Iranian Journal of Mathematical Sciences and Informatics
Vol. 19, No. 2 (2024), pp 111-117
DOI: 10.61186/ijmsi.19.2.111

Behavior of the Solutions of a Single-species Population Model with Piecewise Constant Argument

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Abstract. In this paper, we consider a population model with piecewise constant argument and show that every nonoscillatory solution approaches the equilibrium point as t tends to infinity. Moreover, we investigate every positive solution of the model that oscillates about the positive equilibrium point. Also, we give two examples to support the theorems.

Keywords: Piecewise constant argument, Difference equation, Oscillation.

2020 Mathematics subject classification: 34K11, 39A21.

1. Introduction

In 1982, Busenberg and Cooke [5] first established a mathematical model with a piecewise constant argument for analysing vertically transmitted diseases. After the work, these types of equations have been intensively investigated by many authors [1, 2, 3, 4, 6, 7, 8, 14, 16, 17, 18] and the references cited therein. But such equations as model have considered by few authors [9, 11, 12, 13]. These equations possess the structure of continuous systems in intervals. Continuity of solution of equation at a point linking any two consecutive intervals implies a recurrence relation for the values of the solution at such points. Mathematically, differential equations with piecewise constant arguments are described as hybrid dynamical systems, and hence, this type of equations shows the properties of both differential and difference equations.

Received 3 January 2020; Accepted 16 December 2020

In this paper, we consider the following differential equation with piecewise constant argument to study oscillations of all positive solutions of this equation about its positive equilibrium point and to establish that all nonoscillatory positive solutions of this equation converge to the equilibrium point as $t \to \infty$,

$$N'(t) = N(t)\{a + bN([t-1]) - cN^2([t-1])\}, \tag{1.1}$$

where $a, c \in (0, \infty)$, $b \in \mathbb{R}$ and $[\cdot]$ denotes the greatest integer function. The mathematical model

$$N'(t) = N(t)[a + bN(t-\tau) - cN^2(t-\tau)]$$

is presented to study by Gopalsamy and Ladas [10]. It is modeled the dynamics of a single-species population with a quadratic per-capita growth rate which is a "first order" nonlinear approximation of more general types of plausible nonlinear growth rates with single humps.

2. Preliminaries

Definition 2.1. A function N defined on the set $\{-1\} \cup [0, \infty)$ is a solution of Eq.(1.1) if $N(t)$ is continuous on $(0, \infty)$, and $N(t)$ is differentiable and satisfies (1.1) for any $t \in (0, \infty)$, with the possible exception of the points $[t]$ in $(0, \infty)$, where one-sided derivatives exist.

Definition 2.2. A solution $N(t)$ of Eq.(1.1) is said to be oscillatory about N^* if $N(t) - N^*$ has arbitrarily large zeros. Otherwise, it is said to be nonoscillatory.

Remark 2.3. The solution $N(t)$ oscillates about N^* if and only if $x(t)$ oscillates about zero.

Definition 2.4. A solution $y(n)$ is said to be oscillatory if it is neither eventually positive nor eventually negative. Otherwise, it is said to be nonoscillatory.

3. Main results

The analysis of Eq.(1.1) is done by finding the equilibrium point and its corresponding stability and oscillatory properties. It is noted that an equilibrium point is a state of the equation where the variable do not change over time. Eq.(1.1) has a biologically meaningful equilibrium point

$$N^* = \frac{b + \sqrt{b^2 + 4ac}}{2c}.$$

By taking

$$N(t) = N^* \exp(x(t)),$$

Eq.(1.1) is rewritten as

$$x'(t) = bN^*(\exp(x([t-1])) - 1) - c(N^*)^2(\exp(2x([t-1])) - 1). \tag{3.1}$$

[DOI: 10.61186/ijmsi.19.2.111]

Theorem 3.1. *Eq.(3.1) has a unique solution* $x(t)$ *on* $\{-1\} \cup [0, \infty)$ *with the initial conditions*

$$x(-1) = ln\frac{N_{-1}}{N^*} = x_{-1},\ x(0) = ln\frac{N_0}{N^*} = x_0, \tag{3.2}$$

where $N(-1) = N_{-1} > 0,\ N(0) = N_0 > 0$. *Moreover for* $n \le t < n+1,\ n \in \mathbb{N}$, x *has the form*

$$x(t) = y(n) + [bN^*(\exp(y(n-1)) - 1) - c(N^*)^2(\exp(2y(n-1)) - 1)](t-n), \tag{3.3}$$

where $y(n) = x(n)$ *and the sequence* $y(n)$ *is the unique solution of the difference equation*

$$y(n+1) - y(n) = bN^*(\exp(y(n-1)) - 1) - c(N^*)^2(\exp(2y(n-1)) - 1), \tag{3.4}$$

with the initial conditions

$$y(-1) = x_{-1}, y(0) = x_0. \tag{3.5}$$

Proof. Let $x_n(t) \equiv x(t)$ be a solution of (3.1) on $n \le t < n+1$. Eq.(3.1) is rewritten in the form

$$x'_n(t) = bN^*(\exp(x(n-1)) - 1) - c(N^*)^2(\exp(2x(n-1)) - 1). \tag{3.6}$$

Integrating both sides of Eq.(3.6) from n to t we get that

$$x_n(t) = x(n) + [bN^*(\exp(x(n-1)) - 1) - c(N^*)^2(\exp(2x(n-1)) - 1)](t-n). \tag{3.7}$$

On the other hand, from (3.7) the solution $x_{n-1}(t)$ on $n-1 \le t < n$ can be written as

$$\begin{aligned} x_{n-1}(t) = x(n-1) &+ [bN^*(\exp(x(n-2)) - 1) \\ &- c(N^*)^2(\exp(2x(n-2)) - 1)](t-n+1). \end{aligned}$$

Then, by using the continuity of the solutions at $t = n$, we obtain the difference equation (3.4). Considering the initial conditions (3.5), the solution of Eq.(3.4) can be obtained uniquely. Thus, the unique solution of (3.1), (3.2) is obtained as (3.3). □

Theorem 3.2. *If a solution* $N(t)$ *of (1.1) is nonoscillatory about* N^*, *then*

$$\lim_{t\to\infty} N(t) = N^*.$$

Proof. It is sufficient to prove that for every nonoscillatory solution $x(t)$ of the Eq.(3.1)

$$\lim_{t\to\infty} x(t) = 0.$$

Let $x(t)$ be an eventually positive solution of Eq.(3.1). Set

$$F(u) = cu^2 - bu - a$$

[DOI: 10.61186/ijmsi.19.2.111]

and observe that

$$F(u) > 0 \quad for \quad u > N^*. \tag{3.8}$$

From (3.8) and (3.1) we get that, eventually,

$$x'(t) < 0 \quad if \quad x(t) > 0.$$

Because $x(t) > 0$ is nonincreasing,

$$\lim_{t\to\infty} x(t) = l \geq 0$$

exists. We claim that $l = 0$. On the other hand

$$\lim_{t\to\infty} x'(t) = bN^*(\exp l - 1) - c(N^*)^2(\exp 2l - 1) < 0$$

which implies that

$$x(t) - x(n) = [bN^*(\exp l - 1) - c(N^*)^2(\exp 2l - 1)](t - n)$$

and so, as $t \to n + 1$ and $n \to \infty$, we have $0 = l - l = A < 0$. It is a contradiction. So, $l = 0$. For an eventually negative solution of Eq.(3.1), it is proved as similar. □

Theorem 3.3. *Assume that condition*

$$(2cN^* - b)N^* > \frac{1}{4}. \tag{3.9}$$

Then every positive solution of Eq.(1.1) oscillates about the positive equilibrium point N^*.

Proof. It suffices to show that every solution of Eq.(3.1) oscillates about zero. For the sake of contradiction, we assume that Eq.(3.1) has a nonoscillatory solution $x(t)$. Then, from Theorem 3.2

$$\lim_{t\to\infty} x(t) = 0.$$

At $t = n$, in view of $x(n) = y(n)$,

$$\lim_{n\to\infty} y(n) = 0.$$

We rewrite Eq.(3.4) in the form

$$y(n + 1) - y(n) + p(n)y(n - 1) = 0, \tag{3.10}$$

where

$$p(n) = \frac{-bN^*(\exp(y(n-1)) - 1)}{y(n-1)} + \frac{2c(N^*)^2(\exp(2y(n-1)) - 1)}{2y(n-1)},$$

$$\lim_{n\to\infty} p(n - 1) = (2cN^* - b)N^* > 0.$$

Hence

$$\liminf_{n\to\infty} p(n - 1) = (2cN^* - b)N^* > \frac{1}{4}$$

and by Philos [15] every solution of (3.10) oscillates. This is a contradiction. Hence every positive solution of Eq.(1.1) oscillates and the proof is completed. □

4. Examples

Example 4.1. Taking $a = \frac{1}{8}$, $b = -1$, $c = 1$ in Eq.(1.1),

$$N'(t) = N(t)\{\frac{1}{8} - N([t-1]) - N^2([t-1])\}.$$

From Theorem 3.2, every nonoscillatory solution of Eq.(1.1) approaches to the positive equilibrium point $N^* = \frac{1}{2}(-1 + \sqrt{\frac{3}{2}})$ with the initial conditions $N_{-1} = N^*$, $N_0 = N^* e^{0.1}$ in Fig 1.

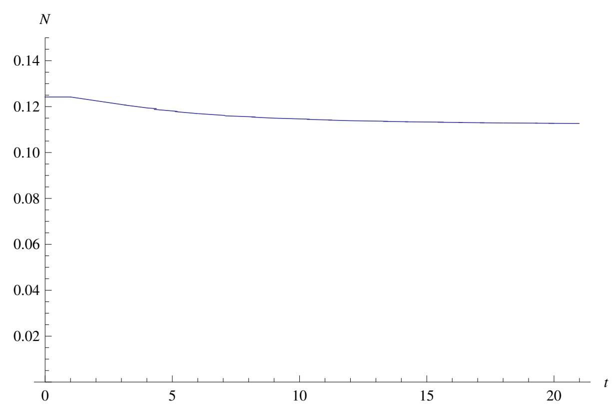

Figure 1. The solution of Eq.(1.1) with the initial conditions $N_{-1} = N^*$, $N_0 = N^* e^{0.1}$ in $t \in [0, 21)$.

Example 4.2. Taking $a = 1$, $b = -5$, $c = 2$ in Eq.(1.1),

$$N'(t) = N(t)\{1 - 5N([t-1]) - 2N^2([t-1])\}.$$

It is noted that $N^* = \frac{1}{4}(-5+\sqrt{33})$ is the positive equilibrium point of Eq.(1.1) and from the condition (3.9) of Theorem 3.3, every positive solution of Eq.(1.1) oscillates about $N^* = \frac{1}{4}(-5+\sqrt{33})$ with the initial conditions $N_{-1} = N^*$, $N_0 = N^* e^{0.1}$ in Fig 2.

Acknowledgments

The author wish to thank the referee for the helpful suggestions and comments.

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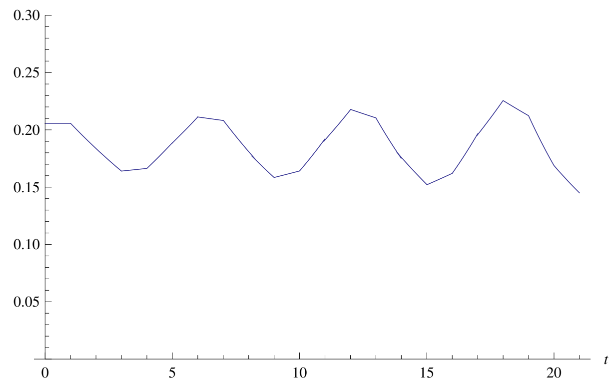

FIGURE 2. The solution of Eq.(1.1) with the initial conditions $N_{-1} = N^*$, $N_0 = N^* e^{0.1}$ in $t \in [0, 21)$.

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[DOI: 10.61186/ijmsi.19.2.111]