B(x_n)$ such that $z_n \in \left(\frac{I-C}{A}\right)(y_n)$. From the definition of the mapping $\frac{I-C}{A}$, we have $z_n u_n = y_n - v_n$ where $v_n \in C(y_n)$ and $u_n \in A(y_n)$. Since B is a weakly compact and $\{x_n\}$ is bounded. By Eberlien-Šmulian's Theorem, $\{z_n\}$ has a subsequence $\{z_{n_k}\}$ which weakly converges to some $z \in B(x)$. Also, A is a weakly compact and $\{y_n\}$ is bounded, then $\{u_n\}$ has a subsequence $\{u_{n_k}\}$ which weakly converges to some $u \in A(y)$. In addition, C has a weakly sequentially closed graph, and E satisfies a condition $(\mathcal{P})$. Then we get

$$v_{n_k} = y_{n_k} - z_{n_k} u_{n_k} \rightharpoonup y - zu \in C(y).$$

Hence, $z \in \left(\frac{I-C}{A}\right)(y)$. Consequently, $\left(\frac{I-C}{A}\right)(y) \cap B(x) \neq \emptyset$ and $y \in \left(\frac{I-C}{A}\right)^{-1} B(x)$. Hence F has a weakly sequentially closed graph. Now, let D be an arbitrary bounded subset of Ω. We claim that $F(D)$ is relatively weakly compact. Let $y_n \in F(D)$. Choose $\{x_n\} \subset D$ such that $y_n \in F(x_n)$, that is $y_n \in A(y_n)B(x_n) + C(y_n)$. Thus there exists $z_n \in B(x_n)$, $u_n \in A(y_n)$ and $v_n \in C(y_n)$ such that $y_n = u_n z_n + v_n$. Since A and B are weakly compact, and

E satisfies a condition $(\mathcal{P})$. Tt follows that $y_{n_k} - v_{n_k} = u_{n_k} z_{n_k} \rightharpoonup uz$. Since C is a hemi-weakly compact, then $\{y_{n_k}\}$ has a weakly convergent subsequence. Hence $F(D)$ is relatively weakly compact. Consequently, F is weakly compact. From Theorem 2.4, F has a fixed point. Then there exists $x \in \Omega$ such that $x \in A(x)B(x) + C(x)$. □

Proposition 3.5. *Let E be a Banach algebra satisfying condition $(\mathcal{P})$ and Φ is a $(MWNC)$ on E. Let Ω be a non-empty closed convex subset of E and $A, C : E \to P_{cv}(E)$ be two multivalued mappings satisfying the following conditions:*

(i)- A and C have weakly sequentially closed graphs,
(ii)- A is a weakly compact, and C is $\beta-$ condensing,
(iii)- $A(E)$ and $C(E)$ are bounded,

Then the multivalued operator $\left(\frac{I-C}{A}\right)^{-1}$ exists in E.

Proof. Fix z in E. Consider,

$$\Gamma_z : E \to P_{cv}(E), \;\; x \longmapsto Ax.z + Cx.$$

Since $Ax.z + Cx$ is convex, it is clear that Γ_z is well defined. We prove that the multivalued mapping Γ_z satisfies all the assumptions in the statement of Theorem 2.5. Let $x_n \rightharpoonup x$ and $y_n \in Ax_n.z + Cx_n$ such that $y_n \rightharpoonup y$. There exists $u_n \in Ax_n$ and $v_n \in Cx_n$ such that $y_n = u_n.z + v_n$. Since A is a weakly compact with weakly sequentially closed graph, and $\{x_n\}$ is bounded. By Eberlien-Šmulian's Theorem, $u_{n_k} \rightharpoonup u \in Ax$. The right hand multiplication operator $R_z(x) = x.z$ is a continuous linear operator, so it is weakly continuous. Accordingly,

$$v_{n_k} = y_{n_k} - u_{n_k}.z \rightharpoonup y - u.z.$$

The operator C has a weakly sequentially closed graph, we deduce that $y - u.z \in Cx$. Hence $y \in Ax.z + Cx = \Gamma_z(x)$. Consequently, Γ_z has a weakly sequentially closed graph.

Now, let D be an arbitrary bounded subset of E with $\beta(D) \neq 0$. It is clear that $\Gamma_x(D)$ is bounded. From Lemma 2.10, we get

$$\beta(\Gamma_z(D)) \leq \beta(A(D).z) + \beta(C(D)) \leq \|z\| \beta(A(D)) + \beta(C(D)) < \beta(D).$$

Hence Γ_z is $\beta-$condensing. Moreover, by assumption (iii) $\Gamma_z(E)$ is bounded. Then the multivalued mapping Γ_z satisfies all the assumptions in the statement of Theorem 2.5, so there exists $x \in E$ such that $x \in \Gamma_z(x) = Ax.z + Cx$. Thus $z \in \left(\frac{I-C}{A}\right)(x)$. Therefore, the multivalued operator $\left(\frac{I-C}{A}\right)^{-1}$ exists in E. □

Theorem 3.6. *Let E be a Banach algebra satisfying condition $(\mathcal{P})$ and Φ a $(MWNC)$ on E. Let Ω be a non-empty closed convex subset of E and*

$A, C : E \to P_{cv}(E)$ and $B : \Omega \to P(E)$ be three multivalued mappings satisfying the following conditions:

(i)- A, B and C have weakly sequentially closed graphs,
(ii)- A and C are weakly compacts, and B is s.w.u.sco.,
(iii)- $A(E)$, $B(\Omega)$ and $C(E)$ are bounded,
(iv)- For all $y \in \Omega$, $\left(\frac{I-C}{A}\right)^{-1} B(y) \in P_{cv}(\Omega)$,
(v)- $\left(\frac{I-C}{A}\right)^{-1} B$ is $\Phi-$ condensing.

Then, there exists $x \in \Omega$ with $x \in A(x)B(x) + C(x)$.

Proof. Let $y \in \Omega$ and fix $z \in B(y)$. Consider,

$$\Gamma_z : E \to P_{cv}(E), \;\; x \longmapsto Ax.z + Cx.$$

Since every weakly compact operator is $\beta-$condensing. It follows that C is $\beta-$condensing, and from Proposition 3.5, the multivalued mapping $\left(\frac{I-C}{A}\right)^{-1}$ exists on $B(\Omega)$. Let $F = \left(\frac{I-C}{A}\right)^{-1} B$, then by assumption (iv) $F : \Omega \rightharpoonup P_{cv}(\Omega)$ is well defined. We show that F has a weakly sequentially closed graph. Let $x_n \rightharpoonup x \in \Omega$ and $y_n \in F(\Omega)$, $y_n \rightharpoonup y$ with $y_n \in F(x_n)$. Then $\left(\frac{I-C}{A}\right)(y_n) \cap B(x_n) \neq \emptyset$. Accordingly, there exists $z_n \in B(x_n)$ such that $z_n \in \left(\frac{I-C}{A}\right)(y_n)$. Hence $u_n z_n = y_n - v_n$ where $v_n \in C(y_n)$ and $u_n \in A(y_n)$. Since B is s.w.u.sco. and has a weakly sequentially closed graph, it follows that there exists a subsequence $\{z_{n_k}\}$ such that $z_{n_k} \rightharpoonup z \in B(x)$. Since A and C are weakly compacts, E satisfies a condition $(\mathcal{P})$, and $\{y_n\}$ is bounded. By Eberlien-Šmulian's Theorem, $\{u_n\}$ and $\{v_n\}$ has convergent subsequences $u_{n_k} \rightharpoonup u \in A(y)$ and $v_{n_k} \rightharpoonup v \in C(y)$ respectively. Hence $y_{n_k} = u_{n_k} z_{n_k} + v_{n_k}$. By the uniqueness of weak limit we can get $y = uz + v \in A(y).z + C(y)$, that is $z \in \left(\frac{I-C}{A}\right) y$. Then $\left(\frac{I-C}{A}\right)(y) \cap B(x) \neq \emptyset$. Hence $y \in \left(\frac{I-C}{A}\right)^{-1} B(x)$, and so F has a weakly sequentially closed graph. Using,

$$F(\Omega) \subset A(F(\Omega))B(\Omega) + C(F(\Omega)),$$

and taking into account the assumption (iii). We conclude that $F(\Omega)$ is bounded. Now by Theorem 2.5 F has a fixed point $x \in \Omega$. □

Theorem 3.7. *Let E be a Banach algebra satisfying condition $(\mathcal{P})$ and Φ a $(MWNC)$ on E. Let Ω be a non-empty closed convex subset of E and $A, C : E \to P_{cv}(E)$ and $B : \Omega \to P(E)$ be three multivalued mappings satisfying the following conditions:*

(i)- A, B and C have weakly sequentially closed graphs,
(ii)- A and C are weakly compacts, and B is s.w.u.sco.,
(iii)- $A(E)$, $B(\Omega)$ and $C(E)$ are bounded,
(iv)- For all $y \in \Omega$, $\left(\frac{I-C}{A}\right)^{-1} B(y) \in P_{cv}(\Omega)$,
(v)- $\|A(\Omega)\| \leq 1$, and B is $\beta-$condensing.

Then, there exists $x \in \Omega$ with $x \in A(x)B(x) + C(x)$.

Proof. According to Theorem 3.6, it suffices to show that the operator $F := \left(\frac{I-C}{A}\right)^{-1} B$ is $\Phi-$condensing. To do this, let D be an arbitrary bounded subset of Ω with $\beta(D) > 0$. Clearly,

$$F(D) \subseteq C(F(D)) + A(F(D)).B(D) \subseteq C(F(D)) + \overline{A(F(D))^w} B(D),$$

and $\overline{A(F(D))^w}$ is weakly compact. For $x \in \overline{A(F(D))^w}$, Eberlein-Šmulian's Theorem says there exists a sequence $\{x_n\} \subset A(F(D))$ with $x_n \rightharpoonup x$. Since for all n, $\|x_n\| \leq 1$ and $\|x\| \leq \liminf \|x_n\|$, then $\|x\| \leq 1$, and hence $\|\overline{A(F(D))^w}\| \leq 1$. By Lemma 2.10 and the hypotheses that C is a weakly compact, we have

$$\beta(F(D)) \leq \beta(C(F(D))) + \beta(\overline{A(F(D))^w}.B(D)) \leq \|\overline{A(F(D))^w}\| \beta(B(D)) < \beta(D).$$

Hence, $\left(\frac{I-C}{A}\right)^{-1} B$ is $\beta-$condensing. The proof is complete. □

Theorem 3.8. *Let E be a Banach algebra satisfying condition $(\mathcal{P})$ and Φ a $(MWNC)$ on E. Let Ω be a non-empty closed convex subset of E and $A, C : E \to P_{cv}(E)$ and $B : \Omega \to P(E)$ be three multivalued mappings satisfying the following conditions:*

(i)- A, B and C have weakly sequentially closed graphs,
(ii)- A and B are weakly compacts, and C is $\beta-$condensing,
(iii)- $A(E)$, $B(\Omega)$ and $C(E)$ are bounded,
(iv)- For all $y \in \Omega$, $\left(\frac{I-C}{A}\right)^{-1} B(y) \in P_{cv}(\Omega)$,

Then, there exists $x \in \Omega$ with $x \in A(x)B(x) + C(x)$.

Proof. From Proposition 3.5, the multivalued mapping $\left(\frac{I-C}{A}\right)^{-1}$ exists on $B(\Omega)$. Let $F = \left(\frac{I-C}{A}\right)^{-1} B$. Then by assumption (iv) $F : \Omega \to P_{cv}(\Omega)$ is well defined. In view of Theorem 3.4, F has a weakly sequentially closed graph. Using

$$F(\Omega) \subset A(F(\Omega))B(\Omega) + C(F(\Omega))$$

and taking into account the assumption (iii), we conclude that $F(\Omega)$ is bounded. Let D be an arbitrary bounded subset of Ω. We claim that F is a weakly compact. From Remark 2.9 we have

$$\beta(F(D)) \leq \beta(\overline{(A(F(D)))^w}.\overline{(B(D))^w}) + \beta(C(F(D))) < \beta(F(D)),$$

a contradiction. Hence $F(D)$ is relatively weakly compact. Consequently, F is weakly compact. From Theorem 2.4, F has a fixed point. □

Recall that, every weakly compact operator is $\beta-$condensing, so an interesting special case of Theorem 3.8 in the applicable form is:

Corollary 3.9. *Let Ω be a non-empty closed convex subset of a Banach algebra E satisfying condition $(\mathcal{P})$. Let $A, C : E \to P_{cv}(E)$ and $B : \Omega \to P(E)$ be three multivalued mappings satisfying the following conditions:*

(i)- A, B and C have weakly sequentially closed graphs,
(ii)- A, B and C are weakly compacts,
(iii)- $A(E)$, $B(\Omega)$ and $C(E)$ are bounded,
(iv)- For each $x \in \Omega$, $\left(\frac{I-C}{A}\right)^{-1} B(x) \in P_{cv}(\Omega)$.

Then, there exists $x \in \Omega$ with $x \in A(x)B(x) + C(x)$.

Proposition 3.10. *Let E be a Banach algebra satisfying condition $(\mathcal{P})$ and Φ a MWNC on E. Let Ω be a non-empty closed convex subset of E and $A, C : E \to P_{cv}(E)$ and $B : \Omega \to P(E)$ be three multivalued mappings satisfying the following conditions:*

(i)- A and C have weakly sequentially closed graphs,
(ii)- C is weakly compact,
(iii)- $A(E)$ and $C(E)$ are bounded,
(iv)- $\|B(\Omega)\| \leq 1$, and A is $\beta-$condensing.

Then the multivalued operator $\left(\frac{I-C}{A}\right)^{-1}$ exists on $B(\Omega)$.

Proof. Let $y \in \Omega$ and fix $z \in B(y)$. Consider,

$$\Gamma_z : E \to P_{cv}(E), \;\; x \longmapsto Ax.z + Cx.$$

Since $Ax.z + Cx$ is convex, it is clear that Γ_z is well defined. We prove that the multivalued mapping Γ_z satisfies all the assumptions in the statement of Theorem 2.5. Let $x_n \rightharpoonup x$ and $y_n \in Ax_n.z + Cx_n$ such that $y_n \rightharpoonup y$. There exists $u_n \in Ax_n$ and $v_n \in Cx_n$ such that $y_n = u_n.z + v_n$. Since C is weakly compact with weakly sequentially closed graph and $\{x_n\}$ is bounded. Then by Eberlien-Šmulian's Theorem $v_{n_k} \rightharpoonup v \in Cx$. Accordingly,

$$u_{n_k}.z = y_{n_k} - v_{n_k} \rightharpoonup y - v.$$

Since A has a weakly sequentially closed graph, it follows that $(y - v) \in Ax.z$. Hence $y \in Ax.z + Cx = \Gamma_z(x)$. Consequently, Γ_z has a weakly sequentially closed graph. Now, let D be an arbitrary bounded subset of E with $\beta(D) \neq 0$. It is clear that $\Gamma_x(D)$ is bounded. From Lemma 2.10, we get

$$\beta(\Gamma_z(D)) \leq \beta(A(D).z) + \beta(C(D)) \leq \|z\|\beta(A(D)) + \beta(C(D)) \leq \beta(A(D)) < \beta(D).$$

Hence Γ_z is $\beta-$condensing. Moreover, by assumption (iii), $\Gamma_z(E)$ is bounded. Then the multivalued mapping Γ_z satisfies all the assumptions in the statement of Theorem 2.5, it follows that there exists $x \in \Omega$ such that $x \in \Gamma_z(x) = Ax.z + Cx$. Thus $z \in \left(\frac{I-C}{A}\right)(x)$, and hence $\left(\frac{I-C}{A}\right)(x) \cap B(y) \neq \emptyset$. Hence, the multivalued operator $\left(\frac{I-C}{A}\right)^{-1}$ exists on $B(\Omega)$. □

Theorem 3.11. *Let E be a Banach algebra satisfying condition $(\mathcal{P})$ and Φ a $(MWNC)$ on E. Let Ω be a non-empty closed convex subset of E and $A, C : E \to P_{cv}(E)$ and $B : \Omega \to P(E)$ be three multivalued mappings satisfying the following conditions:*

(i)- A, B and C have weakly sequentially closed graphs,
(ii)- B and C are weakly compacts, and A is s.w.u.sco.,
(iii)- $A(E)$, $B(\Omega)$ and $C(E)$ are bounded,
(iv)- For all $y \in \Omega$, $\left(\frac{I-C}{A}\right)^{-1} B(y) \in P_{cv}(\Omega)$,
(v)- $\|B(\Omega)\| \leq 1$, and A is $\beta-$condensing.

Then, there exists $x \in \Omega$ with $x \in A(x)B(x) + C(x)$.

Proof. From Proposition 3.10, it follows that $\left(\frac{I-C}{A}\right)^{-1}$ exists on $B(\Omega)$. From assumption (iv) $F := \left(\frac{I-C}{A}\right)^{-1} B : \Omega \to P_{cv}(\Omega)$ is well defined. We show that F has a weakly sequentially closed graph. Let $x_n \rightharpoonup x \in \Omega$ and $y_n \in F(\Omega)$, $y_n \rightharpoonup y$ with $y_n \in F(x_n)$. Then $\left(\frac{I-C}{A}\right)(y_n) \cap B(x_n) \neq \emptyset$, so there exists $z_n \in B(x_n)$ such that $z_n \in \left(\frac{I-C}{A}\right)(y_n)$. Hence $y_n = u_n z_n + v_n$ where $v_n \in C(y_n)$ and $u_n \in A(y_n)$. Since A is s.w.u.sco. and has weakly sequentially closed graph, it follows that there exists a subsequence $\{u_{n_k}\}$ such that $u_{n_k} \rightharpoonup u \in A(y)$. Since B is a weakly compact and $\{x_n\}$ is bounded. By Eberlien-Šmulian's Theorem $\{z_n\}$ has a convergent subsequence $z_{n_k} \rightharpoonup z \in B(x)$. (Similarly, $v_{n_k} \rightharpoonup v \in C(y)$). Keep in your mined that E satisfies a condition $(\mathcal{P})$, hence, $y_{n_k} = u_{n_k} z_{n_k} + v_{n_k}$. By the uniqueness of weak limit we can get $y = uz + v \in A(y).z + C(y)$, that is $z \in \left(\frac{I-C}{A}\right) y$. Consequently, $\left(\frac{I-C}{A}\right)(y) \cap B(x) \neq \emptyset$. Hence $y \in \left(\frac{I-C}{A}\right)^{-1} B(x)$. Accordingly, F has a weakly sequentially closed graph. Next,

$$F(\Omega) \subset A(F(\Omega))B(\Omega) + C(F(\Omega))$$

and taking into account the assumption (ii). We conclude that $F(\Omega)$ is bounded. Now, we claim that $F := \left(\frac{I-C}{A}\right)^{-1} B$ is a weakly compact. To do this, let D be an arbitrary bounded subset of Ω with $\beta(D) > 0$. Using

$$F(D) \subseteq C(F(D)) + A(F(D)).B(D) \subseteq C(F(D)) + A(F(D))\overline{(B(D))}^w$$

Obviously, $\overline{(B(D))}^w$ is a weakly compact. For $x \in \overline{(B(D))}^w$, Eberlein-Šmulian's Theorem implies that there exists a sequence $\{x_n\} \subset B(D)$ with $x_n \rightharpoonup x$. We know that for all n, $\|x_n\| \leq 1$ and $\|x\| \leq \liminf \|x_n\|$, then $\|x\| \leq 1$, and hence $\|\overline{(B(D))}^w\| \leq 1$. By Lemma 2.10 we get,

$$\begin{aligned} \beta(F(D)) &\leq \beta(C(F(D))) + \beta(A(F(D)).\overline{(B(D))}^w) \\ &\leq \|\overline{(B(D))}^w\| \beta(A(F(D))) < \beta(F(D)). \end{aligned}$$

A contradiction. Hence, $\left(\frac{I-C}{A}\right)^{-1} B$ is a weakly compact. Therefore, Theorem 2.4 guarantees that F has a fixed point $x \in \Omega$. ∎

Proposition 3.12. *Let E be a Banach algebra satisfying condition $(\mathcal{P})$ and Ω be a non-empty closed convex subset of E. Assume Φ is a positive homogeneous semi-additive (MWNC) on E. Let $A, C : E \to P_{cv}(E)$ be two multivalued mappings satisfying the following conditions:*

(i)- A and C have weakly sequentially closed graph,
(ii)- A is weakly compact,
(iii)- $A(E)$ and $C(E)$ are bounded,
(iv)- C is $\beta-$non-expansive and hemi-weakly compact,

Then, the multivalued operator $\left(\frac{I-C}{A}\right)^{-1}$ exists in E.

Proof. Fix z in E. Consider for each $m \in \mathbb{N}$,

$$\Gamma_{m,z} : E \to P_{cv}(E), \;\; x \longmapsto Ax.z + \lambda_m Cx,$$

where $\{\lambda_m\} \subseteq (0,1)$ such that $\lambda_m \to 1$. Let m be fixed, and since $Ax.z+\lambda_m Cx$ is convex, it is clear that $\Gamma_{m,z}$ is well defined. We prove that the multivalued mapping $\Gamma_{m,z}$ satisfies all the assumptions in the statement of Theorem 2.5. Let $x_n \rightharpoonup x$ and $y_n \in Ax_n.z+\lambda_m Cx_n$ such that $y_n \rightharpoonup y$. There exists $u_n \in Ax_n$ and $v_n \in Cx_n$ such that $y_n = u_n.z + \lambda_m v_n$. Since A is a weakly compact with weakly sequentially closed graph and $\{x_n\}$ is bounded. By Eberlien-Šmulian's Theorem there exists a subsequence $\{u_{n_k}\}$ such that $u_{n_k} \rightharpoonup u \in Ax$. It well known that the right hand multiplication operator $R_z(x) = x.z$ is a continuous linear operator, consequently, it is weakly continuous. Then we can get

$$\lambda_m v_{n_k} = y_{n_k} - u_{n_k}.z \rightharpoonup y - u.z.$$

From hypotheses C has a weakly sequentially closed graph, then $y - u.z \in \lambda_m Cx$. Hence $y \in Ax.z + \lambda_m Cx = \Gamma_{m,z}(x)$. Hence $\Gamma_{m,z}$ has a weakly sequentially closed graph. Next, we claim that $\Gamma_{m,z}$ is $\beta-$condensing. Let D be an arbitrary bounded subset of E with $\beta(D) \neq 0$. From assumption (iii) $\Gamma_{m,z}(E)$ is bounded. Thus $\Gamma_{m,z}(D)$ is bounded. By Lemma 2.10 and C is $\beta-$non-expansive we can get

$$\beta(\Gamma_{m,z}(D)) \leq \beta(A(D).z) + \beta(\lambda_m C(D)) \leq \|z\|\beta(A(D)) + \lambda_m\beta(C(D)) < \beta(D).$$

Hence $\Gamma_{m,z}$ is $\beta-$ condensing. Then the multivalued mapping $\Gamma_{m,z}$ satisfies all the assumptions in the statement of Theorem 2.5. Then there exists $x_m \in E$ such that $x_m \in \Gamma_{m,z}(x_m) = Ax_m.z + \lambda_m Cx_m$. Also, we can find $u_m \in A(x_m)$ and $v_m \in C(x_m)$ such that

$$x_m = u_m.z + \lambda_m v_m.$$

Obviously $\{x_m\}$ is bounded. We can suppose there exists a subsequence $\{u_{m_k}\}$ with $u_{m_k} \rightharpoonup u$. Since $\{v_m\}$ is bounded and $\lambda_m \to 1$ we can get

$$x_{m_k} - v_{m_k} = u_{m_k}.z + (\lambda_{m_k} - 1)v_{m_k} \rightharpoonup u.z.$$

By assumption (iv) C is hemi-weakly compact, this implies $\{x_{m_k}\}$ has a weakly convergent subsequence, say $\{x_{m_{k_j}}\}$. Hence $v_{m_{k_j}} \rightharpoonup x - u.z$. Also $x - u.z \in C(x)$, that is $x \in A(x).z + C(x)$. Consequently, $z \in \left(\frac{I-C}{A}\right)(x)$. Therefore, $\left(\frac{I-C}{A}\right)^{-1}$ exists in E. □

Theorem 3.13. *Let E be a Banach algebra satisfying condition $(\mathcal{P})$ and Ω be a non-empty closed convex subset of E. Assume Φ is a positive homogeneous semi-additive (MWNC) on E. Let $A, C : E \to P_{cv}(E)$ and $B : \Omega \to P(E)$ be three multivalued mappings satisfying the following conditions:*

(i)- A, B and C have weakly sequentially closed graph,
(ii)- A and B are weakly compacts,
(iii)- $A(E)$, $B(\Omega)$ and $C(E)$ are bounded,
(iv)- For all $y \in \Omega$, $\left(\frac{I-C}{A}\right)^{-1} B(y) \in P_{cv}(\Omega)$,
(v)- C is $\beta-$non-expansive and hemi-weakly compact.

Then, there exists $x \in \Omega$ with $x \in A(x)B(x) + C(x)$.

Proof. This is an immediate consequence of Theorem 3.4 and Proposition 3.12. □

4. Non-linear Leray-Schauder alternatives

Depending on the results of Propositions 3.5, 3.10 and 3.12. We prove some new versions of Leray-Schauder type fixed point theorem for the sum and the product of three multivalued mappings.

Theorem 4.1. *Let Ω be a non-empty closed convex subset of a Banach algebra E satisfying condition $(\mathcal{P})$ and U be a weakly open subset of Ω with $\theta \in U$. Assume $A, C : E \to P(E)$ and $B : \overline{U^w} \to P(E)$ be three multivalued mappings satisfying the following conditions:*

(i)- A, B and C have weakly sequentially closed graphs,
(ii)- A and B are weakly compacts, and C is hemi-weakly compact,
(iii)- $\left(\frac{I-C}{A}\right)^{-1}$ exists on $B(\overline{U^w})$,
(iv)- For each $x \in \overline{U^w}$, $\left(\frac{I-C}{A}\right)^{-1} B(x)$ is convex,
(v)- $x \in A(x)B(y) + C(x)$, $y \in \overline{U^w} \Rightarrow x \in \Omega$,
(vi)- $\left(\frac{I-C}{A}\right)^{-1} B(\overline{U^w})$ is bounded.

Then, either

(A_1) the equation $x \in \lambda A(\frac{x}{\lambda})B(x) + \lambda C(\frac{x}{\lambda})$, has a solution for $\lambda = 1$, or
(A_2) there is a point $u \in \partial_\Omega U$ and $\lambda \in (0, 1)$ with $u \in \lambda A(\frac{u}{\lambda})Bu + \lambda C(\frac{u}{\lambda})$.

Proof. From the assumption (iii), the multivalued mapping $\left(\frac{I-C}{A}\right)^{-1}$ exists on $B(\overline{U^w})$. First we show that $F := \left(\frac{I-C}{A}\right)^{-1} B : \overline{U^w} \to P_{cv}(\Omega)$ is well defined.

Let $x \in \left(\frac{I-C}{A}\right)^{-1} B(y)$ for $y \in \overline{U^w}$. Then $x \in A(x)B(y)+C(x)$. By assumption (v) we have $x \in \Omega$. Hence $\left(\frac{I-C}{A}\right)^{-1} B(y) \subseteq \Omega$ for $y \in \overline{U^w}$. This together with (iv) guarantees that $F : \overline{U^w} \to P_{cv}(\Omega)$. Next, we show that F has a weakly sequentially closed graph. Let $x_n \rightharpoonup x \in \overline{U^w}$ and $y_n \in F(\overline{U^w})$, $y_n \rightharpoonup y$ with $y_n \in F(x_n)$. Then $\left(\frac{I-C}{A}\right)(y_n) \cap B(x_n) \neq \emptyset$, so we can find $z_n \in B(x_n)$ such that $z_n \in \left(\frac{I-C}{A}\right)(y_n)$. Chose $v_n \in C(y_n)$ and $u_n \in A(y_n)$ such that $u_n z_n = y_n - v_n$. Since B is a weakly compact and $\{x_n\}$ is bounded. The Eberlien-Šmulians Theorem implies that $\{z_n\}$ has a subsequence $\{z_{n_k}\}$ which weakly converges to some $z \in B(x)$. Since A is a weakly compact and $\{y_n\}$ is bounded, $\{u_n\}$ has a subsequence $\{u_{n_k}\}$ which weakly converges to some $u \in A(y)$. Also, C has a weakly sequentially closed graph and E satisfies the condition $(\mathcal{P})$. It follows that

$$v_{n_k} = y_{n_k} - z_{n_k} u_{n_k} \rightharpoonup y - zu \in C(y),$$

that is, $z \in \left(\frac{I-C}{A}\right)(y)$. Then $\left(\frac{I-C}{A}\right)(y) \cap B(x) \neq \emptyset$ and consequently, $y \in \left(\frac{I-C}{A}\right)^{-1} B(x)$. Hence F has a weakly sequentially closed graph. Finally, we claim that $F(\overline{U^w})$ is a relatively weakly compact. Let $y_n \in F(\overline{U^w})$. Choose $\{x_n\} \subset \overline{U^w}$ such that $y_n \in F(x_n)$, that is $y_n \in A(y_n)B(x_n) + C(y_n)$. Thus there exists $z_n \in B(x_n)$, $u_n \in A(y_n)$ and $v_n \in C(y_n)$ such that $y_n = u_n z_n + v_n$. Since A and B are weakly compact and E satisfies a condition $(\mathcal{P})$. It follows that $y_{n_k} - v_{n_k} = u_{n_k} z_{n_k} \rightharpoonup uz$. Since C is hemi-weakly compact, then $\{y_{n_k}\}$ has a weakly convergent subsequence, say $\{y_{n_{k_j}}\}$. Hence $F(\overline{U^w})$ is relatively weakly compact. From Theorem 2.6, either F has a fixed point or there exists $u \in \partial_\Omega U$ and $\lambda \in (0,1)$ with $u \in \lambda F(u)$. The proof is complete. □

Theorem 4.2. *Let Ω be a non-empty closed convex subset of a Banach algebra E satisfying condition $(\mathcal{P})$ and Φ a $(MWNC)$ on E. Let U be a weakly open subset of Ω with $\theta \in U$. Assume $A, C : E \to P_{cv}(E)$ and $B : \overline{U^w} \to P(E)$ are three multivalued mappings satisfying the following conditions:*

- (i)- A, B and C have weakly sequentially closed graphs,
- (ii)- A and C are weakly compacts, and B is s.w.u.sco.,
- (iii)- For each $x \in \overline{U^w}$, $\left(\frac{I-C}{A}\right)^{-1} B(x)$ is convex,
- (iv)- $x \in A(x)B(y) + C(x)$, $y \in \overline{U^w} \Rightarrow x \in \Omega$,
- (v)- $A(E)$, $C(E)$ and $B(\overline{U^w})$ are bounded,
- (vi)- $\left(\frac{I-C}{A}\right)^{-1} B(\overline{U^w})$ is relatively weakly compact.

Then, either

- (A_1) the equation $x \in \lambda A(\frac{x}{\lambda})B(x) + \lambda C(\frac{x}{\lambda})$, has a solution for $\lambda = 1$, or
- (A_2) there is a point $u \in \partial_\Omega U$ and $\lambda \in (0,1)$ with $u \in \lambda A(\frac{u}{\lambda})Bu + \lambda C(\frac{u}{\lambda})$.

Proof. From Proposition 3.5, it follows that the multivalued mapping $\left(\frac{I-C}{A}\right)^{-1}$ exists on $B(\overline{U^w})$. By assumption (iii) and (iv), $F := \left(\frac{I-C}{A}\right)^{-1} B : \overline{U^w} \to P_{cv}(\Omega)$

is well defined. In view of Theorem 3.6, it easy to see that F has weakly sequentially closed graph. Using,

$$F(\overline{U^w}) \subseteq A(F(\overline{U^w}))B(\overline{U^w}) + C(F(\overline{U^w})).$$

Hence assumption (v) guarantees that $F(\overline{U^w})$ is bounded. Applying Theorem 2.6. The proof is complete. □

Theorem 4.3. *Let Ω be a non-empty closed convex subset of a Banach algebra E satisfying condition $(\mathcal{P})$ and Φ a $(MWNC)$ on E. Let U be a weakly open subset of Ω with $\theta \in U$. Assume $A, C : E \to P_{cv}(E)$ and $B : \overline{U^w} \to P(E)$ are three multivalued mappings satisfying the following conditions:*

(i)- A, B and C have weakly sequentially closed graphs,
(ii)- A and B are weakly compacts,
(iii)- C is $\Phi-$ condensing,
(iv)- For each $x \in \overline{U^w}$, $\left(\frac{I-C}{A}\right)^{-1} B(x)$ is convex,
(v)- $x \in A(x)B(y) + C(x)$, $y \in \overline{U^w} \Rightarrow x \in \Omega$,
(vi)- $A(E)$, $C(E)$ and $B(\overline{U^w})$ are bounded.

Then, either

(A_1) the equation $x \in \lambda A(\frac{x}{\lambda})B(x) + \lambda C(\frac{x}{\lambda})$, has a solution for $\lambda = 1$, or
(A_2) there is a point $u \in \partial_\Omega U$ and $\lambda \in (0, 1)$ with $u \in \lambda A(\frac{u}{\lambda})Bu + \lambda C(\frac{u}{\lambda})$.

Proof. From Proposition 3.5, and as the same argument of the proof of the Theorem 4.1, $F := \left(\frac{I-C}{A}\right)^{-1} B : \overline{U^w} \to P_{cv}(\Omega)$ is well defined and has a weakly sequentially closed graph. Now we show that $F(\overline{U^w})$ is a relatively weakly compact. By assumptions (vi) we see that $F(\overline{U^w})$ is bounded. Let $\{y_n\} \subset F(\overline{U^w})$ and choose $\{x_n\} \subset \overline{U^w}$ such that $y_n \in F(x_n)$. Accordingly, $y_n \in A(y_n)B(x_n)+C(y_n)$, and hence $(I-C)y_n \in Ay_nBx_n$. Taking into account assumption (ii) and the condition $(\mathcal{P})$. We can get a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ such that $(I-C)y_{n_k} \rightharpoonup z$ in E. Let $D = \{y_{n_k}\}$ and put $D \subseteq (I-C)D+CD \subseteq \overline{(I-C)D^w} + CD$, hence

$$\Phi(D) \le \Phi(\overline{((I-C)D)^w}) + \Phi(C(D)).$$

Obviously, $\overline{((I-C)D)^w}$ is a weakly compact. Since C is Φ-condensing. then we get $D = \{y_{n_k}\}$ is a relatively weakly compact. Accordingly, $\{y_{n_k}\}$ has a subsequence $\{y_{n_{k_j}}\}$ which converges weakly to some y_0 in Ω. Hence $F(\overline{U^w})$ is relatively weakly compact. From Theorem 2.6, either F has a fixed point or there exists $u \in \partial_\Omega U$ and $\lambda \in (0, 1)$ with $u \in \lambda F(u)$. □

Theorem 4.4. *Let Ω be a non-empty closed convex subset of a Banach algebra E satisfying condition $(\mathcal{P})$ and Φ a $(MWNC)$ on E. Let U be a weakly open subset of Ω with $\theta \in U$. Assume $A, C : E \to P_{cv}(E)$ and $B : \overline{U^w} \to P(E)$ are three multivalued mappings with weakly sequentially closed graph and satisfying the following conditions:*

(i)- A, B and C have weakly sequentially closed graphs,
(ii)- B and C are weakly compacts,
(iii)- $\|B(\overline{U^w})\| \leq 1$, and A is $\Phi-$ condensing,
(iv)- For each $x \in \overline{U^w}$, $\left(\frac{I-C}{A}\right)^{-1} B(x)$ is convex,
(v)- $x \in A(x)B(y) + C(x)$, $y \in \overline{U^w} \Rightarrow x \in \Omega$,
(vi)- $A(E)$, $C(E)$ and $B(\overline{U^w})$ are bounded.

Then, either

(A_1) the equation $x \in \lambda A(\frac{x}{\lambda})B(x) + \lambda C(\frac{x}{\lambda})$, has a solution for $\lambda = 1$, or
(A_2) there is a point $u \in \partial_\Omega U$ and $\lambda \in (0,1)$ with $u \in \lambda A(\frac{u}{\lambda})Bu + \lambda C(\frac{u}{\lambda})$.

Proof. From Proposition 3.10 the multivalued operator $\left(\frac{I-C}{A}\right)^{-1}$ exists on $B(\overline{U^w})$. Using the same argument of the proof of the Theorem 3.11 we can get $\left(\frac{I-C}{A}\right)^{-1} B(\overline{U^w})$ is a relatively weakly compact. Then applying Theorem 2.6. The proof is complete. □

Theorem 4.5. *Let Ω be a non-empty closed convex subset of a Banach algebra E satisfying condition $(\mathcal{P})$ and Φ a MWNC on E. Let U be a weakly open subset of Ω with $\theta \in U$. Assume $A, C : E \to P_{cv}(E)$ and $B : \overline{U^w} \to P(E)$ are three multivalued mappings with weakly sequentially closed graph and satisfying the following conditions:*

(i)- A, B and C have weakly sequentially closed graphs,
(ii)- A and B are weakly compacts,
(iii)- C is $\Phi-$non-expansive and hemi-weakly compact,
(iv)- For each $x \in \overline{U^w}$, $\left(\frac{I-C}{A}\right)^{-1} B(x)$ is convex,
(v)- $x \in A(x)B(y) + C(x)$, $y \in \overline{U^w} \Rightarrow x \in \Omega$,
(vi)- $A(E)$ $C(E)$ and $B(\overline{U^w})$ are bounded.

Then, either

(A_1) the equation $x \in \lambda A(\frac{x}{\lambda})B(x) + \lambda C(\frac{x}{\lambda})$, has a solution for $\lambda = 1$, or
(A_2) there is a point $u \in \partial_\Omega U$ and $\lambda \in (0,1)$ with $u \in \lambda A(\frac{u}{\lambda})Bu + \lambda C(\frac{u}{\lambda})$.

Proof. This is an immediate consequence of Proposition 3.12, Theorem 3.4 and Theorem 2.6. □

5. Functional integral inclusion of fractional order

In this section, we prove an existence theorem for the quadratic integral inclusion 1.2. Let $J = [0, T]$ be a closed and bounded interval subset of $\mathbb{R}$, $L^1(J, X)$ the space of all integrable X-valued functions with norm

$$\|x\|_{L^1} = \int_0^T |x(t)| dt.$$

$E := C(J, X)$ denote the Banach algebra of all continuous X-valued functions defined on J endowed with the norm

$$\|x\| = \sup_{t \in J} |x(t)|.$$

Throughout this section X will be a finite dimensional Banach algebra which satisfying a sequential condition $(\mathcal{P})$. Assume that (X, Σ, μ) is a measurable space The multivalued map $F : X \to P(Y)$ (where Y is a separable metric space) with closed values is called measurable if $F^{-1}(V) \in \Sigma$ for each open subset $V \subseteq Y$, and is called weakly measurable if $F^{-1}(U) \in \Sigma$ for each closed subset $U \subseteq Y$. Furthermore, F is weakly measurable if and only if the distance functions $f_y : X \to \mathbb{R}$, $f_y = dist(y, F(x)) = \inf\{|y - z| : z \in F(x)\}$ is measurable for all $y \in Y$.

Definition 5.1. A multivalued map $F : J \times X \to P(X)$ is called L^1_X-carathéodory if

- (i)- For each $x \in X$, $F_x = F(., x)$ is weakly measurable,
- (ii)- For each $t \in J$, $F_t = F(t, .)$ is weakly upper semi-continuous,
- (iii)- There exists a function $\alpha \in L^1(J, X)$ such that $\|F(t, x)\| \leq \alpha(t)$, a.e $t \in J$ and for all $x \in X$, where α is the growth function of F on $J \times X$.

For a function x defined on J we define the set $S_F(x) = \{u \in L^1(J, X) : u(t) \in F(t, x(t))$ for a.e $t \in J\}$ which is known as the set of selection functions. Also, denote that $\|F(t, x(t))\| = \sup\{|u(t)| : u(t) \in F(t, x(t))\}$.

Lemma 5.2 ([16]). *Let X be a Banach space, if* $\dim(X) < \infty$ *and* $F : J \times X \to P_{cl,bd}(X)$ *is* L^1_X*-carathéodory , then* $S_F(x) \neq \phi$ *for all* $x \in X$.

Lemma 5.3 ([12]). *Let $F : X \to P(Y)$, where X, Y be any topological vector space and D be a subset of X then the graph of F is convex if and only if the set D is convex and $\lambda F(x) + (1 - \lambda)F(y) \subseteq F(\lambda x + (1 - \lambda)y)$ for all $x, y \in D$ and $\lambda \in (0, 1)$.*

Now to discus the functional integral inclusion 1.2 we list the following hypotheses,

- (H_1) A multivalued mapping $G : J \times X \to P_{cl,bd,cv}(X)$ is L^1_X-carathéodory with a growth function $\eta \in L^1(J, X)$ such that $\|G(t, x(t))\| \leq \eta(t)$ a.e $t \in J$ for all $x \in E$.
- (H_2) For each $t \in J$, $K_t = K(t, .) : X \longrightarrow P_{cl,bd,cv}(X)$ is L^1_X-carathéodory, and there exist a bounded function $\beta \in L^1(J, X)$ with bound $\|\beta\|$ such that for all $x, y \in E$,

$$\|K(t, x(t)) - K(t, y(t))\| \leq \beta(t)|x(t) - y(t)| \quad a.e. \quad t \in J.$$

- (H_3) For each $t \in J$, $F_t = F(t, .) : X \longrightarrow P_{cl,bd,cv}(X)$ is L^1_X-carathéodory, and there exist a bounded function $\xi \in L^1(J, X)$ with bound $\|\xi\|$ such that for all $x, y \in E$,

$$\|F(t,x(t)) - F(t,y(t))\| \leq \xi(t)|x(t) - y(t)| \quad a.e. \quad t \in J.$$

(H_4) For each $t \in J$, $F_t = F(t,.)$ and $K_t = K(t,.)$ have convex graph.

(H_5) For each $x \in X$, $K_x = K(.,x)$ and $F_x = F(.,x)$ are continuous.

Consider $Q = \{u \in E : \|u\| \leq r\}$. Clearly Q is a non-empty, closed, bounded and convex set. Let us consider the operators A, B and C defined on Q by

$$Ax(t) = K(t,x(t)),$$
$$Cx(t) = F(t,x(t)),$$
$$Bx(t) = I^\alpha G(t,x(t)) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} G(s,x(s))ds.$$

Accordingly, the inclusion 1.2 is equivalent to the operator inclusion

$$x(t) \in Ax(t)Bx(t) + Cx(t).$$

By Lemma 5.2 it is clear that $S_K(x)$, $S_F(x)$, and $S_G(x)$ are non-empty sets then A, B and C are well defined.

Theorem 5.4. *Assume that the hypotheses $(H_1)-(H_5)$ holds. Suppose that there is a real number $r > 0$ such that*

$$r = \frac{F_1\Gamma(\alpha+1) + K_1T^\alpha\|\eta\|_{L1}}{(1-\|\xi\|)\Gamma(\alpha+1) - \|\beta\|T^\alpha\|\eta\|_{L1}},$$

where $F_1 = \sup_{t\in J}|F(t,0)|$, $K_1 = \sup_{t\in J}|K(t,0)|$, and $\frac{\|\beta\|T^\alpha\|\eta\|_{L_1}}{1-\|\xi\|} < \Gamma(\alpha+1)$. Then the inclusion 1.2 has a solution on J.

Proof. Since X is a Banach algebra with a sequential condition $(\mathcal{P})$, then $E = C(J,X)$ is also Banach algebra satisfying the condition $(\mathcal{P})$ [4]. Now, we claim that the mappings A, B and C satisfy all the assumptions of Corollary 3.9. To this end, we split the proof into sequence of steps.

Step1: A has weakly sequentially closed graph and $A(Q)$ is relatively weakly compact. First, we show that A has weakly sequentially closed graph. Let $\{x_n\} \subset Q$ and $x_n \rightharpoonup x \in Q$, $y_n \in A(x_n)$ for all $n \in \mathbb{N}$, $y_n \rightharpoonup y$. Then we get $y_n(t) \rightharpoonup y(t)$ (similarly, $x_n(t) \rightharpoonup x(t)$) [18]. Now for $y_n \in A(x_n)$ there exists $v_n \in S_K(x_n)$ such that

$$y_n(t) = v_n(t).$$

Fix $t \in J$. Without loss of generality we may assume that $y_n(t) \neq 0$. In view of Hahn-Banach theorem there exists $f \in X^*$ such that $f(y_n(t)) = |y_n(t)|$ and $|f|_* = 1$. Since $v_n(t) \in K(t,(x_n(t)))$ a.e. $t \in J$, and K with bounded values. By the reflexivity of X there exists a subsequence $\{v_{n_k}(t)\}$ converges weakly to some $v(t)$. i.e. $f(v_{n_k}(t)) \to f(v(t))$. Then $y(t) = v(t)$. Moreover, by hypotheses (H_2) $K(t,.)$ is weakly upper semi-continuous, so $K(t,.)$ has a weakly closed graph [19]. Then $v_{n_k}(t) \in K(t,x_{n_k}(t))$ implies that $v(t) \in K(t,x(t))$. Hence $v \in S_K(x)$ and therefore $y \in A(x)$ consequently A has a weakly sequentially

closed graph.
Next, we show that $A(Q)$ is relatively weakly compact. Let $t \in J$ be fixed, and $\{u_n\}$ be a sequence in $A(Q)$. Thus there exists $x_n \in Q$ such that $u_n \in A(x_n)$, and hence there exists $v_n \in S_F(x_n)$ such that $u_n(t) = v_n(t)$. Thus

$$\begin{aligned} |u_n(t)| &= |v_n(t)| \leq \|K(t, x_n(t))\| \\ &\leq \|K(t, x_n(t)) - K(t,0)\| + \|K(t,0)\| \\ &\leq \|\beta\| r + K_1. \end{aligned}$$

Therefore $\{u_n(t)\}$ is weakly equi-bounded. For all $t \in J$, the reflexivity of X implies that the set $\{u_n(t) : n \in \mathbb{N}\}$ is relatively weakly sequentially compact [22]. Now we show that $A(Q)$ is weakly equi-continuous. Let $t_1, t_2 \in J$ and assume that $u_n(t_1) \neq u_n(t_2)$. Then there exists $f \in X^*$ such that
$f(u_n(t_1) - u_n(t_2)) = |u_n(t_1) - u_n(t_2)|$ and $|f|_* = 1$. Thus

$$\begin{aligned} |u_n(t_1) - u_n(t_2)| &= |v_n(t_1) - v_n(t_2)| \leq |K(t_1, x_n(t_1)) - K(t_2, x_n(t_2))| \\ &\leq |K(t_1, x_n(t_1)) - K(t_2, x_n(t_1))| \\ &+ |K(t_2, x_n(t_1)) - K(t_2, x_n(t_2))| \\ &\leq |K(t_1, x_n(t_1)) - K(t_2, x_n(t_1))| \\ &+ \beta(t)|x_n(t_1) - x_n(t_2)|. \end{aligned}$$

Since $K_x = K(., x)$ is continuous and so as $t_1 \to t_2$, we get $|u_n(t_1) - u_n(t_2)| \to 0$. Hence $A(Q)$ is weakly equi-continuous. By Arzela Ascoli theorem, $u_{n_j} \rightharpoonup u \in A(Q)$, and hence by Eberlien-Šmulian theorem we conclude that $A(Q)$ is relatively weakly compact.
Step2: As an argument similar to that in Step 1, $C(Q)$ is relatively weakly compact and C has a weakly sequentially closed graph.
Step3: B has weakly sequentially closed graph and $B(Q)$ is relatively weakly compact. First, we show that B has weakly sequentially closed graph. Let $\{x_n\} \subset Q$ and $x_n \rightharpoonup x \in Q$, $y_n \in B(x_n)$ for all $n \in \mathbb{N}$, $y_n \rightharpoonup y$. Then we get $y_n(t) \rightharpoonup y(t)$ (similarly, $x_n(t) \rightharpoonup x(t)$) [18]. Now for $y_n \in B(x_n)$ there exists $v_n \in S_G(x_n)$ such that

$$y_n(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} v_n(s) ds.$$

Fix $t \in J$. Without loss of generality we may assume that $y_n(t) \neq 0$. In view of Hahn-Banach theorem there exists $f \in X^*$ such that $f(y_n(t)) = |y_n(t)|$ and $|f|_* = 1$. Since $v_n(t) \in G(t, (x_n(t)))$ a.e. $t \in J$, and G with bounded values. Then by the reflexivity of X there exists a subsequence $\{v_{n_k}(t)\}$ converges weakly to some $v(t)$. i.e. $f(v_{n_k}(t)) \to f(v(t))$. An application of Lebesgue dominated convergence theorem [21] yields

$$|y_{n_k}(t)| = f\left(\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} v_{n_k}(s) ds\right) \to f\left(\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} v(s) ds\right).$$

Then

$$y(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} v(s)ds.$$

Moreover, by hypotheses (H_2), $K(t,.)$ is weakly upper semi-continuous, so $K(t,.)$ has a weakly closed graph [19]. Then $v_{n_k}(t) \in G(t, x_{n_k}(t))$ implies that $v(t) \in G(t, x(t))$. Hence $v \in S_G(x)$ and therefore $y \in B(x)$ consequently B has a weakly sequentially closed graph.
Next we show that $B(Q)$ is relatively weakly compact. Let $t \in J$ be fixed, and $\{u_n\}$ be a sequence in $B(Q)$. Thus there exists $x_n \in Q$ such that $u_n \in B(x_n)$, and hence there exists $v_n \in S_G(x_n)$ such that

$$u_n(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} v_n(s)ds.$$

Thus

$$\begin{aligned} |u_n(t)| &= f\left(\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} v_n(s)ds\right) \\ &\leq \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |v_n(s)|ds \leq \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \|G(s, x_n(s))\|ds \\ &\leq \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |\eta(s)|ds \leq \frac{T^\alpha \|\eta\|_{L_1}}{\Gamma(\alpha+1)}. \end{aligned}$$

Therefore $\{u_n(t)\}$ is weakly equi-bounded. For all $t \in J$, the reflexivity of X implies that the set $\{u_n(t) : n \in \mathbb{N}\}$ is relatively weakly sequentially compact ([22] P.782). Now we show that $B(Q)$ is weakly equi-continuous. Let $t_1, t_2 \in J$ and assume that $u_n(t_1) \neq u_n(t_2)$. Then there exists $f \in X^*$ such that $f(u_n(t_1) - u_n(t_2)) = |u_n(t_1) - u_n(t_2)|$ and $|f|_* = 1$. Thus
$|u_n(t_1) - u_n(t_2)|$

$$\begin{aligned} &= f\left(\int_0^{t_1} \frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)} v_n(s)ds - \int_0^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} v_n(s)ds\right) \\ &\leq \left|\int_{t_2}^{t_1} \frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)} v_n(s)ds - \int_0^{t_2} \frac{(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}}{\Gamma(\alpha)} v_n(s)ds\right| \\ &\leq \int_{t_2}^{t_1} \frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)} \|G(s, x_n(s))\|ds \\ &+ \int_0^{t_2} \frac{(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}}{\Gamma(\alpha)} \|G(s, x_n(s))\|ds \\ &\leq \frac{\|\eta\|_{L_1}}{\Gamma(\alpha)} \left(\int_{t_2}^{t_1} (t_1-s)^{\alpha-1}ds + \int_0^{t_2} ((t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1})ds\right) \\ &\leq \frac{\|\eta\|_{L_1}}{\Gamma(\alpha+1)} (2(t_1-t_2)^\alpha + t_2^\alpha - t_1^\alpha). \end{aligned}$$

As $t_1 \to t_2$, we get $|u_n(t_1) - u_n(t_2)| \to 0$. Hence $B(Q)$ is weakly equi-continuous. By (Arzela Ascoli Theorem), $u_{n_j} \rightharpoonup u \in B(Q)$ and hence by

Eberlien-Šmulian's Theorem, we conclude that $B(Q)$ is relatively weakly compact.

Step4: If $y \in AyBx + Cy$, $x \in Q$ then $y \in Q$. Let $y \in AyBx + Cy$. Then there exists $k \in S_K(y)$, $f \in S_F(y)$ and $g \in S_G(x)$ such that

$$y(t) = k(t)I^{\alpha}g(t) + f(t).$$

Now

$$\begin{aligned}
|y(t)| &\leq |k(t)| \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |g(s)| ds + |f(t)| \\
&\leq \|K(t, y(t))\| \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \|G(s, x(s))\| ds + \|F(t, y(t))\| \\
&\leq (\|K(t, y(t)) - K(t, 0)\| + \|K(t, 0)\|) \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |\eta(t)| ds \\
&+ \|F(t, y(t)) - F(t, 0)\| + \|F(t, 0)\| \\
&\leq (\|\beta\| |y(t)| + K_1)(\frac{\|\eta\|_{L_1} T^{\alpha}}{\Gamma(\alpha+1)}) + \|\xi\| |y(t)| + F_1,
\end{aligned}$$

and we can do this for all $t \in J$. Then

$$\|y\| \leq (\|\beta\| \|y\| + K_1)(\frac{\|\eta\|_{L_1} T^{\alpha}}{\Gamma(\alpha+1)}) + \|\xi\| \|y\| + F_1.$$

Hence,

$$\|y\| \leq \frac{F_1 \Gamma(\alpha+1) + K_1 T^{\alpha} \|\eta\|_{L1}}{(1 - \|\xi\|)\Gamma(\alpha+1) - \|\beta\| T^{\alpha} \|\eta\|_{L1}}.$$

Therefore $y \in Q$. Thus $\left(\frac{I-C}{A}\right)^{-1} B(Q) \subset Q$.

Step5: $\left(\frac{I-C}{A}\right)^{-1} B(x)$ is convex for each $x \in Q$. Let $u_1, u_2 \in \left(\frac{I-C}{A}\right)^{-1} B(x)$. Then $u_1 \in A(u_1)B(x) + C(u_1)$ and $u_2 \in A(u_2)B(x) + C(u_2)$. Thus there exists $k_1 \in S_K(u_1)$, $k_2 \in S_K(u_2)$, $f_1 \in S_F(u_1)$, $f_2 \in S_F(u_2)$ and $g \in S_G(x)$ such that for all $t \in J$

$$u_1(t) = k_1(t)I^{\alpha}g(t) + f_1(t),$$
$$u_2(t) = k_2(t)I^{\alpha}g(t) + f_2(t).$$

For all $\lambda \in [0, 1]$,

$$\begin{aligned}
\lambda u_1(t) + (1-\lambda)u_2(t) &= \lambda k_1(t) I^{\alpha} g(t) \\
&+ \lambda f_1(t) + (1-\lambda)k_2(t)I^{\alpha}g(t) + (1-\lambda)f_2(t) \\
&= [\lambda k_1(t) + (1-\lambda)k_2(t)]I^{\alpha}g(t) \\
&+ [\lambda f_1(t) + (1-\lambda)f_2(t)] \\
&= k(t)I^{\alpha}g(t) + f(t).
\end{aligned}$$

Note that

$$k(t) = \lambda k_1(t) + (1-\lambda)k_2(t) \in \lambda K(t, u_1(t)) + (1-\lambda)K(t, u_2(t)),$$

$$f(t) = \lambda f_1(t) + (1-\lambda)f_2(t) \in \lambda F(t, u_1(t)) + (1-\lambda)F(t, u_2(t)).$$

From (H_4) $K(t,.)$ and $F(t,.)$ have convex graph and so

$$\lambda K(t,u_1(t)) + (1-\lambda)K(t,u_2(t)) \subseteq K(t,\lambda u_1(t) + (1-\lambda)u_2(t)),$$

$$\lambda F(t,u_1(t)) + (1-\lambda)F(t,u_2(t)) \subseteq F(t,\lambda u_1(t) + (1-\lambda)u_2(t)).$$

Obviously, $k = \lambda k_1 + (1-\lambda)k_2 \in L_1(J,X)$ and $f = \lambda f_1 + (1-\lambda)f_2 \in L_1(J,X)$. Then $k \in S_K(\lambda u_1 + (1-\lambda)u_2)$ and $f \in S_F(\lambda u_1 + (1-\lambda)u_2)$. Hence

$$\lambda u_1 + (1-\lambda)u_2 \in A(\lambda u_1 + (1-\lambda)u_2)B(x) + C(\lambda u_1 + (1-\lambda)u_2),$$

and therefore

$$\lambda u_1 + (1-\lambda)u_2 \in \left(\frac{I-C}{A}\right)^{-1} B(x).$$

□

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