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A Hybrid Method to Systems of Fredholm Integral Differential Equations

Jafar Biazar[a*], Yalda Parvari Moghaddam[b], Khadijeh Sadri[c]

[a]Department of Applied Mathematics, Faculty of Mathematical Science, University of Guilan, Rasht, Iran
[b]Department of Applied Mathematics, University Campus 2, University of Guilan, Rasht, Iran
[c]Mathematics Research Center, Near East University TRNC, Mersin 10, Nicosia 99138, Turkey

E-mail:biazar@guilan.ac.ir
E-mail:yaldaparvari.m@gmail.com
E-mail:khadijeh.sadrikhatouni@neu.edu.tr

Abstract. The method used in this research consists of a hybrid of the Block-Pulse functions and third-kind Chebyshev polynomials for solving systems of Fredholm integral differential equations. Through the use of an operational matrix representing the derivation, the problem is represented by a system of algebraic equations. Some examples are provided to illustrate the simplicity and effectiveness of the utilized method. In addition, results of the presented method have been compared with those obtained from the Tau method and variational iteration method that reveal the proposed scheme to be more applicable.

Keywords: System of Fredholm integral differential equations, Hybrid method, Block-pulse functions, Third-kind Chebyshev polynomials, Operational matrix.

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1. Introduction

Systems of integral differential equations appear in mathematical models of many phenomena. Because it is difficult to obtain the analytical solutions of systems of integral differential equations, numerical methods to obtain approximate solutions are widely considered. During the last years, many different

*Corresponding Author

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orthogonal functions and polynomials, such as Block-Pulse functions [7, 9], Fourier series [10], Legendre polynomials [6], different kinds of hybrid functions [2,3,14] and Legendre wavelets [15], were used to solve these systems, just to mention a few. The first- and second-kind Chebyshev polynomials have widely been used to solve various functional equations [10,11] and the third- and forth-kind Chebyshev polynomials have less been taken into account. On the other hand, the Block-Pulse method has a low accuracy. Hence, in order to increase the accuracy of resultant approximate solutions, a numerical hybrid approach using the Block- Pulse functions and the third-kind Chebyshev polynomials is presented for the following system of Fredholm integral differential equations,

$$\begin{aligned} y_i^{(m)}(\chi) &= f_i(\chi) + p_i(\chi, y_1(\chi), y_1^{'}(\chi), ..., y_1^{(m)}(\chi), ..., y_n^{(m)}(\chi)) \\ &+ \sum_{j=1}^{m_1} \int_a^b k_{ij}(\chi,\xi)\ q_{ij}(y_1(\xi), ..., y_1^{(m)}(\xi), ..., y_n^{(m)}(\xi) d\xi, \qquad a \le \chi < b, \end{aligned} \tag{1.1}$$

with the initial conditions

$$y_i^{(r)}(0) = \alpha_{i,r}, \quad i = 1, 2, ..., n,\ r = 0, 1, ..., m-1, \tag{1.2}$$

where m, m_1 are positive integers, $f_i(\chi)$, $i = 0, 1, 2, \cdots, n$, are known functions, p_i, $i = 1, 2, \cdots, n$, are linear or non-linear operators, $k_{ij}(\chi, \xi) \in L^2([a, b] \times [a, b])$ are the kernels and $y_i(\chi), i = 1, 2, \cdots, n$, are unknown functions [1]. The main target of this study is to prepare an effective approach for solving the system of (1.1) that leads to considerable computational validity and simplicity. Solutions of the given system are approximated by linear combinations of hybrid functions with unknown coefficients. The operational matrices are then utilized to reduce the system (1.1) to a linear or non-linear system of algebraic equations that by solving this algebraic equation, the unknown coefficients are determined and approximate solutions are acquired for $y_i(\chi), i = 1, 2, \cdots, n$. The rest of the current paper is organized as follows, in Section 2, a hybrid method and its construction are explained. Section 3 is devoted to obtaining operational matrices. In Section 4, the hybrid method is applied to approximate solutions of the system of Fredholm integral differential equations. Section 5 is devoted to illustrating the effectiveness of the proposed scheme through giving three objective examples. Finally, the conclusion and discussion appear in Section 6.

2. Construction of the hybrid method

In this section, we recall briefly the Block-Pulse functions, the third-kind Chebyshev polynomials, and represent a hybrid method composing of mentioned functions. It's acronym is as HBV. This method is introduced in [4] and is used to solved Fredholm Integro-differential Equations.

2.1. **HBV functions.** A set of Block-Pulse functions $b_i(\chi), i = 1, 2, ..., N$, on the closed interval $[0, T]$, are defined as follows

$$b_i(\chi) = \begin{cases} 1, & \dfrac{(i-1)T}{N} \le \chi \le \dfrac{iT}{N}, \\ 0, & \text{otherwise.} \end{cases} \tag{2.1}$$

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The Block-Pulse functions on $[0, T)$ are disjoint, so for $i, j = 1, 2, ..., N$, we have $b_i(\chi)b_j(\chi) = \delta_{ij}b_i(\chi)$, also these functions have the property of orthogonality on $[0, T)$ [5]. Using these disjoint functions and the third-kind Chebyshev polynomials, a piecewise orthogonal basis is constructed as hybrid functions. Polynomials considered in the hybrid Block-Pulse functions and the third-kind Chebyshev polynomials HBV on the closed interval $[0, T]$ are defined as follows,

$$H_{ij}(\chi) = \begin{cases} \sqrt{\frac{2T}{N}} v_j\left(\frac{2N\chi}{T} - 2j + 1\right), & \frac{(j-1)T}{N} \leq \chi \leq \frac{jT}{N}, \\ 0, & \text{otherwise.} \end{cases} \tag{2.2}$$

Here, $v_j(\chi), j = 1, 2, \cdots, M - 1$, are the third-kind Chebyshev polynomials that satisfy the following well-known three terms relation,

$$v_j(\chi) = 2\chi\ v_{j-1}(\chi) - v_{j-2}(\chi),\ j = 2, 3, ...,$$
$$v_0(\chi) = 1,\ v_1(\chi) = 2\chi - 1.$$

Since $H_{ij}(\chi)$ is the combination of the third-kind Chebyshev polynomials and the Block-Pulse functions which are both complete and orthogonal, then the set of hybrid functions is a complete orthogonal system in $L^2_\omega[0, 1)$ with the weight functions,

$$w_i(\chi) = w(2N\chi - 2i + 1), \quad i = 1, 2, ..., N.$$

In other words,

$$\int_{-1}^{1} w_i(\chi)v_i(\chi)v_j(\chi)d\chi = \sqrt{\pi}\delta_{ij},$$

where δ_{ij} is the Kronecker function and $w(\chi) = \sqrt{\frac{1+\chi}{1-\chi}}$.

2.2. **The approximation of functions.** The following theorem ensures the expansion of a square integrable function based on the proposed basis.

Theorem 2.1. *Let $y(\chi) \in L^2[0, 1)$ with a bounded second-order deriative, say $| y^{''}(\chi) | \leq A$, for some constant A, then*

(i) *$y(\chi)$ can be expanded as an infinite sum of the HBV and the series converges to $y(\chi)$ uniformly, that is*

$$y(\chi) = \sum_{i=1}^{\infty}\sum_{j=0}^{\infty} c_{ij}H_{ij}(\chi),$$

where $c_{ij} = \langle y(\chi), H_{ij}(\chi)\rangle_{L^2_{w_i}[0,1)}$.

(ii) *The approximation error is bounded, i.e.*

$$\beta_{y,i,M} \leqslant \frac{\pi A^2}{8} \sum_{i=N+1}^{\infty}\sum_{j=M}^{\infty} \frac{1}{i^5(j-1)^4}$$

where

$$\beta_{y,i,M} = \left(\int_0^1 \left|y(\chi) - \sum_{i=1}^{N}\sum_{j=0}^{M-1} c_{ij}H_{ij}(\chi)\right|^2 \omega_i(\chi)\ d\chi\right)^{\frac{1}{2}}.$$

Proof. See [4] on page 352, Theorem 5.1. □

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According to Theorem 2.1, a continuous function $y(\chi) \in L^2[0,1)$ can be expanded as,

$$y(\chi) = \sum_{i=1}^{\infty}\sum_{j=0}^{\infty} c_{ij}H_{ij}(\chi), \tag{2.3}$$

where,

$$c_{ij} = \frac{(y(\chi), H_{ij}(\chi))}{(H_{ij}(\chi), H_{ij}(\chi))} = \frac{N^2}{\pi}\int_0^1 H_{ij}(\chi)y(\chi)w_i(\chi)d\chi, \tag{2.4}$$

where $(.,.)$ stands for inner product on $L^2 \in [0,1]$ with the weight function $w_i(\chi)$. In practice, infinite series (2.3) will be reduced into the following finite form,

$$y(\chi) \approx \sum_{i=1}^{N}\sum_{j=0}^{M-1} c_{ij}H_{ij}(\chi) = C^T HBV(\chi), \tag{2.5}$$

where C and $HBV(x)$ are the following $(NM \times 1)$ vectors

$$\begin{aligned} &C = [c_{1,0}, ..., c_{1,M-1}, c_{2,0}, ..., c_{2,M-1}, ..., c_{N,0}, ..., c_{N,M-1}]^T,\\ &HBV(\chi) = [H_{1,0}, ..., H_{1,M-1}, H_{2,0}, ..., H_{2,M-1}, ..., H_{N,0}, ..., H_{N,M-1}]^T. \end{aligned} \tag{2.6}$$

The kernel $k(\chi,\xi) \in L^2([0,1] \times [0,1])$ can be approximated as follows,

$$K_{ij} = \frac{(H_i(\chi), (k(\chi,\xi), H_j(\chi))}{(H_i(\chi), H_j(\chi))(H_i(\xi), H_j(\xi))}, \; i,j = 1,2,...,NM. \tag{2.7}$$

3. Operational matrix of derivative

In this section, we will compute the operational matrix of derivative, which is important for solving the Fredholm integral differential equations [7]. To clarify first consider the six basis functions in (2.6) as the following (for $N = 2, M = 3$),

$$\begin{cases} H_{10}(\chi) = 1 \\ H_{11}(\chi) = 8\chi - 3 \\ H_{12}(\chi) = 64\chi^2 - 40\chi + 5 \end{cases} \quad 0 \le \chi < \frac{1}{2} \tag{3.1}$$

$$\begin{cases} H_{20}(\chi) = 1 \\ H_{21}(\chi) = 8\chi - 7 \\ H_{22}(\chi) = 64\chi^2 - 104\chi + 41 \end{cases} \quad \frac{1}{2} \le \chi < 1 \tag{3.2}$$

So, $HBV_6(\chi) = [H_{10}, H_{11}, H_{12}, H_{20}, H_{21}, H_{22}]$. By differentiating (3.1)–(3.2), we obtain

$$\begin{cases} \dfrac{dH_{10}}{d\chi} = 0 \\ \dfrac{dH_{11}}{d\chi} = 8 = 8H_{10} \\ \dfrac{dH_{12}}{d\chi} = 128\chi - 40 = 16H_{11} + 8H_{10} \\ \dfrac{dH_{20}}{d\chi} = 0 \\ \dfrac{dH_{21}}{d\chi} = 8 = 8H_{20} \\ \dfrac{dH_{22}}{d\chi} = 128\chi - 104 = 16H_{21} + 8H_{20} \end{cases}$$

Thus,

$$\frac{dHBV_6(\chi)}{d\chi} \approx \Delta_{6\times 6}\; HBV_6(\chi) \tag{3.3}$$

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where,

$$\Delta_{6\times 6} = 2\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 & 0 & 0 \\ 4 & 8 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 4 & 8 & 0 \end{bmatrix}.$$

The matrix $\Delta_{6\times 6}$ is called the operational matrix of derivative and can be written as

$$\Delta_{6\times 6} = 2\begin{bmatrix} \varrho_{3\times 3} & O_{3\times 3} \\ O_{3\times 3} & \varrho_{3\times 3} \end{bmatrix}$$

where

$$\varrho_{3\times 3} = \begin{bmatrix} 0 & 0 & 0 \\ 4 & 0 & 0 \\ 4 & 8 & 0 \end{bmatrix}.$$

Therefore, the differentiations of basis functions are written as linear combinations of basis functions themselves. In general, we have for arbitrary M, N

$$\frac{dHBV(\chi)}{d\chi} \approx \Delta \; HBV(\chi), \tag{3.4}$$

where Δ is the $NM \times NM$ operational matrix of the derivative as follows,

$$\Delta = \begin{bmatrix} \varrho & 0 & \dots & 0 \\ 0 & \varrho & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \varrho \end{bmatrix}$$

and $\varrho = [\alpha_{ij}]_{M\times M}$ whose elements are as the following,

$$\alpha_{ij} = \begin{cases} 2(i+j-1), & i > j \text{ and } (i+j) \text{ odd}, \\ 2(i-j), & i > j \text{ and } (i+j) \text{ even}, \\ 0, & \text{otherwise}. \end{cases}$$

As a result, we have

$$\frac{d^m HBV(\chi)}{d\chi^m} \approx \Delta^m \; HBV(\chi).$$

As an example, the matrix ϱ for $M = 5$ is as follows

$$\varrho = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 & 0 \\ 4 & 8 & 0 & 0 & 0 \\ 8 & 4 & 12 & 0 & 0 \\ 8 & 12 & 4 & 16 & 0 \end{bmatrix}$$

Remark 3.1. The integral of the product of two basis vectors in (2.6) can be obtained as

$$E = \int_0^1 HBV(\chi) HBV^T(\chi) d\chi, \tag{3.5}$$

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where E is a $NM \times NM$ diagonal matrix that can be obtained as the following,

$$E = \frac{1}{N^2}\begin{bmatrix} L_1 & 0 & \cdots & 0 \\ 0 & L_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & L_N \end{bmatrix} \tag{3.6}$$

where $L_n, n = 1, 2, ..., N$, are $M \times M$ symmetric matrices. For example, if $N = 2$ and $M = 3$, we have

$$E = \frac{1}{4}\begin{bmatrix} 2 & -2 & \frac{2}{3} & 0 & 0 & 0 \\ -2 & \frac{14}{3} & -\frac{10}{3} & 0 & 0 & 0 \\ \frac{2}{3} & -\frac{10}{3} & \frac{86}{15} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{674}{105} & -\frac{494}{105} & \frac{922}{315} \\ 0 & 0 & 0 & -\frac{494}{105} & \frac{2182}{315} & -\frac{1622}{315} \\ 0 & 0 & 0 & \frac{922}{315} & -\frac{1622}{315} & \frac{25402}{3465} \end{bmatrix} \tag{3.7}$$

4. Methodology

In this section the mentioned matrices in Section 3 are utilized to reduce system of (1.1) to a system of algebraic equations. For this purpose, consider system (1.1) with the initial conditions (1.2). The following approximations are proposed,

$$y_i(\chi) \approx C_i^T HBV(\chi), \; i = 1, 2, ..., n, \tag{4.1}$$

where the vectors $C_i, i = 1, 2, ..., n$, and $HBV(\chi)$ are determined in (2.6). So, we have

$$y_i^{(r)}(\chi) \approx C_i^T \Delta^r HBV(\chi), \quad i = 1, 2, ..., n, \; r = 1, 2, ..., m. \tag{4.2}$$

Therefore, initial conditions (1.2) are approximated as follows

$$C_i^T \Delta^r HBV(0) - \alpha_{i,r} \approx 0, \quad i = 1, 2, ..., n, \; r = 0, 1, ..., m-1. \tag{4.3}$$

So, nm algebraic equations are achieved. Now, using approximations (4.2) and resultant matrices, other terms of system (1.1) are approximated as

$$\begin{aligned} & f_i(\chi) \approx F_i^T HBV(\chi), \\ & p_i(\chi, y_1(\chi), ..., y_1^{(m)}(\chi), ..., y_n^{(m)}) \approx P_i^T HBV(\chi), \\ & k_{ij}(\chi, \xi) \approx HBV^T(\chi) K_{ij} HBV(\xi), \\ & q_{ij}^{(m)}(\chi, y_1(\chi), ..., y_1^{(m)}(\chi), ..., y_n^{(m)}) \approx Q_{ij}^T HBV(\chi), \quad i = 1, 2, ..., n, \; j = 1, 2, ..., m_1. \end{aligned} \tag{4.4}$$

where F_i and P_i are known vectors, K_{ij} are known matrices of the order $NM \times NM$, and C_i are unknown vectors that must be determined. Substituting

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approximations (4.2) and (4.4) into system (1.1) leads to the following equations

$$\begin{aligned} C_i^T HBV(\chi) &= F_i^T HBV(\chi) + P_i^T HBV(\chi) \\ &+ \sum_{j=1}^{m_1} \int_0^1 Q_{ij}^T HBV(\xi) HBV^T(\xi) K_{ij} HBV(\chi) d\xi \\ &= F_i^T HBV(\chi) + P_i^T HBV(\chi) + \Big\{ \sum_{j=1}^{m_1} Q_{ij}^T \ E \ K_{ij} \Big\} HBV^T(\chi). \end{aligned} \tag{4.5}$$

Multiplying both sides of (4.5) by $HBV^T(\chi)w_i(\chi)$ and applying $\int_0^1 (.)d\chi$, the following linear or non-linear system of algebraic equations will be obtained,

$$C_i^T = F_i^T + P_i^T + \sum_{j=1}^{m_1} Q_{ij}^T \ E \ K_{ij}, \quad i = 1, 2, ..., n, \ j = 1, 2, ..., m_1. \tag{4.6}$$

To determine n unknown vectors C_i, nm equations of (4.3) and $n(NM) - nm$ equations from (4.6) are considered. Finally, by solving the resultant system of algebraic equations including nNM equations, the vectors of coefficients $C_i, i = 1, 2, ..., n$, will be determined and approximate solutions are obtained from (4.1). Also, the absolute error function $e(y_i(\chi))$ is constructed as follows,

$$e(y_i(\chi)) = \Big| y_i(\chi) - \sum_{i=1}^{N} \sum_{j=0}^{M-1} c_{ij} \ H_{ij}(\chi) \Big|, \quad i = 1, 2, ..., n. \tag{4.7}$$

5. Numerical examples

For showing the efficiency of the proposed numerical approach, we consider the following examples. In the tables, the absolute error of y_i is denoted by AE_{y_i}. The results will be compared with those of some existing methods.

Example 5.1. As the first example, we consider the following linear system of Fredholm integral differential equations [8],

$$\begin{cases} y_1^{''}(\chi) + y_2^{'}(\chi) + \int_0^1 2\chi\xi(y_1(\xi) - 3y_2(\xi))d\xi = 3\chi^2 + \frac{3}{10}\chi + 8, \\ y_2^{''}(\chi) + y_1^{'}(\chi) + \int_0^1 3(2\chi + \xi^2)(y_1(\xi) - 2y_2(\xi))d\xi = 21\chi + \frac{4}{5}, \end{cases}$$

subject to the initial conditions $y_1(0) = 1, y_1^{'}(0) = 0, y_2(0) = -1$, and $y_2^{'}(0) = 2$. The exact solutions are $y_1(\chi) = 3\chi^2 + 1$ and $y_2(\chi) = \chi^3 + 2\chi - 1$. We can approximate the functions as follows,

$$\begin{aligned} & y_i(\chi) \approx HBV^T(\chi) \ C_i, \quad i = 1, 2, \\ & y_i^{'}(\chi) \approx HBV^T(\chi)\Delta \ C_i, \quad i = 1, 2, \\ & y_i^{''}(\chi) \approx HBV^T(\chi)\Delta^2 C_i, \quad i = 1, 2, \\ & 2\chi\xi \approx HBV^T(\chi) K_1 HBV(\xi), \\ & 3(2\chi + \xi^2) \approx HBV^T(\chi) K_2 HBV(\xi), \\ & 3\chi^2 + \frac{3}{10}x + 8 \approx HBV^T(\chi) F_1, \\ & 21\chi + \frac{4}{5} \approx HBV^T(\chi) F_2, \end{aligned}$$

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and the initial conditions can be approximated as

$$HBV^T(0)\ C_1 - 1 \approx 0, \quad HBV^T(0)\Delta\ C_1 \approx 0,$$
$$HBV^T(0)\ C_2 + 1 \approx 0, \quad HBV^T(0)\Delta\ C_2 - 2 \approx 0.$$

Substituting these approximations into the given equations and solving the resultant linear algebraic system, the coefficients will be obtained for $N = 1$ and $M = 4$ as follows

$$C1 = \begin{bmatrix} 2.0394 & 0.6629 & 0.1326 & -1.2663 \times 10^{-13} \end{bmatrix}^T,$$
$$C2 = \begin{bmatrix} 0.74025 & 0.5856 & 0.0773 & 0.01104 \end{bmatrix}^T.$$

Therefore, the following approximate solutions are resulted,

$$\begin{cases} y_1(\chi) \approx -1.1461 \times 10^{-11}\chi^3 + 3.0000\chi^2 + 1.9643 \times 10^{-16}\chi + 1, \\ y_2(\chi) \approx \chi^3 + 1.3900 \times 10^{-11}\chi^2 + 2\chi - 1. \end{cases}$$

The values of the exact solutions and absolute errors of approximate solutions are computed at the points $\chi_i = 0.2i, i = 0, 1, ..., 5$, for $N = 1$ and $M = 5$ that are seen in Table 1. Also, the results of the proposed approach are compared with those of the Tau method [8] in Table 1. The plots of the exact and approximate solutions are depicted in Figure 1. The results show the good agreement of the approximate solutions with the exact solutions.

χ_i	Exact solution (y_1, y_2)	HBV method (AE_{y_1}, AE_{y_2})	Tau method in [8] (AE_{y_1}, AE_{y_2})
0	$(1, -1)$	$(4.44e-16, 2.22e-16)$	$(3.0e-14, 3.1e-14)$
0.2	$(1.12, -0.592)$	$(2.66e-16, 5.42e-16)$	$(2.0e-14, 2.7e-14)$
0.4	$(1.48, -0.136)$	$(5.99e-15, 2.64e-15)$	$(1.0e-14, 3.4e-14)$
0.6	$(2.08, 0.416)$	$(1.86e-16, 7.03e-16)$	$(2.0e-14, 2.3e-14)$
0.8	$(2.92, 1.112)$	$(6.70e-16, 1.45e-16)$	$(1.0e-14, 2.0e-14)$
1.0	$(4, 2)$	$(1.89e-15, 2.59e-15)$	$(1.0e-14, 2.0e-14)$

Table 1. Numerical results for $N = 1$, $M = 5$ of Example 5.1.

Example 5.2. The second example is the following non-linear system of Fredholm integro-differential equations [16],

$$\begin{cases} y_1'(\chi) = 2\chi + \dfrac{149}{64} + \dfrac{1}{64}\displaystyle\int_0^1 (y_1^2(\xi) + y_2^2(\xi))d\xi, \quad y_1(0) = 1, \\ y_2'(\chi) = 2\chi - \dfrac{67}{64} + \dfrac{1}{64}\displaystyle\int_0^1 (y_1^2(\xi) - y_2^2(\xi))d\xi, \quad y_2(0) = 1, \end{cases}$$

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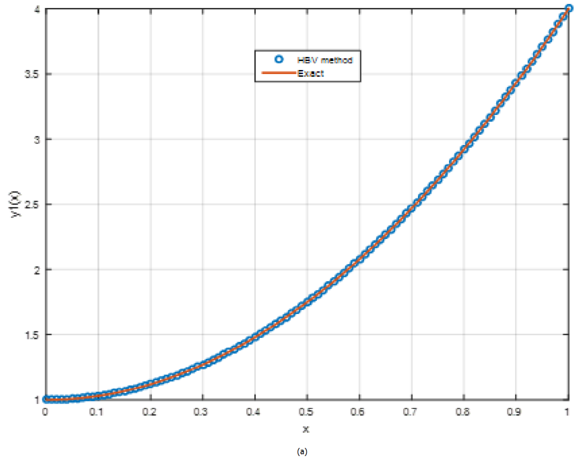

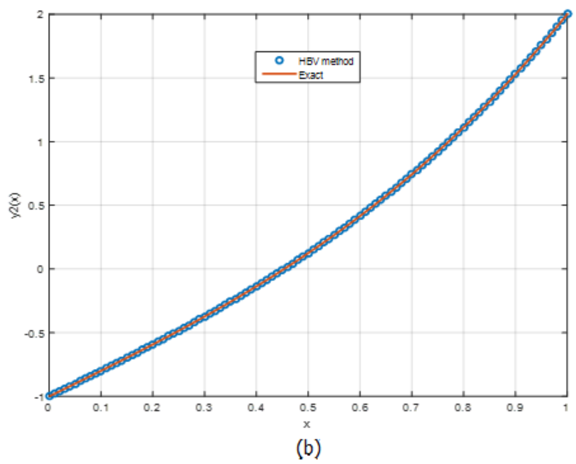

FIGURE 1. Plots of the exact and approximate solutions: (a) $y_1(\chi)$, (b) $y_2(\chi)$ for Example 5.1.

with the exact solutions $y_1(\chi) = 1 + \chi + \chi^2$, $y_2(\chi) = 1 - \chi + \chi^2$. We can approximate the functions as the following,

$$y_i(\chi) \approx HBV^T(\chi)C_i, \quad i = 1, 2,$$
$$y_i^{'}(\chi) \approx HBV^T(\chi)\Delta C_i, \quad i = 1, 2,$$
$$2\chi + \frac{149}{64} \approx HBV^T(\chi)F_1, \quad 2\chi - \frac{67}{64} \approx HBV^T(\chi)F_2,$$
$$\int_0^1 y_i^2(\xi)d\xi \approx \int_0^1 C_i^T HBV(\xi)HBV^T(\xi)C_i d\xi \approx C_i^T EC_i, \ i = 1, 2,$$

and the initial conditions can be approximated as

$$HBV^T(0)\ C_1 - 1 \approx 0, \quad HBV^T(0)\ C_2 - 1 \approx 0.$$

Substituting the above approximations into the given system leads to a nonlinear system of algebraic equations. By solving this system, the following

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coefficients would be achieved for $N = 1$ and $M = 3$,

$$C1 = \begin{bmatrix}1.6794 & 0.39775 & 0.044194\end{bmatrix}^T,$$
$$C2 = \begin{bmatrix}0.61872 & 0.44159 & 0.044194\end{bmatrix}^T.$$

The values of the exact solutions and absolute errors are computed at the points $\chi_i = 0.2i, i = 0, 1, ..., 5$, for $N = 1$ and $M = 3$ which are listed in Table 2. The maximum absolute errors of the proposed approach are $(\| e(y_1) \|_\infty, \| e(y_2) \|_\infty) = (4.44\times10^{-16}, 1.11\times10^{-16})$, while errors of the variational iteration method reported in [16] are $(\| e(y_1) \|_\infty, \| e(y_2) \|_\infty) = (6.26 \times 10^{-7}, 3.43 \times 10^{-6})$. The results show the more accuracy of the proposed method. The exact and approximate solutions are depicted in Figure 2.

χ_i	Exact solution (y_1, y_2)	Absolute errors (AE_{y_1}, AE_{y_2})
0	$(1.11, 0.91)$	$(0, 0)$
0.2	$(1.24, 0.84)$	$(0, 1.11e-16)$
0.4	$(1.56, 0.76)$	$(0, 0)$
0.6	$(1.96, 0.76)$	$(0, 0)$
0.8	$(2.44, 0.84)$	$(0, 0)$
1.0	$(3, 1)$	$(0, 0)$

Table 2. Values of exact solution and absolute errors for $N = 1$, $M = 3$ of Example 5.2.

Example 5.3. We study the following non-linear system of the Fredholm integral differential equations [16],

$$\begin{cases} y_1''(\chi) = -\cos(\chi) + \dfrac{3\pi}{128} + \dfrac{1}{64}\displaystyle\int_0^{\frac{\pi}{2}} (y_1^2(\xi) + y_2^2(\xi))d\xi, & y_1(0) = 2, y_1'(0) = 0, \\ y_2''(\chi) = \sin(\chi) - \dfrac{1}{64} + \dfrac{1}{64}\displaystyle\int_0^{\frac{\pi}{2}} (y_1^2(\xi) - y_2^2(\xi))d\xi, & y_2(0) = 1, y_2'(0) = -1, \end{cases}$$

with the exact solutions $y_1(\chi) = 1 + \cos(\chi)$, $y_2(\chi) = 1 - \sin(\chi)$. Implementing the hybrid method and solving the non-linear system, the following results will be achieved for $N = 1$ and $M = 5$,

$$C1 = \begin{bmatrix}1.2048 & -0.10261 & -0.018523 & 0.00099 & 0.00013\end{bmatrix}^T,$$
$$C2 = \begin{bmatrix}0.23375 & -0.14198 & 0.01174 & 0.00147 & -0.00011\end{bmatrix}^T.$$

Thus, the approximate solutions are as

$$\begin{cases} y_1(\chi) \approx \sqrt{2}(0.033795\chi^4 - 0.012251\chi^3 - 0.35096\chi^2 + 2.7756e - 17\chi + 1.4142), \\ y_2(\chi) \approx \sqrt{2}(-0.028135\chi^4 + 0.15757\chi^3 - 0.024557\chi^2 - 0.70711\chi + 0.70711). \end{cases}$$

Table 3 shows the values of the exact solutions and the absolute errors at the points $\chi_i = \pi/10i, i = 0, 1, ..., 5$, for $N = 1, M = 5$ and $N = 3, M = 4$.

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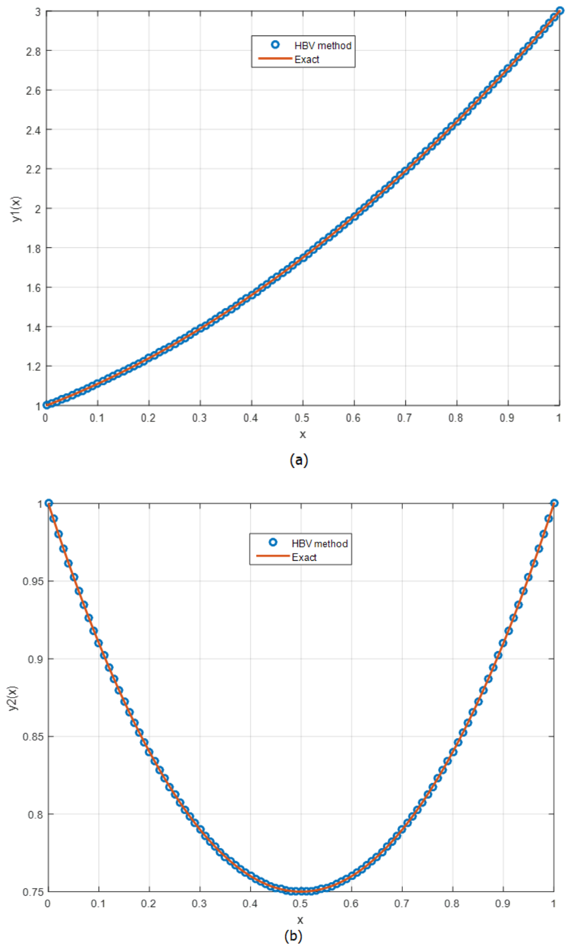

FIGURE 2. Plots of the exact and approximate solutions: (a) $y_1(\chi)$, (b) $y_2(\chi)$ for Example 5.2.

The maximum absolute errors of the proposed technique are $(\| e(y_1) \|_\infty, \| e(y_2) \|_\infty) = (6.4 \times 10^{-8}, 8.7 \times 10^{-8})$ and the errors reoprted in [16] are $(\| e(y_1) \|_\infty, \| e(y_2) \|_\infty) = (5.4 \times 10^{-7}, 3.5 \times 10^{-7})$. The plots of the exact and approximate solutions are shown in Figure 3. Since the trigonometric solutions of the system are approximated by a polynomial basis, the absolute errors increase in compared to the previous two examples. It is excepted the errors decrease when N, M increase for this non-linear system.

6. Conclusion

A Hybrid method to approximate the solutions of systems of Fredholm integral differential equations is proposed. This method is a combination of the Chebyshev polynomials of the third kind and the Block-Pulse functions. The numerical results show that the approximate solutions are in good agreement with the exact solutions. The comparison of the results with other methods, such as the Tau method [8] and variational iteration method [16], reveals that

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	Exact solution	$N=1, M=5$	$N=3, M=4$
χ_i	(y_1, y_2)	(AE_{y_1}, AE_{y_2})	$(AE_{y_1}, AE_{y_2}$
0	$(2,1)$	$(0, 2.22e-16)$	$(0,0)$
$\pi/10$	$(1.9801, 0.80133)$	$(7.54e-6, 6.93e-4)$	$(6.88e-8,\ 1.96e-8)$
$\pi/5$	$(1.9211, 0.61058)$	$(8.38e-4, 1.16e-3)$	$(1.86e-8,\ 3.52e-8)$
$3\pi/10$	$(1.8253, 0.43536)$	$(2.13e-3, 3.39e-3)$	$(3.39e-8, 1.81e-8)$
$2\pi/5$	$(1.6967, 0.28264)$	$(3.89e-3, 5.92e-3)$	$(5.77e-8, 3.54e-8)$
$\pi/2$	$(1.5403, 0.15853)$	$(6.49e-3, 8.71e-3)$	$(8.86e-8, 5.73e-8)$

Table 3. Values of exact solution and absolute errors at selected points for different values of N, M of Example 5.3.

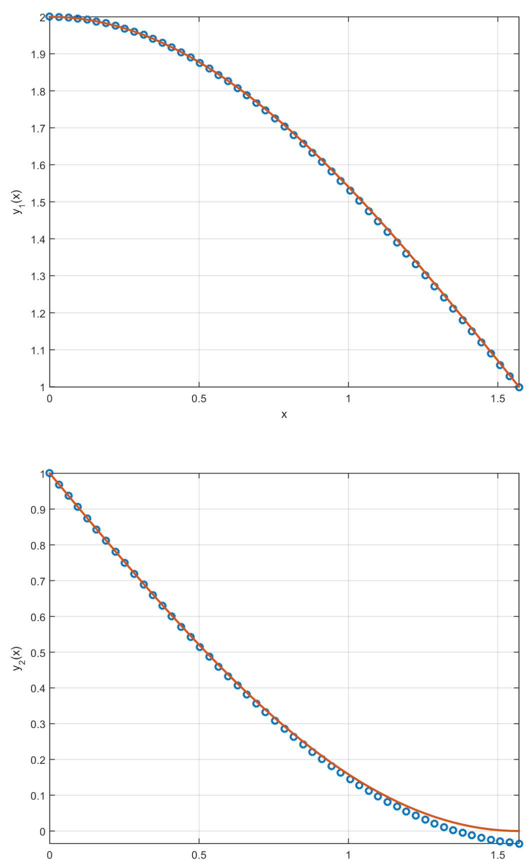

Figure 3. Plots of the exact and approximate solutions: (a) $y_1(\chi)$, (b) $y_2(\chi)$ for Example 5.3.

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the proposed approach has very good accuracy. Also, to get the best approximate solutions of the system, the values of N and M must be chosen large enough. The proposed method may be also applicable for the approximate solutions of the system of integral equations and other systems of functional equations.

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