Iranian Journal of Mathematical Sciences and Informatics
Vol. 19, No. 2 (2024), pp 41-50
DOI: 10.61186/ijmsi.19.2.41

On bi–bases of Γ–semihypergroups

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Abstract. This paper focuses on the Γ–semihypergroups. Our goal seeks to find the conditions of sub–Γ–semihypergroup using bi–bases properties. We provide definitions and explain some properties of bi–bases in Γ–semihypergroups. The findings extend the results from bi–bases of Γ–semigroups. The findings demonstrate that if B is a bi–bases of a Γ–semihypergroup H; then, B is a sub–Γ–semihypergroup of H if and only if for any $b, c \in B$ and $\gamma \in \Gamma, b \in b\gamma c$ or $c \in b\gamma c$.

Keywords: Γ–semihypergroup, bi–bases, sub–Γ–semihypergroup.

2020 Mathematics subject classification: 20N20.

1. Introduction and preliminaries

F. Marty [1] created hyperstructure theory. Algebraic hyperstructures are a suitable generalization of classical algebraic structures. In a classical algebraic structure, the composition of elements is an element, while in an algebraic hyperstructure, the composition of two elements is a set. Fabrici [2, 3] introduced the concepts of one–sided and two–sided bases of a semigroup which were extended to ordered semigroups by T. Changpas and P. Summaprab [9]. Later,

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Received 19 November 2019; Accepted 31 July 2022

P. Kummoon and T. Changpas [6] defined and identified some properties of bi–bases in semigroups, and they [7] extended the results to Γ–semigroups. This paper attempts to find the condition of sub–Γ–semihypergroup using bi–bases properties in Γ–semihypergroups and begins by introducing the concept of bi–bases of Γ–semihypergroup and extend the results of bi–bases in Γ–semigroups to Γ–semihypergroups. In this section, the authors begin by recalling terminologies of Γ–semihypergroups as follows:

Let H be a nonempty set. Then, the map $\circ : H \times H \to P^*(H)$ where $P^*(H)$ is the family of nonempty subset of H. The system $(H, \circ)$ is called a semihypergroup if for every $x, y, z \in H$, $x \circ (y \circ z) = (x \circ y) \circ z$. If A and B are two nonempty subsets of H, then, we denote

$$A \circ B = \bigcup_{a\in A, b\in B} a \circ b;\ x \circ A = \{x\} \circ A \text{ and } A \circ x = A \circ \{x\} \text{ for all } x \in H.$$

A nonempty subset A of semihypergroup H is called a subsemihypergroup of H if $A \circ A \subseteq A$.

Definition 1.1. [8] Let H and Γ be two nonempty sets. Then, H is called a Γ–semihypergroup if Γ is a set of hyperoperation on H and for every $\alpha, \beta \in \Gamma$, $x\alpha(y\beta z) = (x\alpha y)\beta z$ for all $x, y, z \in H$.

If A and B are two nonempty subsets of H, we denote

$$A\Gamma B = \bigcup_{\gamma\in\Gamma} A\gamma B = \cup\{a\gamma b \mid a \in A, b \in B \text{ and } \gamma \in \Gamma\}.$$

Let $(H, \circ)$ be a semihypergroup and $\Gamma = \{\circ\}$. Then, H is a Γ–semihypergroup. Clearly, every semihypergroup is a Γ–semihypergroup.

Definition 1.2. [5] Let H be a Γ–semihypergroup. A nonempty subset A of H is called a sub–Γ–semihypergroup of H if $A\Gamma A \subseteq A$. A sub–Γ–semihypergroup A of H is called a bi–Γ–hyperideal of H if $A\Gamma H\Gamma A \subseteq A$.

Proposition 1.3. [8] *Let H be a Γ–semihypergroup and B_i be a bi–Γ–hyperideal of H for any $i \in I$. If $\bigcap_{i\in I} B_i \neq \emptyset$; then, $\bigcap_{i\in I} B_i$ is a bi-Γ-hyperideal of H.*

Let A be a nonempty subset of a Γ-semihypergroup H and define the set of all bi–Γ–hyperideal of H containing A as follows:

$$K = \{B \mid B \text{ is a bi–}\Gamma\text{–hyperideal of } H \text{ containing } A\}.$$

Clearly, $K \neq \emptyset$, because $H \in K$. Suppose $(A)_b = \bigcap_{B\in K} B$. This indicates seen that $A \subseteq (A)_b$. By proposition 1.3, $(A)_b$ is a bi–Γ–hyperideal of H. Moreover, $(A)_b$ is the smallest bi–Γ–hyperideal of H containing A.

Proposition 1.4. *Let A be a nonempty subset of a Γ–semihypergroup H. Then,*

$$(A)_b = A \cup A\Gamma A \cup A\Gamma H\Gamma A$$

[DOI: 10.61186/ijmsi.19.2.41]

Proof. Suppose $B = A \cup A\Gamma A \cup A\Gamma H\Gamma A$. Clearly, $A \subseteq B$. Consider $B\Gamma B = (A \cup A\Gamma A \cup A\Gamma H\Gamma A)\Gamma(A \cup A\Gamma A \cup A\Gamma H\Gamma A) \subseteq A\Gamma A \cup A\Gamma H\Gamma A \subseteq B$. Hence, B is a sub–Γ–semihypergroup of H containing A. Consider $B\Gamma H\Gamma B = (A \cup A\Gamma A \cup A\Gamma H\Gamma A)\Gamma H\Gamma(A \cup A\Gamma A \cup A\Gamma H\Gamma A) \subseteq A\Gamma H\Gamma A \subseteq B$. Therefore, B is a bi–Γ–hyperideal of H containing A. Let C be any bi–Γ–hyperideal of H containing A. Thus, $A \subseteq C$. Since C is a sub–Γ–semihypergroup of H; so, $A\Gamma A \subseteq C\Gamma C \subseteq C$. $A\Gamma H\Gamma A \subseteq C\Gamma H\Gamma C \subseteq C$ because C is bi–Γ–hyperideal of H. Hence, $B = A \cup A\Gamma A \cup A\Gamma H\Gamma A \subseteq C$. Thus, B is the smallest bi–Γ–hyperideal of H containing A. Therefore, $(A)_b = A \cup A\Gamma A \cup A\Gamma H\Gamma A$. □

$(A)_b$ is called the bi–Γ–hyperideal of H generated by A.

Definition 1.5. Let H be a Γ–semihypergroup. A nonempty subset B of H is called a bi–bases of H if it satisfies the following two conditions.

1. $H = (B)_b$ (i.e., $H = B \cup B\Gamma B \cup B\Gamma H\Gamma B$).
2. If A is a nonempty subset of B such that $H = (A)_b$; then, $A = B$.

Example 1.6. Let $H = \{x, y, z, w\}$ and $\Gamma = \{\beta, \alpha\}$ be the sets of hyperoperations defined below

β	x	y	z	w
x	$\{x\}$	$\{x,y\}$	$\{z,w\}$	$\{w\}$
y	$\{x,y\}$	$\{x,y\}$	$\{z,w\}$	$\{w\}$
z	$\{z,w\}$	$\{z,w\}$	$\{z\}$	$\{w\}$
w	$\{w\}$	$\{w\}$	$\{w\}$	$\{w\}$

α	x	y	z	w
x	$\{x,y\}$	$\{x,y\}$	$\{z,w\}$	$\{w\}$
y	$\{x,y\}$	$\{y\}$	$\{z,w\}$	$\{w\}$
z	$\{z,w\}$	$\{z,w\}$	$\{z\}$	$\{w\}$
w	$\{w\}$	$\{w\}$	$\{w\}$	$\{w\}$

N. Yaqoob [4] showed that H is a Γ-semihypergroup. Consider $(A)_1 = \{x\}$ and $(A)_2 = \{y\}$; so, $(A)_1$ and $(A)_2$ are bi–bases of H.

2. Main results

In this section, we characterize bi–bases of Γ–semihypergroups and show conditions of sub–Γ–semihypergroup using bi–bases properties.

Lemma 2.1. *Let B be a bi-bases of a Γ–semihypergroup H and $a, b \in B$. If $a \in b\Gamma b \cup b\Gamma H\Gamma b$; then, $a = b$.*

Proof. Let $a, b \in B$. Suppose $a \in b\Gamma b \cup b\Gamma H\Gamma b$ and $a \neq b$. Setting $A = B\backslash\{a\}$; then, $A \subseteq B$. From $a \neq b$, so $b \in A$. Hence, $(A)_b \subseteq (B)_b = H$. Let $x \in H = (B)_b$. Then, $x \in B \cup B\Gamma B \cup B\Gamma H\Gamma B$. There are three cases to consider.

Case 1: $x \in B$.

Subcase 1.1: $x \neq a$. Then, $x \in B\backslash\{a\} = A \subseteq (A)_b$.

Subcase 1.2: $x = a$. So, $x = a \in b\Gamma b \cup b\Gamma H\Gamma b \subseteq A\Gamma A \cup A\Gamma H\Gamma A \subseteq (A)_b$.

Case 2 : $x \in B\Gamma B$. Hence, $x \in b_1\gamma b_2$ for some $b_1, b_2 \in B$ and $\gamma \in \Gamma$.

[DOI: 10.61186/ijmsi.19.2.41]

Subcase 2.1: $b_1 = a$ and $b_2 = a$. By assumption,

$$\begin{aligned} x \in b_1\gamma b_2 = a\gamma a \\ &\subseteq (b\Gamma b \cup b\Gamma H\Gamma b)\Gamma(b\Gamma b \cup b\Gamma H\Gamma b) \\ &= b\Gamma b\Gamma b\Gamma b \cup b\Gamma b\Gamma b\Gamma H\Gamma b \cup b\Gamma H\Gamma b\Gamma b\Gamma b \cup b\Gamma H\Gamma b\Gamma b\Gamma H\Gamma b \\ &\subseteq A\Gamma A\Gamma A\Gamma A \cup A\Gamma A\Gamma A\Gamma H\Gamma A \cup A\Gamma H\Gamma A\Gamma A\Gamma A \cup A\Gamma H\Gamma A\Gamma A\Gamma H\Gamma A \\ &\subseteq A\Gamma H\Gamma A \subseteq (A)_b \end{aligned}$$

Subcase 2.2: $b_1 \neq a$ and $b_2 = a$. By assumption and $A = B\backslash\{a\}$,

$$\begin{aligned} x \in b_1\gamma b_2 = b_1\gamma a &\subseteq (B\backslash\{a\})\Gamma(b\Gamma b \cup b\Gamma H\Gamma b) \\ &= (B\backslash\{a\})\Gamma b\Gamma b \cup (B\backslash\{a\})\Gamma b\Gamma H\Gamma b \\ &\subseteq A\Gamma A\Gamma A \cup A\Gamma A\Gamma H\Gamma A \subseteq A\Gamma H\Gamma A \subseteq (A)_b \end{aligned}$$

Subcase 2.3: $b_1 = a$ and $b_2 \neq a$. By assumption and $A = B\backslash\{a\}$; then,

$$\begin{aligned} x \in b_1\gamma b_2 = a\gamma b_2 &\subseteq (b\Gamma b \cup b\Gamma H\Gamma b)\Gamma(B\backslash\{a\}) \\ &= b\Gamma b\Gamma(B\backslash\{a\}) \cup b\Gamma H\Gamma b\Gamma(B\backslash\{a\}) \\ &\subseteq A\Gamma A\Gamma A \cup A\Gamma A\Gamma H\Gamma A \subseteq A\Gamma H\Gamma A \subseteq (A)_b. \end{aligned}$$

Subcase 2.4: $b_1 \neq a$ and $b_2 \neq a$. From $A = B\backslash\{a\}$, so $x \in b_1\gamma b_2 \subseteq (B\backslash\{a\})\Gamma(B\backslash\{a\}) = A\Gamma A \subseteq (A)_b$.

Case 3: $x \in B\Gamma H\Gamma B$. Hence, $x \in b_3\gamma_1 h\gamma_2 b_4$ for some $b_3, b_4 \in B$, $\gamma_1, \gamma_2 \in \Gamma$ and $h \in H$.

Subcase 3.1: $b_3 = a$ and $b_4 = a$, consider

$$\begin{aligned} x &\in b_3\gamma_1 h\gamma_2 b_4 \\ &= a\gamma_1 h\gamma_2 a \\ &\subseteq (b\Gamma b \cup b\Gamma H\Gamma b)\Gamma H\Gamma(b\Gamma b \cup b\Gamma H\Gamma b) \\ &= b\Gamma b\Gamma H\Gamma b\Gamma b \cup b\Gamma b\Gamma H\Gamma b\Gamma H\Gamma b \cup b\Gamma H\Gamma b\Gamma H\Gamma b\Gamma b \cup b\Gamma H\Gamma b\Gamma H\Gamma b\Gamma b \\ &\subseteq A\Gamma A\Gamma H\Gamma A\Gamma A \cup A\Gamma A\Gamma A\Gamma A\Gamma H\Gamma A \cup A\Gamma H\Gamma A\Gamma H\Gamma A\Gamma A \cup A\Gamma H\Gamma A\Gamma H\Gamma A\Gamma A \\ &\subseteq A\Gamma H\Gamma A \subseteq (A)_b. \end{aligned}$$

Subcase 3.2: $b_3 \neq a$ and $b_4 = a$. By assumption and $A = B\backslash\{a\}$; so,

$$\begin{aligned} x \in b_3\gamma_1 h\gamma_2 b_4 &= b_3\gamma_1 h\gamma_2 a \\ &\subseteq (B\backslash\{a\})\Gamma H\Gamma(b\Gamma b \cup b\Gamma H\Gamma b) \\ &= (B\backslash\{a\})\Gamma H\Gamma b\Gamma b \cup (B\backslash\{a\})\Gamma H\Gamma b\Gamma H\Gamma b \\ &\subseteq A\Gamma H\Gamma A\Gamma A \cup A\Gamma H\Gamma A\Gamma H\Gamma A \subseteq A\Gamma H\Gamma A \subseteq (A)_b. \end{aligned}$$

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Subcase 3.3 : $b_3 = a$ and $b_4 \neq a$. By assumption and $A = B\backslash\{a\}$; hence,

$$\begin{aligned} x \in b_3\gamma_1 h\gamma_2 b_4 &= a\gamma_1 h\gamma_2 b_4 \\ &\subseteq (b\Gamma b \cup b\Gamma H\Gamma b)\Gamma H\Gamma(B\backslash\{a\}) \\ &= b\Gamma b\Gamma H\Gamma(B\backslash\{a\}) \cup b\Gamma H\Gamma b\Gamma H\Gamma(B\backslash\{a\}) \\ &\subseteq A\Gamma A\Gamma H\Gamma A \cup A\Gamma H\Gamma A\Gamma H\Gamma A \subseteq A\Gamma H\Gamma A \subseteq (A)_b. \end{aligned}$$

Subcase 3.4: $b_3 \neq a$ and $b_4 \neq a$. From $A = B\backslash\{a\}$; then, $x \in b_3\gamma_1 h\gamma_2 b_4 \subseteq (B\backslash\{a\})\Gamma H\Gamma(B\backslash\{a\}) = A\Gamma H\Gamma A \subseteq (A)_b$. Thus, $(A)_b = H$. This is a contradiction. Therefore, $a = b$. □

Lemma 2.2. *Let B be a bi–bases of a Γ–semihypergroup H and $a, b, c \in B$. If $a \in c\Gamma b \cup c\Gamma H\Gamma b$, then $a = b$ or $a = c$.*

Proof. Let $a, b, c \in B$ and $h \in H$. Assume $a \in c\Gamma b \cup c\Gamma H\Gamma b$ such that $a \neq b$ and $a \neq c$. Setting $A = (B\backslash\{a\})$. Hence, $A \subseteq B$, then $(A)_b \subseteq (B)_b = H$. From $a \neq b$ and $a \neq c$; so, $b, c \in A$. Let $x \in H$. Since $(B)_b = H$; thus, $x \in B \cup B\Gamma B \cup B\Gamma H\Gamma B$. There are three cases to consider.

Case 1: $x \in B$.

Subcase 1.1: $x \neq a$. Then, $x \in B\backslash\{a\} = A \subseteq (A)_b$.

Subcase 1.2: $x = a$. Thus, $x = a \in c\Gamma b \cup c\Gamma H\Gamma b \subseteq A\Gamma A \cup A\Gamma H\Gamma A \subseteq (A)_b$.

Case 2: $x \in B\Gamma B$. Then, $x \in b_1\gamma b_2$ for some $b_1, b_2 \in B$ and $\gamma \in \Gamma$.

Subcase 2.1: $b_1 = a$ and $b_2 = a$. By assumption,
$x \in b_1\gamma b_2 = a\gamma a \subseteq (c\Gamma b \cup c\Gamma H\Gamma b)\Gamma(c\Gamma b \cup c\Gamma H\Gamma b) \subseteq A\Gamma H\Gamma A \subseteq (A)_b$.

Subcase 2.2: $b_1 \neq a$ and $b_2 = a$. By assumption and $A = B\backslash\{a\}$; then,
$x \in b_1\gamma b_2 = b_1\gamma a \subseteq (B\backslash\{a\})\Gamma(c\Gamma b \cup c\Gamma H\Gamma b) \subseteq A\Gamma H\Gamma A \subseteq (A)_b$.

Subcase 2.3: $b = a$ and $b \neq a$. By assumption and $A = B\backslash\{a\}$; so,
$x \in b_1\gamma b_2 = a\gamma b_2 \subseteq (c\Gamma b \cup c\Gamma H\Gamma b)\Gamma(B\backslash\{a\}) \subseteq A\Gamma H\Gamma A \subseteq (A)_b$.

Subcase 2.4: $b_1 \neq a$ and $b_2 \neq a$. By assumption and $A = B\backslash\{a\}$; thus,
$x \in b_1\gamma b_2 \subset (B\backslash\{a\})\Gamma(B\backslash\{a\}) = A\Gamma A \subseteq (A)_b$.

Case 3: $x \in B\Gamma H\Gamma B$. Hence, $x \in b_3\gamma_1 h\gamma_2 b_4$ for some $b_3, b_4 \in B$, $\gamma_1, \gamma_2 \in \Gamma$ and $h \in H$.

Subcase 3.1: $b_3 = a$ and $b_4 = a$. Then,
$x \in b_3\gamma_1 h\gamma_2 b_4 = a\gamma_1 h\gamma_2 a \subseteq (c\Gamma b \cup c\Gamma H\Gamma b)\Gamma H\Gamma(c\Gamma b \cup c\Gamma H\Gamma b) \subseteq A\Gamma H\Gamma A \subseteq (A)_b$.

Subcase 3.2: $b_3 \neq a$ and $b_4 = a$. By assumption and $A = B\backslash\{a\}$; so,
$x \in b_3\gamma_1 h\gamma_2 b_4 = b_3\gamma_1 h\gamma_2 a \subseteq (B\backslash\{a\})\Gamma H\Gamma(c\Gamma b \cup c\Gamma H\Gamma b) \subseteq A\Gamma H\Gamma A \subseteq (A)_b$.

Subcase 3.3: $b_3 = a$ and $b_4 \neq a$. By assumption and $A = B\backslash\{a\}$; thus,
$x \in b_3\gamma_1 h\gamma_2 b_4 = a\gamma_1 h\gamma_2 b_4 \subseteq (c\Gamma b \cup c\Gamma H\Gamma b)\Gamma H\Gamma(B\backslash\{a\}) \subseteq A\Gamma H\Gamma A \subseteq (A)_b$.

Subcase 3.4: $b_3 \neq a$ and $b_4 \neq a$. From $A = B\backslash\{a\}$; then,
$x \in b_3\gamma_1 h\gamma_2 b_4 \in (B\backslash\{a\})\Gamma H\Gamma(B\backslash\{a\}) = A\Gamma H\Gamma A \subseteq (A)_b$.

From all cases, $(A)_b = H$. This is a contradiction. Therefore, $a = b$ or $a = c$. □

[DOI: 10.61186/ijmsi.19.2.41]

Definition 2.3. Let H be a Γ–semihypergroup. Define a quasi-order on H by, for any $a, b \in H$, $a \leq_b b \Leftrightarrow (a)_b \subseteq (b)_b$.

In example 1.6, $A_1 = \{x\}$ and $A_2 = \{y\}$ are bi–bases of H. Since $(x)_b \subseteq (y)_b$, so $x \leq_b y$ and since $(y)_b \subseteq (x)_b$, then $y \leq_b x$. From $x \leq_b y$ and $y \leq_b x$, but $x \neq y$. Therefore, $\leq_b$ is not a partial order on H. This shows that the order $\leq_b$ defined above is not, in general, a partial order.

Lemma 2.4. *Let B be a bi–bases of a Γ–semihypergroup H and $a, b \in B$. If $a \neq b$, then neither $a \leq_b b$ or $b \leq_b a$.*

Proof. Let $a, b \in B$. Suppose $a \neq b$. If $a \leq_b b$; hence, $(a)_b \subseteq (b)_b$. Thus, $a \in (a)_b \subseteq (b)_b = \{b\} \cup b\Gamma b \cup b\Gamma H\Gamma b$. By Lemma 2.1, $a = b$. This is a contradiction. If $b \leq_b a$, can be proved similarly. □

Lemma 2.5. *Let B be a bi-bases of a Γ–semihypergroup H. For all $a, b, c \in B$, $\gamma_1, \gamma_2 \in \Gamma$ and $h \in H$.*

1. *If $a \in b\gamma_1 c \cup b\gamma_1 c\Gamma b\gamma_1 c \cup b\gamma_1 c\Gamma H\Gamma b\gamma_1 c$; then, $a = b$ or $a = c$.*
2. *If $a \in b\gamma_1 h\gamma_2 c \cup b\gamma_1 h\gamma_2 c\Gamma b\gamma_1 h\gamma_2 c \cup b\gamma_1 h\gamma_2 c\Gamma H\Gamma b\gamma_1 h\gamma_2 c$; then, $a = b$ or $a = c$.*

Proof. Let $a, b, c \in B$, $\gamma \in \Gamma$ and $h \in H$.
(1.) Assume $a \in b\gamma_1 c \cup b\gamma_1 c\Gamma b\gamma_1 c \cup b\gamma_1 c\Gamma H\Gamma b\gamma_1 c$ such that $a \neq b$ and $a \neq c$. Consider $A = B\backslash\{a\}$. Clearly, $A \subseteq B$, thus $(A)_b \subseteq (B)_b = H$. From $a \neq b$ and $a \neq c$; so, $b, c \in A$. Let $x \in H$. Since $(B)_b = H$, so $x \in B \cup B\Gamma B \cup B\Gamma H\Gamma B$. There are three cases to consider.
Case 1: $x \in B$.
Subcase 1.1: $x \neq a$. Then, $x \in B\backslash\{a\} = A \subseteq (A)_b$.
Subcase 1.2: $x = a$. By assumption, so
$x = a \in b\gamma_1 c \cup b\gamma_1 c\Gamma b\gamma_1 c \cup b\gamma_1 c\Gamma H\Gamma b\gamma_1 c \subseteq A\Gamma A \cup A\Gamma H\Gamma A \subseteq (A)_b$.
Case 2: $x \in B\Gamma B$. Thus, $x \in b_1\gamma b_2$ for some $b_1, b_2 \in B$ and $\gamma \in \Gamma$.
Subcase 2.1 : $b_1 = a$ and $b_2 = a$. By assumption, then

$$\begin{aligned} x \in b_1\gamma b_2 &= a\gamma a \\ &\subseteq (b\gamma_1 c \cup b\gamma_1 c\Gamma b\gamma_1 c \cup b\gamma_1 c\Gamma H\Gamma b\gamma_1 c)\Gamma(b\gamma_1 c \cup b\gamma_1 c\Gamma b\gamma_1 c \cup b\gamma_1 c\Gamma H\Gamma b\gamma_1 c) \\ &\subseteq A\Gamma H\Gamma A \subseteq (A)_b \end{aligned}$$

Subcase 2.2: $b_1 \neq a$ and $b_2 = a$. By assumption and $A = B\backslash\{a\}$; then,

$$\begin{aligned} x \in b_1\gamma b_2 = b_1\gamma a &\subseteq (B\backslash\{a\})\Gamma(b\gamma_1 c \cup b\gamma_1 c\Gamma b\gamma_1 c \cup b\gamma_1 c\Gamma H\Gamma b\gamma_1 c) \\ &\subseteq A\Gamma H\Gamma A \subseteq (A)_b. \end{aligned}$$

Subcase 2.3: $b_1 = a$ and $b_2 \neq a$. By assumption and $A = B\backslash\{a\}$; so,

$$\begin{aligned} x \in b_1\gamma b_2 = a\gamma b_2 &\subseteq (b\gamma_1 c \cup b\gamma_1 c\Gamma b\gamma_1 c \cup b\gamma_1 c\Gamma H\Gamma b\gamma_1 c)\Gamma(B\backslash\{a\}) \\ &\subseteq A\Gamma H\Gamma A \subseteq (A)_b. \end{aligned}$$

[DOI: 10.61186/ijmsi.19.2.41]

Subcase 2.4: $b_1 \neq a$ and $b_2 \neq a$. By assumption and $A = B \backslash \{a\}$; thus, $x \subseteq b_1 \gamma b_2 \subseteq (B\backslash\{a\})\Gamma(B\backslash\{a\}) = A\Gamma A \subseteq (A)_b$.
Case 3: $x \in B\Gamma H \Gamma B$. Hence, $x \in b_3\gamma_1 h\gamma_2 b_4$ for some $b_3, b_4 \in B$ and for some $\gamma_1, \gamma_2 \in \Gamma$ and for some $h \in H$.
Subcase 3.1: $b_3 = a$ and $b_4 = a$. By assumption, then

$$\begin{aligned} x &\in b_3\gamma_1 h\gamma_2 b_4 \\ &= a\gamma_1 h\gamma_2 a \\ &\subseteq (b\gamma_1 c \cup b\gamma_1 c\Gamma b\gamma_1 c \cup b\gamma_1 c\Gamma H\Gamma b\gamma_1 c)\Gamma H\Gamma(b\gamma_1 c \cup b\gamma_1 c\Gamma b\gamma_1 c \cup b\gamma_1 c\Gamma H\Gamma b\gamma_1 c) \\ &\subseteq A\Gamma H\Gamma A \subseteq (A)_b. \end{aligned}$$

Subcase 3.2: $b_3 \neq a$ and $b_4 = a$. By assumption and $A = B\backslash\{a\}$; so,

$$\begin{aligned} x \in b_3\gamma_1 h\gamma_2 b_4 = b_3\gamma_1 h\gamma_2 a &\subseteq (B\backslash\{a\})\Gamma H\Gamma(b\gamma_1 c \cup b\gamma_1 c\Gamma b\gamma_1 c \cup b\gamma_1 c\Gamma H\Gamma b\gamma_1 c) \\ &\subseteq A\Gamma H\Gamma A \subseteq (A)_b. \end{aligned}$$

Subcase 3.3: $b_3 = a$ and $b_4 \neq a$. By assumption and $A = B\backslash\{a\}$; thus,

$$\begin{aligned} x \in b_3\gamma_1 h\gamma_2 b_4 = b_3\gamma_1 h\gamma_2 a &\subseteq (b\gamma_1 c \cup b\gamma_1 c\Gamma b\gamma_1 c \cup b\gamma_1 c\Gamma H\Gamma b\gamma_1 c)\Gamma H\Gamma(B\backslash\{a\}) \\ &\subseteq A\Gamma H\Gamma A \subseteq (A)_b. \end{aligned}$$

Subcase 3.4: $b_3 \neq a$ and $b_4 \neq a$. By assumption and $A = B\backslash\{a\}$; hence, $x \in b_3\gamma b_4 \subseteq (B\backslash\{a\})\Gamma(B\backslash\{a\}) = A\Gamma A \subseteq (A)_b$.
From all cases, $(A)_b = H$. This is a contradiction. Therefore, $a = b$ and $a = c$.
(2.) Assume $a \in b\gamma_1 h\gamma_2 c \cup b\gamma_1 h\gamma_2 c\Gamma b\gamma_1 h\gamma_2 c \cup b\gamma_1 h\gamma_2 c\Gamma H\Gamma b\gamma_1 h\gamma_2 c$ such that $a \neq b$ and $a \neq c$. Setting $A = B\backslash\{a\}$. Then, $A \subseteq B$. Thus, $(A)_b \subseteq (B)_b = H$. From $a \neq b$ and $a \neq c$; so, $b, c \in A$. Let $x \in H$. Since $(B)_b = H$; then, $x \in B \cup B\Gamma B \cup B\Gamma H\Gamma B$. There are three cases to consider.
Case 1: $x \in B$.
Subcase 1.1: $x \neq a$. Then, $x \in B\backslash\{a\} = A \subseteq (A)_b$.
Subcase 1.2: $x = a$. By assumption, then
$x = a \in b\gamma_1 h\gamma_2 c \cup b\gamma_1 h\gamma_2 c\Gamma b\gamma_1 h\gamma_2 c \cup b\gamma_1 h\gamma_2 c\Gamma H\Gamma b\gamma_1 h\gamma_2 c \subseteq A\Gamma H\Gamma A \subseteq (A)_b$.
Case 2: $x \in B\Gamma B$. Thus, $x \in b_1\gamma b_2$ for some $b_1, b_2 \in B$ and $\gamma \in \Gamma$.
Subcase 2.1: $b_1 = a$ and $b_2 = a$. By assumption, thus

$$\begin{aligned} x \in b_1\gamma b_2 &= a\gamma a \\ &\subseteq (b\gamma_1 h\gamma_2 c \cup b\gamma_1 h\gamma_2 c\Gamma b\gamma_1 h\gamma_2 c \cup b\gamma_1 h\gamma_2 c\Gamma H\Gamma b\gamma_1 h\gamma_2 c)\Gamma \\ &\quad (b\gamma_1 h\gamma_2 c \cup a\gamma b\gamma_1 h\gamma_2 c\Gamma a\gamma b\gamma_1 h\gamma_2 c \cup a\gamma b\gamma_1 h\gamma_2 c\Gamma H\Gamma a\gamma b\gamma_1 h\gamma_2 c) \\ &\subseteq A\Gamma H\Gamma A \subseteq (A)_b. \end{aligned}$$

Subcase 2.2: $b_1 \neq a$ and $b_2 = a$. By assumption and $A = B\backslash\{a\}$; so,

$$\begin{aligned} x \in b_1\gamma b_2 &= b_1\gamma a \\ &\subseteq (B\backslash\{a\})\Gamma(b\gamma_1 h\gamma_2 c \cup a\gamma b\gamma_1 h\gamma_2 c\Gamma a\gamma b\gamma_1 h\gamma_2 c \cup a\gamma b\gamma_1 h\gamma_2 c\Gamma H\Gamma a\gamma b\gamma_1 h\gamma_2 c) \\ &\subseteq A\Gamma H\Gamma A \subseteq (A)_b. \end{aligned}$$

[DOI: 10.61186/ijmsi.19.2.41]

Subcase 2.3: $b_1 = a$ and $b_2 \neq a$. By assumption $A = B\backslash\{a\}$; hence,

$$\begin{aligned} x \in b_1\gamma b_2 &= a\gamma b_2 \\ &\subseteq (b\gamma_1 h\gamma_2 c \cup a\gamma b\gamma_1 h\gamma_2 c\Gamma a\gamma b\gamma_1 h\gamma_2 c \cup a\gamma b\gamma_1 h\gamma_2 c\Gamma H\Gamma a\gamma b\gamma_1 h\gamma_2 c)\Gamma(B\backslash\{a\}) \\ &\subseteq A\Gamma H\Gamma A \subseteq (A)_b. \end{aligned}$$

Subcase 2.4: $b_1 \neq a$ and $b_2 \neq a$. From $A = B\backslash\{a\}$; then,
$x \in b_1\gamma b_2 \in (B\backslash\{a\})\Gamma(B\backslash\{a\}) = A\Gamma A \subseteq (A)_b$.
Case 3: $x \in B\Gamma H\Gamma B$. Hence, $x \in b_3\gamma h\gamma b_4$ for some $b_3, b_4 \in B$, $\gamma_1, \gamma_2 \in \Gamma$ and $h \in H$.

Subcase 3.1: $b_1 = a$ and $b_2 = a$. By assumption, so

$$\begin{aligned} x \in b_3\gamma_1 h\gamma_2 b_4 &= a\gamma_1 h\gamma_2 a \\ &\subseteq (b\gamma_1 h\gamma_2 c \cup b\gamma_1 h\gamma_2 c\Gamma b\gamma_1 h\gamma_2 c \cup b\gamma_1 h\gamma_2 c\Gamma H\Gamma b\gamma_1 h\gamma_2 c)\Gamma H\Gamma \\ &\quad (b\gamma_1 h\gamma_2 c \cup b\gamma_1 h\gamma_2 c\Gamma b\gamma_1 h\gamma_2 c \cup b\gamma_1 h\gamma_2 c\Gamma H\Gamma b\gamma_1 h\gamma_2 c) \\ &\subseteq A\Gamma H\Gamma A \subseteq (A)_b. \end{aligned}$$

Subcase 3.2: $b_3 \neq a$ and $b_4 = a$. By assumption and $A = B\backslash\{a\}$; thus,

$$\begin{aligned} x \in b_3\gamma_1 h\gamma_2 b_4 &= b_3\gamma h\gamma_2 a \\ &\subseteq (B\backslash\{a\})\Gamma H\Gamma(b\gamma_1 h\gamma_2 c \cup b\gamma_1 h\gamma_2 c\Gamma b\gamma_1 h\gamma_2 c \cup b\gamma_1 h\gamma_2 c\Gamma H\Gamma b\gamma_1 h\gamma_2 c) \\ &\subseteq A\Gamma H\Gamma A \subseteq (A)_b. \end{aligned}$$

Subcase 3.3: $b_3 = a$ and $b_4 \neq a$. By assumption $A = B\backslash\{a\}$; hence,

$$\begin{aligned} x \in b_3\gamma h\gamma_2 b_4 &= a\gamma_1 h\gamma_2 b_4 \\ &\subseteq (b\gamma_1 h\gamma_2 c \cup b\gamma_1 h\gamma_2 c\Gamma b\gamma_1 h\gamma_2 c \cup b\gamma_1 h\gamma_2 c\Gamma H\Gamma b\gamma_1 h\gamma_2 c)\Gamma H\Gamma(B\backslash\{a\}) \\ &\subseteq A\Gamma H\Gamma A \subseteq (A)_b. \end{aligned}$$

Subcase 3.4: $b_3 \neq a$ and $b_4 \neq a$. From $A = B\backslash\{a\}$; then,
$x \in b_3\gamma_1 h\gamma_2 b_4 \in (B\backslash\{a\})\Gamma H\Gamma(B\backslash\{a\}) = A\Gamma H\Gamma A \subseteq (A)_b$.
From all cases, $(A)_b = H$. This is a contradiction. Therefore, $a = b$ or $a = c$. □

Lemma 2.6. *Let B be a bi–bases of a Γ–semihypergroup H.*

1. *For any $a, b, c \in B, \gamma_1 \in \Gamma$, if $a \neq b$ and $a \neq c$; then, $a \nleq_b b\gamma_1 c$.*
2. *For any $a, b, c \in B, \gamma_2, \gamma_3 \in \Gamma$ and $h \in H$, if $a \neq b$ and $a \neq c$; then, $a \nleq_b b\gamma_2 h\gamma_3 c$.*

Proof. (1.) Assume $a \neq b$ and $a \neq c$. Suppose $a \leq_b b\gamma_1 c$, thus $(a)_b \subseteq (b\gamma_1 c)_b$. Hence, $a \in (a)_b \subseteq (b\gamma_1 c)_b = b\gamma_1 c \cup b\gamma_1 c\Gamma b\gamma_1 c \cup b\gamma_1 c\Gamma H\Gamma b\gamma_1 c$. By Lemma 2.5(1.), it follows $a = b$ and $a = c$. This contradicts the assumption.
(2.) Assume $a \neq b$ and $a \neq c$. Suppose $a \leq_b b\gamma_2 h\gamma_3 c$, then $(a)_b \subseteq (b\gamma_2 h\gamma_3 c)_b$. Thus, $a \in (a)_b \subseteq (b\gamma_2 h\gamma_3 c)_b = b\gamma_2 h\gamma_3 c \cup b\gamma_2 h\gamma_3 c\Gamma b\gamma_2 h\gamma_3 c \cup b\gamma_2 h\gamma_3 c\Gamma H\Gamma b\gamma_2 h\gamma_3 c$. By Lemma 2.5(2.), it follows $a = b$ or $a = c$. This contradicts the assumption. □

[DOI: 10.61186/ijmsi.19.2.41]

Theorem 2.7. *A nonempty subset B of a Γ–semihypergroup H is a bi–bases of H if and only if B satisfies the following conditions.*

1. *For any $x \in H$,*
   1.1. *there exists $b \in B$ such that $x \leq_b b$ or*
   1.2. *there exists $b_1, b_2 \in B$ and $\gamma \in \Gamma$ such that $x \leq_b b_1\gamma b_2$ or*
   1.3. *there exists $b_3, b_4 \in B$ and $\gamma_1, \gamma_2 \in \Gamma$such that $x \leq_b b_3\gamma_1 h\gamma_2 b_4$.*
2. *For any $a, b, c \in B$ and $\gamma_1 \in \Gamma$, if $a \neq b$ and $a \neq c$; then, $a \nleq_b b\gamma_1 c$.*
3. *For any $a, b, c \in B, \gamma_2, \gamma_3 \in \Gamma$ and $h \in H$, if a $a \neq b$ and $a \neq c$; then, $a \nleq_b b\gamma_2 h\gamma_3 c$.*

Proof. Assume B is a bi–bases of H. Then, $H = (B)_b$. To show that (1.) holds, let $x \in H$. Thus, $x \in B \cup B\Gamma B \cup B\Gamma H\Gamma B$. We consider three cases.

case 1: $x \in B$. Then, $x = b$ for some $b \in B$. This implies $(x)_b \subseteq (b)_b$. Therefore, $x \leq_b b$.

case 2: $x \in B\Gamma B$. Then, $x \in b_1\gamma b_2$ for some $b_1, b_2 \in B$ and $\gamma \in \Gamma$. This implies $(x)_b \subseteq (b_1\gamma b_2)_b$. Hence, $x \leq_b b_1\gamma b_2$.

case 3: $x \in B\Gamma H\Gamma B$. Then, $x \in b_3\gamma_1 h\gamma_2 b_4$ for some $b_3, b_4 \in B, h \in H$ and $\gamma_1, \gamma_2 \in \Gamma$. This implies $(x)_b \subseteq (b_3\gamma_1 h\gamma_2 b_4)_b$. Hence, $x \leq_b b_3\gamma_1 h\gamma_2 b_4$.

The validity of (2.) and (3.) follow from Lemma 2.6(1.) and Lemma 2.6(2.), respectively. Conversely, we will show that B is a bi–bases of H. Clearly, $(B)_b \subseteq H$. Let $x \in H$. By assumption, there exists $b \in B$ such that $x \leq_b b$, thus $(x)_b \subseteq (b)_b$. Thus, $x \in (x)_b \subseteq (b)_b = b \cup b\Gamma b \cup b\Gamma H\Gamma b \subseteq B \cup B\Gamma B \cup B\Gamma H\Gamma B = (B)_b$. Then, $H \subseteq (B)_b$. Hence, $H = (B)_b$. Suppose $H = (A)_b$. Since $A \subset B$, there exists $b \in B\backslash A$. Since $b \in B \subseteq H = (A)_b$, so $b \in (A)_b$. Thus, $b \in A \cup A\Gamma A \cup A\Gamma H\Gamma A$. Since $b \notin A$, we have $b \in A\Gamma A \cup A\Gamma H\Gamma A$. There are two cases to consider.

case 1: $b \in A\Gamma A$. Thus, $b \in a_1\gamma_1 a_2$ for some $a_1, a_2 \in A$ and $\gamma_1 \in \Gamma$. From $A \subseteq B$, so $a_1, a_2 \in B$. Since $b \notin A$; hence, $b \neq a_1$ and $b \neq a_2$. From $b \in a_1\gamma_1 a_2$; then, $(b)_b \subseteq (a_1\gamma_1 a_2)_b$. Hence, $b \leq_b a_1\gamma_1 a_2$. This contradicts to (2).

case 2: $b \in A\Gamma H\Gamma A$. Hence, $b \in a_3\gamma_2 h\gamma_3 a_4$ for some $a_3, a_4 \in A$, $h \in H$ and $\gamma_2, \gamma_3 \in \Gamma$. From $A \subset B$, so $a_3, a_4 \in B$. Since $b \notin A$, then $b \neq a_3$ and $b \neq a_4$. From $b \in a_3\gamma_2 h\gamma_3 a_4$, so $(b)_b \subseteq (a_3\gamma_2 h\gamma_3 a_4)_b$. Therefore, $b \leq_b a_3\gamma_2 h\gamma_3 a_4$. This contradicts to (3.). $\square$

Theorem 2.8. *Let B be a bi–baes of a Γ–semihypergroup H. Then B is a sub–Γ–semihypergeoup of H if and only if for any $b, c \in B$ and $\gamma \in \Gamma, b \in b\gamma c$ or $c \in b\gamma c$*

Proof. Suppose B is a sub–Γ–semihypergroup of H and $b \notin b\gamma c$ and $c \notin b\gamma c$ for any $b, c \in B$ and $\gamma \in \Gamma$. Assume $a \in b\gamma c$, then $a \neq b$ and $a \neq c$. Hence, $a \in b\gamma c \cup b\gamma c\Gamma b\gamma c \cup b\gamma c\Gamma H\Gamma b\gamma c$. By Lemma 2.5(1.), $a = b$ or $a = c$. This is a contradiction. Conversely, assume $b \in b\gamma c$ or $c \in b\gamma c$ for any $b, c \in B$. We will show that B is a sub–Γ–semihypergroup of H. Let $a \in B\Gamma B$. Hence, $a \in b\gamma c$ for some $b, c \in B$ and $\gamma \in \Gamma$. This implies $a \in b\gamma c \cup b\gamma c\Gamma b\gamma c \cup b\gamma c\Gamma H\Gamma b\gamma c$.

[DOI: 10.61186/ijmsi.19.2.41]

By Lemma 2.5(1.), $a = b$ or $a = c$. Hence, $a \in \{b, c\} \subseteq B$. Therefore B is a sub–Γ–semihypergroup of H. □

Acknowledgments

The authors are highly grateful to the committee for their valuable comments and suggestions which have improved the presentation of this paper. This research was supported by Buriram Rajabhat University.

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