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First and Second Order Optimality Conditions using Approximations for Fractional Multiobjective Bilevel Problems under Fractional Constraints

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Abstract. In this paper, first and second order optimality conditions using the concept of approximations are developed for an optimistic fractional multiobjective bilevel problem with non-convex lower level problem. Our idea is based on using the properties of approximations in nonsmooth analysis and a separation theorem in convex analysis. All over the article, the data is assumed to be continuous but not necessarily Lipschitz.

Keywords: Fractional Bilevel programming, Optimal value function, Second order approximation, Optimality conditions, Multiobjective optimization.

2000 Mathematics subject classification: 90C29, 90C30, 49J52, 49K99.

1. Introduction

Bilevel programming problems are hierarchical optimization problems with two levels. They are characterized by the existence of two optimization problems where the constraint region of the upper level problem is determined implicitly by the solution set to the lower level problem. They play an important role not only in theoretical studies but also in practical applications. This motivated an intensive investigation of these problems by many mathematicians, economists and engineers. For applications and recent developments on

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the subject one can see [17, 30]. The most important challenge is to develop optimality conditions for the problem. A lot of research has been carried out in this direction [3, 6, 7, 8, 20, 21, 34].

For many optimization problems, notably in bilevel mathematical programming, the characterization of optimal solutions with the help of second order conditions was always of a great interest in order to refine first order optimality conditions. The second order informations complement first order conditions in constructing numerical algorithms for finding optimal solutions (see, e.g. [5, 27]) and also in convergence analysis for numerical algorithms (see, e.g. [10, 14]). Considerable works exist on second order conditions including the papers [1, 9, 11] for C^2 and C^1 data, and [12, 23] for problems with only C^1 data.

In this paper, we are concerned with the following fractional multiobjective bilevel problem

$$(P)\begin{cases} \mathbb{R}^p_+ - \min\limits_{x,y} \left(\frac{f_1(x,y)}{g_1(x,y)}, \cdots, \frac{f_p(x,y)}{g_p(x,y)}\right) \\ \text{subject to } \frac{F_j(x,y)}{G_j(x,y)} \leq 0, \quad j = 1, \cdots, q \\ y \in \Psi(x) \\ (x,y) \in \mathbb{R}^n \times \mathbb{R}^m, \end{cases} \tag{1.1}$$

where for each $x \in \mathbb{R}^n$, $\Psi(x)$ is the solution set of the following parametric optimization problem

$$(P_x)\begin{cases} \min\limits_{y} h(x,y) \\ \text{subject to } \frac{H_s(x,y)}{K_s(x,y)} \leq 0, \quad s = 1, \cdots, r \\ (x,y) \in \mathbb{R}^n \times \mathbb{R}^m, \end{cases} \tag{1.2}$$

where $f_i, g_i, F_j, G_j, H_s, K_s, h : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ with $g_i, i = 1, \cdots, p$, $G_j, j = 1, \cdots, q$, $K_s, s = 1, \cdots, r$ being continuous and nonzero-valued. For the sake of simplicity, we set $I = \{1, \cdots, p\}$, $J = \{1, \cdots, q\}$, $S = \{1, \cdots, r\}$.

The point $(\overline{x}, \overline{y})$ is said to be a local weak efficient solution with respect to $\mathbb{R}^p_+$ of the problem (P) if it is a local weak efficient solution with respect to $\mathbb{R}^p_+$ of the problem

$$\begin{cases} \mathbb{R}^p_+ - \min\limits_{x,y} \left(\frac{f_1(x,y)}{g_1(x,y)}, \cdots, \frac{f_p(x,y)}{g_p(x,y)}\right) \\ \text{subject to } (x,y) \in \overline{S}, \end{cases}$$

where

$$\overline{S} = \left\{ (x,y) \in \mathbb{R}^n \times \mathbb{R}^m : \frac{F_j(x,y)}{G_j(x,y)} \leq 0, \quad \forall\, j \in J,\ y \in \Psi(x) \right\}.$$

Such problem has been discussed by many authors at various levels of generality (see [4, 22, 29]). In [22], the authors gave sufficient optimality conditions and duality results for a special class of (P) (where $p = 1$, $q = 1$ and $r = 1$). Using the quotient rule of generalized differentiation, Bao et al. [4] derived optimality conditions for a multiobjective fractional program with equilibrium constraints.

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Recently, multiobjective fractional control problems involving multiple integrals was considered by many researchers and we make a dishonesty by mentioning only Mititelu and Treanţă [25, 26, 33]. In [25], the authors derive necessary and sufficient optimality conditions for a multiobjective fractional control problem which involve multiple integrals. Broadly speaking, a (ρ, b)-quasiinvexity notion is used to provide sufficient efficiency conditions for a feasible solution. The problem is also investigated in [26]. Using the notion of (ρ, b)-quasiinvexity, weak, strong and converse duality are derived.

In this work, we develop optimality conditions in terms of approximations for the optimistic fractional bilevel programming problem (P) without any convexity assumption on the lower level problem and without the assumption that the solution set $\Psi(x)$ is a singleton. Approximations are important tools of nonsmooth analysis which were introduced by Thibault [32], then further studied by Sweetser [31] and Ioffe [13] and later enhanced by Jourani and Thibault [16]. The importance of approximations lies in the fact that they may exist even for a discontinuous mapping and are useful even when they are nonconvex or unbounded.

The paper is organized as follows : Section 2 gives basic definitions and preliminary results. Section 3 and 4 present the second order necessary and sufficient optimality conditions. A special case is studied in Section 5. Some final comments are then provided in the last section.

2. Preliminaries

In this section, we give some definitions, notations and results, which will be used in the sequel. Recall that X, Y and Z are finite dimensional spaces. For a given $l \in \mathbb{N}$, a mapping $f : X \to Y$ is said to be l-calm at $\overline{x}$ (see [19]) if there exist $L > 0$ and a neighborhood U of $\overline{x}$ such that, for all $x \in U$,

$$\| f(x) - f(\overline{x}) \| \leq L \| x - \overline{x} \|^l .$$

Remark 2.1. [19]

(1) If f is l-calm at $\overline{x}$, then f is continuous at $\overline{x}$, for any $l \in \mathbb{N}$.
(2) L is called the coefficient of calmness of f.

Next we give the definition and some properties of approximations [2, 15]. Let $f : X \longrightarrow Y$ be a given vector function and $\bar{x} \in X$. We denote by $L(X,Y)$ the set of all continuous linear operators mapping X to Y, $B(X,Y,Z)$ the set of all continuous bi-linear operators mapping $X \times Y$ to Z, and $\mathbb{B}_X$ denotes the closed unit ball of X at the origin.

Definition 2.2. [2] The set $A_f(\bar{x}) \subset L(X,Y)$ is said to be a first order approximation of f at $\bar{x}$ if for all $\varepsilon > 0$, there exists $\delta > 0$ such that

$$f(x) - f(\bar{x}) \in A_f(\bar{x})(x - \bar{x}) + \varepsilon \|x - \bar{x}\| \mathbb{B}_Y,$$

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for all $x \in \bar{x} + \delta\mathbb{B}_X$.

Note that, if f is locally Lipschitz at $\bar{x}$, then, the Clarke subdifferential of f at $\bar{x}$ is a first order approximation [15]. Also, when $f : X \longrightarrow Y$ is continuous and admits an approximate Jacobian $\partial_f(\cdot)$ which is upper semi continuous at $\bar{x}$, then $\partial_f(\bar{x})$ is a first order approximation of f at $\bar{x}$ [18].

Remark 2.3. [15]

(1) Let $f : X \longrightarrow Y$ be a vector function. If f is k-Lipschitz on $\bar{x} + \delta\mathbb{B}_X$, then f admits the bounded closed convex set $k \cdot \mathbb{B}_{L(X,Y)}$ as an approximation.
(2) Suppose that $A_f(\overline{x})$ is a first-order approximation of f at $\overline{x}$. If $A_f(\overline{x})$ is bounded, then, f is calm at $\overline{x}$.

In general, approximations are not closed. However they may exist even for a discontinuous mapping as illustrated in the following example.

Example 2.4. [18] Consider the function $g : \mathbb{R} \to \mathbb{R}$ given by

$$g(x) = \begin{cases} \sqrt{x}, & \text{if } x > 0, \\ 0, & \text{if } x = 0, \\ x^{-1}, & \text{if } x < 0. \end{cases}$$

Then g is discontinuous at zero. For any $\alpha > 0$, the set $A_g(0) = (\alpha, +\infty)$ is a first order approximation for g at 0.

Definition 2.5. [2] A couple $(A_f(\bar{x}), B_f(\bar{x}))$, with $A_f(\bar{x}) \subseteq L(X,Y)$ and $B_f(\bar{x}) \subseteq B(X,X,Y)$ is said to be a second order approximation of f at $\bar{x}$ if $A_f(\bar{x}) \subset L(X,Y)$ is a first order approximation of f at $\bar{x}$ and for all $\varepsilon > 0$, there exists $\delta > 0$ such that

$$f(x) - f(\bar{x}) \in A_f(\bar{x})(x - \bar{x}) + B_f(\bar{x})(x - \bar{x})(x - \bar{x}) + \varepsilon \parallel x - \bar{x} \parallel^2 \mathbb{B}_Y,$$

for all $x \in \bar{x} + \delta\mathbb{B}_X$.

Every C^2 mapping $f : X \to Y$ at $\overline{x}$ admits $\left(\nabla f(\overline{x}), \nabla^2 f(\overline{x})\right)$ as a second order approximation, where $\nabla f(\overline{x})$ and $\nabla^2 f(\overline{x})$ are, respectively, the first and second order Fréchet derivatives of f at $\overline{x}$.

Proposition 2.6. [2, 16, 18] *Let $\overline{x} \in X$ and $f : X \to Y$.*

(1) *If f is $C^{1,1}$ then $\left(f'(\overline{x}), \frac{1}{2}\partial_C^2 f(\overline{x})\right)$ is a second order approximation of f at $\overline{x}$, where $\partial_C^2 f(\overline{x})$ is the Clarke Hessian of f at $\overline{x}$.*
(2) *If f is continuously Fréchet differentiable in a neighborhood of $\overline{x}$ and has an approximate Hessian mapping $\partial^2 f(\cdot)$ which is upper semicontinuous at $\overline{x}$, then $\left(f'(\overline{x}), \frac{1}{2}\partial^2 f(\overline{x})\right)$ is a second order approximation of f at $\overline{x}$.*

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We will now recall some algebraic chain rules already stated in [19], which concern approximations of the sum, product and quotient. For this, note that for $f, g : X \to \mathbb{R}$, we define $f \cdot g$ and f/g as usual: $(f \cdot g)(x) := f(x) g(x)$ and $(f/g)(x) := f(x)/g(x)$ for $x \in X$.

Proposition 2.7. *Let $f, g : X \to \mathbb{R}$ and $A_f(\overline{x})$, $A_g(\overline{x})$ be first order approximations of f and g, respectively, at $\overline{x}$. Then, the following assertions hold.*

(1) *If g is continuous at $\overline{x}$ and $A_f(\overline{x})$ is bounded, then $g(\overline{x}) A_f(\overline{x}) + f(\overline{x}) A_g(\overline{x})$ is a first order approximation of $f \cdot g$ at $\overline{x}$.*

(2) *If $A_f(\overline{x})$ and $A_g(\overline{x})$ are bounded with $g(\overline{x}) \neq 0$, then $\frac{g(\overline{x})A_f(\overline{x}) - f(\overline{x})A_g(\overline{x})}{g^2(\overline{x})}$ is a first order approximation of f/g at $\overline{x}$.*

Proposition 2.8. (1) *Let $f_i : X \to Y$, $\tau_i \in \mathbb{R}$ and $(A_{f_i}(\overline{x}), B_{f_i}(\overline{x}))$ be a second order approximation of f_i at $\overline{x}$ for $i = 1, \cdots, l$. Then,*

$$\left(\sum_{i=1}^{l} \tau_i A_{f_i}(\overline{x}), \sum_{i=1}^{l} \tau_i B_{f_i}(\overline{x}) \right)$$

is a second order approximation of $\sum_{i=1}^{l} \tau_i f_i(\overline{x})$ at $\overline{x}$.

(2) *Let $f_i : X \to Y_i$, $i = 1, \cdots, l$ $f = (f_1, \cdots, f_l)$ and $(A_{f_i}(\overline{x}), B_{f_i}(\overline{x}))$ be a second order approximation of f_i at $\overline{x}$ for $i = 1, \cdots, l$. Then, $\Big(A_{f_1}(\overline{x}) \times \cdots \times A_{f_l}(\overline{x}), B_{f_1}(\overline{x}) \times \cdots \times B_{f_l}(\overline{x})\Big)$ is a second order approximation of f at that point.*

Proposition 2.9. *Let $f, g : X \to \mathbb{R}$, g be 2-calm at $\overline{x}$, and $(A_f(\overline{x}), B_f(\overline{x}))$, $(0, B_g(\overline{x}))$ be second order approximations of f and g, respectively, at $\overline{x}$. Then,*

(1) *If $A_f(\overline{x})$, $B_f(\overline{x})$ are bounded, then*

$$\Big(f(\overline{x}) A_g(\overline{x}) + g(\overline{x}) A_f(\overline{x}), f(\overline{x}) B_g(\overline{x}) + g(\overline{x}) B_f(\overline{x}) \Big)$$

is a second order approximation of $f \cdot g$ at $\overline{x}$.

(2) *If $A_f(\overline{x})$, $B_f(\overline{x})$, $B_g(\overline{x})$ are bounded and $g(\overline{x}) \neq 0$, then*

$$\left(\frac{A_f(\overline{x})}{g(\overline{x})}, \frac{g(\overline{x}) B_f(\overline{x}) - f(\overline{x}) B_g(\overline{x})}{g^2(\overline{x})} \right)$$

is a second order approximation of f/g at $\overline{x}$.

Let S be an arbitrary nonempty set of $\mathbb{R}^n \times \mathbb{R}^m$. The contingent cone to S at $\overline{u}$ is

$$K(S, \overline{u}) := \left\{ d \in \mathbb{R}^n \times \mathbb{R}^m : \exists (t_k) \downarrow 0 \text{ and } (d_k) \to d \text{ such that } \overline{u} + t_k d_k \in S, \quad \forall k \in \mathbb{N} \right\}.$$

The second order set to S at $\overline{u}$ in the direction $d \in \mathbb{R}^n \times \mathbb{R}^m$ is given by

$$K^2(S, \overline{u}, d) := \left\{ e \in \mathbb{R}^n \times \mathbb{R}^m : \exists (t_k) \downarrow 0 \text{ and } (e_k) \to e \text{ such that } \overline{u} + t_k d + t_k^2 e_k \in S, \quad \forall k \in \mathbb{N} \right\}.$$

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Remark 2.10. The set $K^2(S,\overline{u},d)$ is not necessarily a cone, and it might be empty when S is not a polyhedral set, see [28].

We recall that the cone of weak feasible directions to $S \subseteq \mathbb{R}^n \times \mathbb{R}^m$ at $\overline{u} \in S$ is

$$D\left(S,\overline{u}\right) = \left\{ u \in \mathbb{R}^n \times \mathbb{R}^m : \exists \left(t_k\right) \downarrow 0, \quad \overline{u} + t_k u \in S, \quad \forall k \in \mathbb{N} \right\}.$$

To close the list of assumptions, we recall the notion of weakly efficient solutions needed in this paper. Let $C \subset \mathbb{R}^n$ be a pointed closed convex cone with nonempty interior introducing a partial order $\preceq_C$ in $\mathbb{R}^n$.

Definition 2.11. Let A be a nonempty set of $\mathbb{R}^n$. The point $x \in A$ is said to be a Pareto (resp. weak Pareto) minimal vector of A w.r.t. C if

$$A \subset x + \left[\left(\mathbb{R}^n \setminus (-C)\right) \cup \{0\}\right] \quad \left(\text{resp., } A \subset x + \left(\mathbb{R}^n \setminus -intC\right)\right), \tag{2.1}$$

where "int" denotes the topological interior.

Let us now consider the multiobjective optimization problem with respect to the partial order introduced by the pointed, closed and convex cone C:

$$C - \min f(x) \ \text{ s.t. } x \in X, \tag{2.2}$$

where f represents a vector-valued function and X the nonempty feasible set.

Definition 2.12. A point $\overline{x} \in X$ is said to be an efficient (resp. weakly efficient) solution of problem (2.2) if $f\left(\overline{x}\right)$ is a Pareto (resp. weak Pareto) minimal vector of $f\left(X\right)$.

Definition 2.13. The point $\overline{x} \in X$ is said to be a local efficient (resp. weakly local efficient) solution of problem (2.2) if there exists a neighborhood U of $\overline{x}$ such that $f\left(\overline{x}\right)$ is a Pareto (resp. weak Pareto) minimal vector of $f\left(U \cap X\right)$.

3. Second order necessary optimality conditions

In this section, we maintain the notations given in the previous section and we give necessary optimality conditions for the fractional multiobjective bilevel problem (P) without any convexity assumption on the lower level problem and without the assumption that the set $\Psi\left(x\right)$ is a singleton.
A classical way to convert problem (P) into an optimization problem with tractable constraints is the so-called value function reformulation. Hence, according to [6], problem (P) can be replaced by

$$\left(\widehat{P}\right) \begin{cases} \mathbb{R}^p_+ - \min\limits_{x,y} \left(\frac{f_1(x,y)}{g_1(x,y)}, \cdots, \frac{f_p(x,y)}{g_p(x,y)}\right) \\ \text{subject to } \frac{F_j(x,y)}{G_j(x,y)} \leq 0, \quad j \in J \\ \frac{H_s(x,y)}{K_s(x,y)} \leq 0, \quad s \in S \\ h\left(x,y\right) - V\left(x\right) \leq 0 \end{cases} \tag{3.1}$$

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provided that $\left(\widehat{P}\right)$ has an optimal solution [24], where

$$V(x) = \min_{y} \left\{ h(x,y) \ : \ \frac{H_s(x,y)}{K_s(x,y)} \leq 0, \ \ \forall\ s \in S,\ y \in \mathbb{R}^m \right\} \tag{3.2}$$

denotes the optimal value function of the lower level problem (P_x).

Remark 3.1. Under the following hypotheses (H_1), (H_2), (H_3) and (H_4) the optimization problem $\left(\widehat{P}\right)$ has at least one optimal solution.

$H(1)$: $f_i(\cdot,\cdot)$ is upper semicontinuous on $\mathbb{R}^n \times \mathbb{R}^m$ and $g_i(\cdot,\cdot)$ is lower semicontinuous on $\mathbb{R}^n \times \mathbb{R}^m$, $\forall i \in I$.
$H(2)$: $V(\cdot)$ is upper semicontinuous on $\mathbb{R}^n$.
$H(3)$: $h(\cdot,\cdot)$, $F_j(\cdot,\cdot)$, $G_j(\cdot,\cdot)$, $H_s(\cdot,\cdot)$, $K_s(\cdot,\cdot)$ are continuous on $\mathbb{R}^n \times \mathbb{R}^m$ for all $j \in J$ and $s \in S$
$H(4)$: The feasible set of $\left(\widehat{P}\right)$ is nonempty and bounded.
Let

$$\overline{V}(x,y) = V(x)$$

and let $t = q + r + 1$. Consider the next problem (P^*) with respect to $\mathbb{R}^p_+$

$$(P^\star) \begin{cases} \min\limits_{x,y} \Upsilon(x,y) \\ \text{subject to } \ (x,y) \in E \end{cases} \tag{3.3}$$

where,

$$\begin{array}{l} \Upsilon(x,y) = (\Upsilon_1(x,y), \cdots, \Upsilon_p(x,y)) \\ \Upsilon_i(x,y) = \frac{f_i(x,y)}{g_i(x,y)} \quad i \in I \end{array}$$

and

$$E = \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^m : \ \Gamma(x,y) \in -\mathbb{R}^t_+\},$$

with

$$\begin{array}{l} \Gamma(x,y) = (\psi(x,y), \phi(x,y), \varphi(x,y)) \\ \psi(x,y) = \left(\frac{F_1(x,y)}{G_1(x,y)}, \cdots, \frac{F_q(x,y)}{G_q(x,y)}\right) \\ \phi(x,y) = \left(\frac{H_1(x,y)}{K_1(x,y)}, \cdots, \frac{H_r(x,y)}{K_r(x,y)}\right) \\ \varphi(x,y) = h(x,y) - \overline{V}(x,y). \end{array}$$

Here,

$$\Gamma_i(x,y) = \begin{cases} \psi_i(x,y) = \frac{F_i(x,y)}{G_i(x,y)}, & i = 1, \cdots, q \\ \phi_{i-q}(x,y) = \frac{H_{i-q}(x,y)}{K_{i-q}(x,y)}, & i = q+1, \cdots, q+r \\ \varphi_t(x,y) = h(x,y) - \overline{V}(x,y). \end{cases}$$

Finally, for a given $(\overline{x}, \overline{y}) \in E$, we set

$$\overline{p}_i = \frac{f_i(\overline{x},\overline{y})}{g_i(\overline{x},\overline{y})}, \quad i \in I, \qquad \overline{q}_j = \frac{F_j(\overline{x},\overline{y})}{G_j(\overline{x},\overline{y})}, \quad j \in J \ \text{ and } \quad \overline{r}_s = \frac{H_s(\overline{x},\overline{y})}{K_s(\overline{x},\overline{y})}, \ s \in S. \tag{3.4}$$

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Theorem 3.2. *Let $\overline{u} = (\overline{x}, \overline{y})$ be a local weak efficient solution of (P). Assume that f_i, g_i, with $i \in I$, F_j, G_j, with $j \in J$, H_s, K_s, with $s \in S$, h and $\overline{V}$ admit bounded first order approximations $A_{f_i}(\overline{u})$, $A_{g_i}(\overline{u})$, with $i \in I$, $A_{F_j}(\overline{u})$, $A_{G_j}(\overline{u})$, with $j \in J$, $A_{H_s}(\overline{u})$, $A_{K_s}(\overline{u})$, with $s \in S$, $A_h(\overline{u})$ and $A_V(\overline{x}) \times \{0\}$. Moreover, assume that $g_i(\overline{u}) > 0$, with $i \in I$, $G_j(\overline{u}) > 0$, with $j \in J$, $K_s(\overline{u}) > 0$, with $s \in S$. Then, for all $u \in \mathbb{R}^n \times \mathbb{R}^m$, there exist $\alpha \in \mathbb{R}^p_+$, $(\beta, \gamma) \in \mathbb{R}^{q+r}_+$, $\delta \in \mathbb{R}_+$ and $A^f_i \in clA_{f_i}(\overline{u})$, $A^g_i \in clA_{g_i}(\overline{u})$, with $i \in I$, $A^F_j \in clA_{F_j}(\overline{u})$, $A^G_j \in clA_{G_j}(\overline{u})$, with $j \in J$, $A^H_s \in clA_{H_s}(\overline{u})$, $A^K_s \in clA_{K_s}(\overline{u})$, with $s \in S$, $A^h \in clA_h(\overline{u})$ and $A^V \in clA_V(\overline{x}) \times \{0\}$ such that $(\alpha, \beta, \gamma, \delta) \neq (0_{\mathbb{R}^p}, 0_{\mathbb{R}^q}, 0_{\mathbb{R}^r}, 0)$ with*

$$\begin{aligned}&\sum_{i=1}^{p} \alpha_i \left(A^f_i(u) - \overline{p}_i A^g_i(u)\right) + \sum_{j=1}^{q} \beta_j \left(A^F_j(u) - \overline{q}_j A^G_j(u)\right) \\ &+\sum_{s=1}^{r} \gamma_s \left(A^H_s(u) - \overline{r}_s A^K_s(u)\right) + \delta\left(A^h(u) - A^V(u)\right) \qquad \geq 0,\end{aligned} \tag{3.5}$$

where "cl" denotes the topological closure of the set in question.

Proof. Let $\overline{u} = (\overline{x}, \overline{y})$ be a local weak efficient solution of (P). Then, by definition of the mappings Υ and Γ, $\overline{u} = (\overline{x}, \overline{y})$ is a local weak efficient solution of $(P^\star)$.

Let $\epsilon > 0$ and $u \in \mathbb{R}^n \times \mathbb{R}^m$ be arbitrarily chosen. Two cases have to be considered.

- **•:** $u \in D(E, \overline{u})$. Then, there exists a sequence $t_k \downarrow 0$ such that $\overline{u} + t_k u \in E$. Hence, for k large enough, one has

$$\Upsilon(\overline{u} + t_k u) - \Upsilon(\overline{u}) \notin -int\left(\mathbb{R}^p_+\right). \tag{3.6}$$

Since, $A_{g_i}(\overline{u})$, $i = 1, \cdots, p$, are bounded, from Remark 2.3, all g_i are calm at $\overline{u}$. Hence, by Proposition 2.7 and Proposition 2.8

$$A_\Upsilon(\overline{u}) = \prod_{i=1}^{p} \frac{1}{g_i(\overline{u})}\left(A_{f_i}(\overline{u}) - \overline{p}_i A_{g_i}(\overline{u})\right) \tag{3.7}$$

is a first order approximation of Υ at $\overline{u}$, while $\overline{p}_i = \frac{f_i(\overline{u})}{g_i(\overline{u})}$, $i = 1, \cdots, p$. Consequently, there exist $A_k \in A_\Upsilon(\overline{u})$ and $b_k \in \mathbb{B}_{\mathbb{R}^p}$ such that

$$\Upsilon(\overline{u} + t_k u) - \Upsilon(\overline{u}) = t_k A_k(u) + \epsilon t_k \parallel u \parallel b_k.$$

Thus, $A_k(u) + \epsilon \parallel u \parallel b_k \notin -int\left(\mathbb{R}^p_+\right)$. Due to the boundedness of the first order approximation, one can assume that $\{A_k\}$ converge to some $A \in clA_\Upsilon(\overline{u})$. Moreover, by compactness of $\mathbb{B}_{\mathbb{R}^p}$, extracting a subsequence if necessary, one may assume that there exists $b \in \mathbb{B}_{\mathbb{R}^p}$ such that $A(u) + \epsilon \parallel u \parallel b \notin -int\left(\mathbb{R}^p_+\right)$. Letting $\epsilon \to 0$, we obtain

$$A(u) \notin -int\left(\mathbb{R}^p_+\right). \tag{3.8}$$

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- **•:** $u \notin D(E, \overline{u})$. Then, for all $t_k \downarrow 0$ there exists k such that $\Gamma(\overline{u} + t_k u) \notin -\mathbb{R}^t$. Since, $A_{G_j}(\overline{u})$, $j = 1, \cdots, q$, $A_{K_s}(\overline{u})$, $s = 1, \cdots, r$ are bounded, all G_j and K_s are calm at $\overline{u}$. Again, by Proposition 2.7 and Proposition 2.8

$$A_\Gamma(\overline{u}) = A_\psi(\overline{u}) \times A_\phi(\overline{u}) \times A_\varphi(\overline{u}) \tag{3.9}$$

is a first order approximation of Γ at $\overline{u}$, with

$$\begin{aligned} A_\psi(\overline{u}) &= \prod_{j=1}^{q} \tfrac{1}{G_j(\overline{u})} \left(A_{F_j}(\overline{u}) - \overline{q}_j A_{G_j}(\overline{u})\right), \\ A_\phi(\overline{u}) &= \prod_{s=1}^{r} \tfrac{1}{K_s(\overline{u})} \left(A_{H_s}(\overline{u}) - \overline{r}_s A_{K_s}(\overline{u})\right) \quad \text{and} \\ A_\varphi(\overline{u}) &= A_h(\overline{u}) - A_V(\overline{x}) \times \{0\}, \end{aligned}$$

where $\overline{q}_j = \frac{F_j(\overline{u})}{G_j(\overline{u})}$, $j \in J$ and $\overline{r}_s = \frac{H_s(\overline{u})}{K_s(\overline{u})}$, $s \in S$.
Taking $t_k = \frac{1}{k}$, one has

$$\Gamma\left(\overline{u} + \frac{1}{k}u\right) \in \Gamma(\overline{u}) + \frac{1}{k}A_\Gamma(\overline{u})(u) + \epsilon\frac{1}{k} \parallel u \parallel \mathbb{B}_{\mathbb{R}^t}. \tag{3.10}$$

Thus,

$$\Gamma\left(\overline{u} + \frac{1}{k}u\right) \in \left(1 - \frac{1}{k}\right)\Gamma(\overline{u}) + \frac{1}{k}\left[\Gamma(\overline{u}) + A_\Gamma(\overline{u})(u) + \epsilon \parallel u \parallel \mathbb{B}_{\mathbb{R}^t}\right]. \tag{3.11}$$

Now, we assert that $A_\Gamma(\overline{u})(u) \nsubseteq -int\left(\mathbb{R}^t_+\right) - \Gamma(\overline{u})$. Indeed, suppose on the contrary, by assumption one gets for ϵ small enough

$$\frac{1}{k}\left[\Gamma(\overline{u}) + A_\Gamma(\overline{u})(u) + \epsilon \parallel u \parallel \mathbb{B}_{\mathbb{R}^t}\right] \subseteq -\mathbb{R}^t_+. \tag{3.12}$$

Hence, (3.11) and (3.12) yield

$$\Gamma\left(\overline{u} + \frac{1}{k}u\right) \in \left(1 - \frac{1}{k}\right)\Gamma(\overline{u}) - \mathbb{R}^t_+.$$

Hence, $\Gamma\left(\overline{u} + \frac{1}{k}u\right) \in -\mathbb{R}^t_+$, which is a contradiction to the assumption that $u \notin D(E, \overline{u})$. Consequently, there exists $B \in clA_\Gamma(\overline{u})$ such that

$$B(u) \notin -int\left(\mathbb{R}^t_+\right) - \Gamma(\overline{u}). \tag{3.13}$$

Applying separation theorem, it follows from (3.8) and (3.13) that for all $u \in \mathbb{R}^n \times \mathbb{R}^m \setminus \{(0,0)\}$, there exists $(\mu, \nu) \in \mathbb{R}^p_+ \times \mathbb{R}^t_+$ such that,

$$\begin{cases} \langle \mu, A(u) \rangle + \langle \nu, B(u) \rangle \geq 0 \\ \langle \nu, \Gamma(\overline{u}) \rangle = 0. \end{cases} \tag{3.14}$$

The first order approximation definition together with boundedness property give us $A^f_i \in clA_{f_i}(\overline{u})$, $A^g_i \in clA_{g_i}(\overline{u})$, with $i \in I$, $A^F_j \in clA_{F_j}(\overline{u})$, $A^G_j \in$

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$clA_{G_j}(\overline{u})$, with $j \in J$, $A_s^H \in clA_{H_s}(\overline{u})$, $A_s^K \in clA_{K_s}(\overline{u})$, with $s \in S$, $A^h \in clA_h(\overline{u})$ and $A^V \in clA_V(\overline{x}) \times \{0\}$ such that

$$\begin{aligned} &\sum_{i=1}^{p} \tfrac{\mu_i}{g_i(\overline{u})} \left(A_i^f(u) - \overline{p}_i A_i^g(u)\right) + \sum_{j=1}^{q} \tfrac{\nu_j}{G_j(\overline{u})} \left(A_j^F(u) - \overline{q}_j A_j^G(u)\right) \\ &+\sum_{s=1}^{r} \tfrac{\nu_s}{K_s(\overline{u})} \left(A_s^H(u) - \overline{r}_s A_r^K(u)\right) + \nu_t \left(A^h(u) - A^V(u)\right) \qquad \geq 0. \end{aligned} \tag{3.15}$$

Setting

$$\alpha_i = \frac{\mu_i}{g_i(\overline{u})},\ \forall\, i \in I,\ \beta_j = \frac{\nu_j}{G_j(\overline{u})},\ \forall\, j \in J,\ \gamma_s = \frac{\nu_s}{K_s(\overline{u})},\ \forall\, s \in S,\ \text{and } \delta = \nu_t$$

complete the proof of the claimed optimality conditions. □

Remark 3.3. By a suitable choice of the constraint qualification, we can show that $\alpha \neq 0$. Such a necessary condition with $\alpha \neq 0$ is usually referred to as a Karush-Kuhn-Tucker (KKT) type optimality condition.

Remark 3.4. Let

$$\Psi(\overline{x}) = \{y \in \mathbb{R}^m : h(\overline{x}, y) = V(\overline{x}),\ \phi(\overline{x}, y) \in -\mathbb{R}_+^r\}$$

stands for the set of optimal solution of the lower-level problem. Then

$$co\{A_h(\cdot, y)(\overline{x}) : y \in \Psi(\overline{x})\}$$

can be taken as a first order approximation of V at $\overline{x}$. where "co" stands for the convex hull of the set in question.

Next, we give the second order necessary optimality conditions of problem (P). To proceed, we first admit the following notations. For $w = (\beta, \gamma, \delta) \in \mathbb{R}_+^q \times \mathbb{R}_+^r \times \mathbb{R}_+$, we consider the set

$$E_w = \left\{u = (x, y) \in E\ :\ \sum_{l=1}^{q+r+1} w_l \Gamma_l(u) = 0\right\}. \tag{3.16}$$

Given further a point $\overline{u} = (\overline{x}, \overline{y}) \in E$ and $\alpha \in \mathbb{R}_+^p$ we define the following sets

$$\Delta(\overline{u}) = \left\{w = (\beta, \gamma, \delta) \in \mathbb{R}_+^q \times \mathbb{R}_+^r \times \mathbb{R}_+ \text{ such that } \parallel w \parallel \leq 1 \text{ and } \sum_{l=1}^{q+r+1} w_l \Gamma_l(u) = 0\right\}$$

and

$$\begin{aligned} \Pi_{\alpha,w}(\overline{u}) = \Big\{ \quad & u \in \mathbb{R}^n \times \mathbb{R}^m : \langle \alpha, A(u)\rangle + \langle \beta, B(u)\rangle + \langle \gamma, R(u)\rangle + \langle \delta, S(u)\rangle = 0, \\ & \forall\ A \in A_\Upsilon(\overline{u}), \forall\ B \in A_\psi(\overline{u}), \forall\ R \in A_\phi(\overline{u}), \forall\ S \in A_\varphi(\overline{u}) \Big\}. \end{aligned}$$

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Theorem 3.5. *Let $\overline{u} = (\overline{x}, \overline{y})$ be a local weak efficient solution of (P). Assume that hypotheses of theorem 3.2 are satisfied. Assume that g_i, $i \in I$, G_j, $j \in J$ and K_s, $s \in S$ are 2-calm at $\overline{u}$. Additionaly, suppose that $(A_{f_i}(\overline{u}), B_{f_i}(\overline{u}))$, $(0, B_{g_i}(\overline{u}))$, with $i \in I$, $\left(A_{F_j}(\overline{u}), B_{F_j}(\overline{u})\right)$, $\left(0, B_{G_j}(\overline{u})\right)$, with $j \in J$, $(A_{H_s}(\overline{u}), B_{H_s}(\overline{u}))$, $(0, B_{K_s}(\overline{u}))$, with $s \in S$, $(A_h(\overline{u}), B_h(\overline{u}))$ and $(A_V(\overline{x}) \times \{0\}, B_V(\overline{x}) \times \{0\})$ are bounded second order approximations of f_i, g_i, $i \in I$, F_j, G_j, $j \in J$, H_s, K_s, $s \in S$, h and $\overline{V}$ respectively at $\overline{u}$. Moreover, let $\alpha \in \mathbb{R}^p_+$ and $w \in \Delta(\overline{u})$. Then, for all $d \in \Pi_{\overline{\alpha}, \overline{w}}(\overline{u})$, with $\frac{\overline{\alpha_i}}{g_i(\overline{u})}$ for $i \in I$ and $\overline{w} = (\overline{\beta}, \overline{\gamma}, \delta)$, where $\overline{\beta_j} = \frac{\beta_j}{G_j(\overline{u})}$ for $j \in J$ and $\overline{\beta_j} = \frac{\gamma_s}{K_s(\overline{u})}$ for $s \in S$. $e \in K^2(E_w, \overline{u}, d)$, there exist $A_i^f \in clA_{f_i}(\overline{u})$, $B_i^f \in clB_{f_i}(\overline{u})$, $B_i^g \in clB_{g_i}(\overline{u})$ with $i \in I$, $A_j^F \in clA_{F_j}(\overline{u})$, $B_j^F \in clB_{F_j}(\overline{u})$, $B_j^G \in clB_{G_j}(\overline{u})$ with $j \in J$, $A_s^H \in clA_{H_s}(\overline{u})$, $B_s^H \in clB_{H_s}(\overline{u})$, $B_s^K \in clB_{K_s}(\overline{u})$, with $s \in S$, $A^h \in clA_h(\overline{u})$, $A^V \in clA_V(\overline{x}) \times \{0\}$, $B^h \in clB_h(\overline{u})$, $B^V \in clB_V(\overline{x}) \times \{0\}$ such that*

$$\begin{aligned}
&\sum_{i=1}^{p} \tfrac{\alpha_i}{g_i(\overline{u})} A_i^f(e) + \sum_{j=1}^{q} \tfrac{\beta_j}{G_j(\overline{u})} A_j^F(e) + \sum_{s=1}^{r} \tfrac{\gamma_s}{K_s(\overline{u})} A_s^H(e) \\
&+\delta\left(A^h - A^V\right)(e) + \delta\left(B^h - B^V\right)(d,d) + \sum_{i=1}^{p} \tfrac{\alpha_i}{g_i(\overline{u})} \left(B_i^f - \overline{p}_i B_i^g\right)(d,d) \\
&+\sum_{j=1}^{q} \tfrac{\beta_j}{G_j(\overline{u})} \left(B_j^F - \overline{q}_j B_j^G\right)(d,d) + \sum_{s=1}^{r} \tfrac{\gamma_s}{K_s(\overline{u})} \left(B_s^H - \overline{r}_s B_s^K\right)(d,d) \qquad \geq 0
\end{aligned} \tag{3.17}$$

Proof. Let $d \in \Pi_{\alpha,w}(\overline{u})$ and $e \in K^2(E_w, \overline{u}, d)$. By definition of second order set, there exists $(t_k, e_k) \to (0^+, e)$ such that

$$u_k = \overline{u} + t_k d + t_k^2 e_k \in E_w, \quad \forall\, k \in \mathbb{N}.$$

Therefore, considering (3.16) and the fact that $\overline{u} = (\overline{x}, \overline{y})$ is a local weak solution of $(P^\star)$ one has

$$\sum_{i=1}^{p} \alpha_i \left(\Upsilon_i(u_k) - \Upsilon_i(\overline{u})\right) + \sum_{l=1}^{q+r+1} w_l \Gamma_l(u_k) \geq 0,$$

for k large enough. Since the functions g_i, G_j and K_s involved in (P) are assumed to be 2-calm, then by Proposition 2.8 and Proposition 2.9

$$\begin{aligned}
&\left(\prod_{i=1}^{p} \tfrac{A_{f_i}(\overline{u})}{g_i(\overline{u})}, \prod_{i=1}^{p} \tfrac{1}{g_i(\overline{u})} \left(B_{f_i}(\overline{u}) - \overline{p}_i B_{g_i}(\overline{u})\right)\right), \\
&\left(\prod_{j=1}^{q} \tfrac{A_{F_j}(\overline{u})}{G_j(\overline{u})}, \prod_{j=1}^{q} \tfrac{1}{G_j(\overline{u})} \left(B_{F_j}(\overline{u}) - \overline{q}_j B_{G_j}(\overline{u})\right)\right), \\
&\left(\prod_{s=1}^{r} \tfrac{A_{H_s}(\overline{u})}{K_s(\overline{u})}, \prod_{s=1}^{r} \tfrac{1}{K_s(\overline{u})} \left(B_{H_s}(\overline{u}) - \overline{r}_s B_{K_s}(\overline{u})\right)\right)
\end{aligned}$$

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and

$$\Big(A_h(\overline{u}) - (A_V(\overline{x}) \times \{0\}), B_h(\overline{u}) - (B_V(\overline{x}) \times \{0\})\Big)$$

are second order approximations of Υ, ψ, ϕ and φ, respectively at $\overline{u}$.
Let $\epsilon > 0$ be arbitrarily chosen. Then, using the approximation definition, it follows that there exist $A^f_{i,k} \in A_{f_i}(\overline{u})$, $B^f_{i,k} \in B_{f_i}(\overline{u})$, $B^g_{i,k} \in B_{g_i}(\overline{u})$ with $i \in I$, $A^F_{j,k} \in A_{F_j}(\overline{u})$, $B^F_{j,k} \in B_{F_j}(\overline{u})$, $B^G_{j,k} \in B_{G_j}(\overline{u})$ with $j \in J$, $A^H_{s,k} \in A_{H_s}(\overline{u})$, $B^H_{s,k} \in B_{H_s}(\overline{u})$, $B^K_{s,k} \in B_{K_s}(\overline{u})$, with $s \in S$, $A^h_k \in A_h(\overline{u})$, $A^V_k \in A_V(\overline{x}) \times \{0\}$, $B^h_k \in B_h(\overline{u})$ and $B^V_k \in B_V(\overline{x}) \times \{0\}$ such that

$$\begin{aligned}
&\sum_{i=1}^{p} \tfrac{\alpha_i}{g_i(\overline{u})} \Big(A^f_{i,k}\left(t_k d + t_k^2 e_k\right)\Big) + \sum_{j=1}^{q} \tfrac{\beta_j}{G_j(\overline{u})} \Big(A^F_{j,k}\left(t_k d + t_k^2 e_k\right)\Big) \\
&+\sum_{s=1}^{r} \tfrac{\gamma_s}{K_s(\overline{u})} \Big(A^H_{s,k}\left(t_k d + t_k^2 e_k\right)\Big) + \sum_{i=1}^{p} \tfrac{\alpha_i}{g_i(\overline{u})} \Big(B^f_{i,k} - \overline{p}_i B^g_{i,k}\Big)\left(t_k d + t_k^2 e_k, t_k d + t_k^2 e_k\right) \\
&+\sum_{j=1}^{q} \tfrac{\beta_j}{G_j(\overline{u})} \Big(B^F_{j,k} - \overline{q}_j B^G_{j,k}\Big)\left(t_k d + t_k^2 e_k, t_k d + t_k^2 e_k\right) \\
&+\sum_{s=1}^{r} \tfrac{\gamma_s}{K_s(\overline{u})} \Big(B^H_{s,k} - \overline{r}_s B^K_{s,k}\Big)\left(t_k d + t_k^2 e_k, t_k d + t_k^2 e_k\right) + \delta\left(A^h_k - A^V_k\right)\left(t_k d + t_k^2 e_k\right) \\
&+\epsilon \parallel t_k d + t_k^2 e_k \parallel^2 \left(\sum_{i=1}^{p} b^{f,g}_{i,k} + \sum_{j=1}^{q} b^{F,G}_{j,k} + \sum_{s=1}^{r} b^{H,K}_{s,k} + b^{h,V}_k\right) \\
&+\delta\left(B^h_k - B^V_k\right)\left(t_k d + t_k^2 e_k, t_k d + t_k^2 e_k\right) \qquad \geq 0,
\end{aligned}$$

where, $b^{f,g}_{i,k}$, $i \in I$, $b^{F,G}_{j,k}$, $j \in J$, $b^{H,K}_{s,k}$, $s \in S$ and $b^{h,V}_k$ are elements of the closed unit ball $\mathbb{B}_{\mathbb{R}}$.
Dividing the last inequality by t_k, we obtain the following one

$$\begin{aligned}
&\sum_{i=1}^{p} \tfrac{\alpha_i}{g_i(\overline{u})} \Big(A^f_{i,k}(e_k)\Big) + \sum_{j=1}^{q} \tfrac{\beta_j}{G_j(\overline{u})} \Big(A^F_{j,k}(e_k)\Big) + \sum_{s=1}^{r} \tfrac{\gamma_s}{K_s(\overline{u})} \Big(A^H_{s,k}(e_k)\Big) \\
&+\sum_{i=1}^{p} \tfrac{\alpha_i}{g_i(\overline{u})} \Big(B^f_{i,k} - \overline{p}_i B^g_{i,k}\Big)(d + t_k e_k, d + t_k e_k) \\
&+\sum_{j=1}^{q} \tfrac{\beta_j}{G_j(\overline{u})} \Big(B^F_{j,k} - \overline{q}_j B^G_{j,k}\Big)(d + t_k e_k, d + t_k e_k) \\
&+\sum_{s=1}^{r} \tfrac{\gamma_s}{K_s(\overline{u})} \Big(B^H_{s,k} - \overline{r}_s B^K_{s,k}\Big)(d + t_k e_k, d + t_k e_k) + \delta\left(A^h_k - A^V_k\right)(e_k) \\
&+\epsilon \parallel d + t_k e_k \parallel^2 \left(\sum_{i=1}^{p} b^{f,g}_{i,k} + \sum_{j=1}^{q} b^{F,G}_{j,k} + \sum_{s=1}^{r} b^{H,K}_{s,k} + b^{h,V}_k\right) \\
&+\delta\left(B^h_k - B^V_k\right)(d + t_k e_k, d + t_k e_k) \qquad \geq 0.
\end{aligned}$$

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while taking into account the fact that $d \in \Pi_{\overline{\alpha},\overline{\omega}}(\overline{u})$ which ensures that

$$\sum_{i=1}^{p} \frac{\alpha_i}{g_i(\overline{u})}\left(A^f_{i,k}(d)\right)+\sum_{j=1}^{q} \frac{\beta_j}{G_j(\overline{u})}\left(A^F_{j,k}(d)\right)+\sum_{s=1}^{r} \frac{\gamma_s}{K_s(\overline{u})}\left(A^H_{s,k}(d)\right)+\delta\left(A^h_k - A^V_k\right)(d) = 0, \tag{3.18}$$

Consequently,

$$\begin{aligned}
&\sum_{i=1}^{p} \tfrac{\alpha_i}{g_i(\overline{u})}\left(A^f_{i,k}(e_k)\right)+\sum_{j=1}^{q} \tfrac{\beta_j}{G_j(\overline{u})}\left(A^F_{j,k}(e_k)\right)+\sum_{s=1}^{r} \tfrac{\gamma_s}{K_s(\overline{u})}\left(A^H_{s,k}(e_k)\right)\\
&+\delta\left(A^h_k - A^V_k\right)(e_k)+\sum_{i=1}^{p} \tfrac{\alpha_i}{g_i(\overline{u})}\left(B^f_{i,k}-\overline{p}_i B^g_{i,k}\right)(d,d)\\
&+\sum_{j=1}^{q} \tfrac{\beta_j}{G_j(\overline{u})}\left(B^F_{j,k}-\overline{q}_j B^G_{j,k}\right)(d,d)+\sum_{s=1}^{r} \tfrac{\gamma_s}{K_s(\overline{u})}\left(B^H_{s,k}-\overline{r}_s B^K_{s,k}\right)(d,d)\\
&+\epsilon \parallel d+t_k e_k \parallel^2\left(\sum_{i=1}^{p} b^{f,g}_{i,k}+\sum_{j=1}^{q} b^{F,G}_{j,k}+\sum_{s=1}^{r} b^{H,K}_{s,k}+b^{h,V}_k\right)+t_k\Theta(k)\\
&+\delta\left(B^h_k - B^V_k\right)(d,d) \qquad\qquad \geq 0,
\end{aligned} \tag{3.19}$$

where

$$\begin{aligned}
\Theta(k) = \;& \sum_{i=1}^{p} \tfrac{\alpha_i}{g_i(\overline{u})}\left(B^f_{i,k}-\overline{p}_i B^g_{i,k}\right)(d,e_k)+\sum_{i=1}^{p} \tfrac{\alpha_i}{g_i(\overline{u})}\left(B^f_{i,k}-\overline{p}_i B^g_{i,k}\right)(e_k,d)\\
&+\sum_{j=1}^{q} \tfrac{\beta_j}{G_j(\overline{u})}\left(B^F_{j,k}-\overline{q}_j B^G_{j,k}\right)(d,e_k)+\sum_{j=1}^{q} \tfrac{\beta_j}{G_j(\overline{u})}\left(B^F_{j,k}-\overline{q}_j B^G_{j,k}\right)(e_k,d)\\
&+\sum_{s=1}^{r} \tfrac{\gamma_s}{K_s(\overline{u})}\left(B^H_{s,k}-\overline{r}_s B^K_{s,k}\right)(d,e_k)+\sum_{s=1}^{r} \tfrac{\gamma_s}{K_s(\overline{u})}\left(B^H_{s,k}-\overline{r}_s B^K_{s,k}\right)(e_k,d)\\
&+\delta\left(B^h_k - B^V_k\right)(d,e_k)+\delta\left(B^h_k - B^V_k\right)(e_k,d)+t_k\Big(\delta\left(B^h_k - B^V_k\right)(e_k,e_k)\\
&+\sum_{i=1}^{p} \tfrac{\alpha_i}{g_i(\overline{u})}\left(B^f_{i,k}-\overline{p}_i B^g_{i,k}\right)(e_k,e_k)+\sum_{j=1}^{q} \tfrac{\beta_j}{G_j(\overline{u})}\left(B^F_{j,k}-\overline{q}_j B^G_{j,k}\right)(e_k,e_k)\\
&+\sum_{s=1}^{r} \tfrac{\gamma_s}{K_s(\overline{u})}\left(B^H_{s,k}-\overline{r}_s B^K_{s,k}\right)(e_k,e_k)\Big).
\end{aligned}$$

Due to the boundedness of the first order and second order approximations, extracting a subsequence if necessary and passing to the limit in (3.19) for k tending to infinity, we derive the existence of $A^f_i \in clA_{f_i}(\overline{u})$, $B^f_i \in clB_{f_i}(\overline{u})$, $B^g_i \in clB_{g_i}(\overline{u})$ with $i \in I$, $A^F_j \in clA_{F_j}(\overline{u})$, $B^F_j \in clB_{F_j}(\overline{u})$, $B^G_j \in clB_{G_j}(\overline{u})$ with $j \in J$, $A^H_s \in clA_{H_s}(\overline{u})$, $B^H_s \in clB_{H_s}(\overline{u})$, $B^K_s \in clB_{K_s}(\overline{u})$, with $s \in S$, $A^h \in clA_h(\overline{u})$, $A^V \in clA_V(\overline{x}) \times \{0\}$, $B^h \in clB_h(\overline{u})$, $B^V \in clB_V(\overline{x}) \times \{0\}$ and

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$b_i^{f,g}$, $i \in I$, $b_j^{F,G}$, $j \in J$, $b_s^{H,K}$, $s \in S$ and $b^{h,V}$ elements of $\mathbb{B}_{\mathbb{R}}$ such that

$$\begin{aligned}
&\sum_{i=1}^{p} \tfrac{\alpha_i}{g_i(\overline{u})} \left(A_i^f(e)\right) + \sum_{j=1}^{q} \tfrac{\beta_j}{G_j(\overline{u})} \left(A_j^F(e)\right) + \sum_{s=1}^{r} \tfrac{\gamma_s}{K_s(\overline{u})} \left(A_s^H(e)\right) + \delta\left(A^h - A^V\right)(e) \\
&+\sum_{i=1}^{p} \tfrac{\alpha_i}{g_i(\overline{u})} \left(B_i^f - \overline{p}_i B_i^g\right)(d,d) + \sum_{j=1}^{q} \tfrac{\beta_j}{G_j(\overline{u})} \left(B_j^F - \overline{q}_j B_j^G\right)(d,d) \\
&+\sum_{s=1}^{r} \tfrac{\gamma_s}{K_s(\overline{u})} \left(B_s^H - \overline{r}_s B_s^K\right)(d,d) + \delta\left(B^h - B^V\right)(d,d) \\
&+\epsilon \parallel d \parallel^2 \left(\sum_{i=1}^{p} b_i^{f,g} + \sum_{j=1}^{q} b_j^{F,G} + \sum_{s=1}^{r} b_s^{H,K} + b^{h,V}\right) \geq 0.
\end{aligned} \tag{3.20}$$

Letting $\epsilon \to 0$, we derive the desired inequality

$$\begin{aligned}
&\sum_{i=1}^{p} \tfrac{\alpha_i}{g_i(\overline{u})} A_i^f(e) + \sum_{j=1}^{q} \tfrac{\beta_j}{G_j(\overline{u})} A_j^F(e) + \sum_{s=1}^{r} \tfrac{\gamma_s}{K_s(\overline{u})} A_s^H(e) + \delta\left(A^h - A^V\right)(e) \\
&+\delta\left(B^h - B^V\right)(d,d) + \sum_{i=1}^{p} \tfrac{\alpha_i}{g_i(\overline{u})} \left(B_i^f - \overline{p}_i B_i^g\right)(d,d) \\
&+\sum_{j=1}^{q} \tfrac{\beta_j}{G_j(\overline{u})} \left(B_j^F - \overline{q}_j B_j^G\right)(d,d) + \sum_{s=1}^{r} \tfrac{\gamma_s}{K_s(\overline{u})} \left(B_s^H - \overline{r}_s B_s^K\right)(d,d) \geq 0.
\end{aligned}$$

□

4. Second order sufficient optimality conditions

Let $\overline{u} \in E$. Assume that f_i, g_i, $i \in I$, F_j, G_j, $j \in J$, H_s, K_s, $s \in S$, h and $\overline{V}$ admit compact first order approximations $A_{f_i}(\overline{u})$, $A_{g_i}(\overline{u})$, with $i \in I$, $A_{F_j}(\overline{u})$, $A_{G_j}(\overline{u})$, with $j \in J$, $A_{H_s}(\overline{u})$, $A_{K_s}(\overline{u})$, with $s \in S$, $A_h(\overline{u})$ and $A_V(\overline{x}) \times \{0\}$, respectively at $\overline{u}$.

Theorem 4.1. *Assume that there exist $\alpha \in \mathbb{R}_+^p$ and $w \in \Delta(\overline{u})$ such that for all directions $d \in K(E,\overline{u}) \setminus \{0\}$ and for all $A_i^f \in A_{f_i}(\overline{u})$, $A_i^g \in A_{g_i}(\overline{u})$, with $i \in I$, $A_j^F \in A_{F_j}(\overline{u})$, $A_j^G \in A_{G_j}(\overline{u})$, with $j \in J$, $A_s^H \in A_{H_s}(\overline{u})$, $A_s^K \in A_{K_s}(\overline{u})$, with $s \in S$, $A^h \in A_h(\overline{u})$ and $A^V \in A_V(\overline{x}) \times \{0\}$ one has*

$$\begin{aligned}
&\sum_{i=1}^{p} \tfrac{\alpha_i}{g_i(\overline{u})} \left(A_i^f(d) - \overline{p}_i A_i^g(d)\right) + \sum_{j=1}^{q} \tfrac{\beta_j}{G_j(\overline{u})} \left(A_j^F(d) - \overline{q}_j A_j^G(d)\right) \\
&+\sum_{s=1}^{r} \tfrac{\gamma_s}{K_s(\overline{u})} \left(A_s^H(d) - \overline{r}_s A_s^K(d)\right) + \delta\left(A^h(d) - A^V(d)\right) > 0.
\end{aligned} \tag{4.1}$$

Then, $\overline{u} = (\overline{x}, \overline{y})$ is a local weak efficient solution of (P).

Proof. Suppose that $\overline{u}$ is not a local weakly efficient solution of (P), Then, there exist a feasible solution $u_k = (x_k, y_k) \in E$ and a neighborhood U of $\overline{u}$ such that

$$u_k \in U \quad \text{and} \quad \Upsilon(u_k) - \Upsilon(\overline{u}) \in -int\ \left(\mathbb{R}_+^p\right).$$

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Setting, $t_k = \| u_k - \overline{u} \|$ and $d_k = \frac{u_k - \overline{u}}{\| u_k - \overline{u} \|}$, one has

$$u_k = \overline{u} + t_k d_k \in E, \ t_k \downarrow 0 \ \text{ and } \ d_k \to d, \ \text{ with } \ \| d \| = 1.$$

Thus, $d \in K(E, \overline{u}) \setminus \{0\}$.
On the one hand, since, $\alpha = (\alpha_1, ..., \alpha_p) \in \mathbb{R}^p_+$, it follows that

$$\sum_{i=1}^{p} \alpha_i \left(\Upsilon_i(u_k) - \Upsilon_i(\overline{u}) \right) \le 0. \tag{4.2}$$

Moreover, considering the fact that $u_k \in E$, while noting that $w \in \Delta(\overline{u})$ we have

$$\sum_{j=1}^{q} \beta_j \left(\Gamma_j(u_k) - \Gamma_j(\overline{u}) \right) + \sum_{s=1}^{r} \gamma_s \left(\Gamma_s(u_k) - \Gamma_s(\overline{u}) \right) + \delta \left(h(u_k) - V(x_k) \right) = 0. \tag{4.3}$$

Hence, (4.2) and (4.3) yield,

$$\begin{aligned} &\sum_{i=1}^{p} \alpha_i \left(\Upsilon_i(u_k) - \Upsilon_i(\overline{u}) \right) + \sum_{j=1}^{q} \beta_j \left(\Gamma_j(u_k) - \Gamma_j(\overline{u}) \right) \\ &+ \sum_{s=1}^{r} \gamma_s \left(\Gamma_s(u_k) - \Gamma_s(\overline{u}) \right) + \delta \left(h(u_k) - V(x_k) \right) \qquad \le 0. \end{aligned} \tag{4.4}$$

On the other hand, let $\epsilon > 0$ be arbitrarily chosen. Then by the definition of first order approximations and definition of the mappings Υ_i, ψ_j, ϕ_s and φ there exist $A^f_{i,k} \in A_{f_i}(\overline{u})$, $A^g_{i,k} \in A_{g_i}(\overline{u})$, with $i \in I$, $A^F_{j,k} \in A_{F_j}(\overline{u})$, $A^G_{j,k} \in A_{G_j}(\overline{u})$, with $j \in J$, $A^H_{s,k} \in A_{H_s}(\overline{u})$, $A^K_{s,k} \in A_{K_s}(\overline{u})$, with $s \in S$, $A^h_k \in A_h(\overline{u})$ and $A^V_k \in A_V(\overline{x}) \times \{0\}$ and there exist $b^{f,g}_{i,k}$, $i \in I$, $b^{F,G}_{j,k}$, $j \in J$, $b^{H,K}_{s,k}$, $s \in S$ and $b^{h,V}_k$ in the closed unit ball $\mathbb{B}_{\mathbb{R}}$ such that

$$\begin{aligned} &\sum_{i=1}^{p} \tfrac{\alpha_i}{g_i(\overline{u})} \left(A^f_{i,k}(d_k) - \overline{p}_i A^g_{i,k}(d_k) \right) + \sum_{j=1}^{q} \tfrac{\beta_j}{G_j(\overline{u})} \left(A^F_{j,k}(d_k) - \overline{q}_j A^G_{j,k}(d_k) \right) \\ &+ \sum_{s=1}^{r} \tfrac{\gamma_s}{K_s(\overline{u})} \left(A^H_{s,k}(d_k) - \overline{r}_s A^K_{s,k}(d_k) \right) + \delta \left(A^h_k(d_k) - A^V_k(d_k) \right) \\ &+ \epsilon \| d_k \| \left(\sum_{i=1}^{p} b^{f,g}_{i,k} + \sum_{j=1}^{q} b^{F,G}_{j,k} + \sum_{s=1}^{r} b^{H,K}_{s,k} + b^{h,V}_k \right) \qquad \le 0. \end{aligned} \tag{4.5}$$

Under the compactness property of the first order approximations, while passing to the limit, it follows that there exist $A^f_i \in A_{f_i}(\overline{u})$, $A^g_i \in A_{g_i}(\overline{u})$, with $i \in I$, $A^F_j \in A_{F_j}(\overline{u})$, $A^G_j \in A_{G_j}(\overline{u})$, with $j \in J$, $A^H_s \in A_{H_s}(\overline{u})$, $A^K_s \in A_{K_s}(\overline{u})$,

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with $s \in S$, $A^h \in A_h(\overline{u})$ and $A^V \in A_V(\overline{x}) \times \{0\}$ such that

$$\begin{aligned}&\sum_{i=1}^{p} \tfrac{\alpha_i}{g_i(\overline{u})} \left(A_i^f(d) - \overline{p}_i A_i^g(d)\right) + \sum_{j=1}^{q} \tfrac{\beta_j}{G_j(\overline{u})} \left(A_j^F(d) - \overline{q}_j A_j^G(d)\right) \\ &+\sum_{s=1}^{r} \tfrac{\gamma_s}{K_s(\overline{u})} \left(A_s^H(d) - \overline{r}_s A_s^K(d)\right) + \delta\left(A^h(d) - A^V(d)\right) \\ &+\epsilon \parallel d \parallel \left(\sum_{i=1}^{p} b_i^{f,g} + \sum_{j=1}^{q} b_j^{F,G} + \sum_{s=1}^{r} b_s^{H,K} + b^{h,V}\right) \qquad \leq 0.\end{aligned} \tag{4.6}$$

with, $b_i^{f,g}$, $i \in I$, $b_j^{F,G}$, $j \in J$, $b_s^{H,K}$, $s \in S$ and $b^{h,V}$ are elements of the closed unit ball $\mathbb{B}_{\mathbb{R}}$. For $\epsilon \to 0$ we derive

$$\begin{aligned}&\sum_{i=1}^{p} \tfrac{\alpha_i}{g_i(\overline{u})} \left(A_i^f(d) - \overline{p}_i A_i^g(d)\right) + \sum_{j=1}^{q} \tfrac{\beta_j}{G_j(\overline{u})} \left(A_j^F(d) - \overline{q}_j A_j^G(d)\right) \\ &+\sum_{s=1}^{r} \tfrac{\gamma_s}{K_s(\overline{u})} \left(A_s^H(d) - \overline{r}_s A_s^K(d)\right) + \delta\left(A^h(d) - A^V(d)\right) \qquad \leq 0.\end{aligned}$$

which is a contradiction to the assumptions of the theorem. □

Let $\overline{u} \in E$. Assume that f_i, g_i, $i \in I$, F_j, G_j, $j \in J$, H_s, K_s, $s \in S$, h and $\overline{V}$ admit bounded second order approximations $(A_{f_i}(\overline{u}), B_{f_i}(\overline{u}))$, $(0, B_{g_i}(\overline{u}))$, with $i \in I$, $\left(A_{F_j}(\overline{u}), B_{F_j}(\overline{u})\right)$, $\left(0, B_{G_j}(\overline{u})\right)$, with $j \in J$, $(A_{H_s}(\overline{u}), B_{H_s}(\overline{u}))$, $(0, B_{K_s}(\overline{u}))$, with $s \in S$, $(A_h(\overline{u}), B_h(\overline{u}))$ and $(A_V(\overline{x}) \times \{0\}, B_V(\overline{x}) \times \{0\})$, respectively, at $\overline{u}$ such that $A_{f_i}(\overline{u})$, with $i \in I$, $A_{F_j}(\overline{u})$, with $j \in J$, $A_{H_s}(\overline{u})$, with $s \in S$, $A_h(\overline{x})$ and $A_V(\overline{x}) \times \{0\}$ are compact sets.

Theorem 4.2. *Suppose that there exist $\alpha \in \mathbb{R}_+^p$, $w \in \Delta(\overline{u})$ and $\sigma > 0$ such that for all $A_i^f \in A_{f_i}(\overline{u})$, $A_i^g \in A_{g_i}(\overline{u})$, with $i \in I$, for all $A_j^F \in A_{F_j}(\overline{u})$, $A_j^G \in A_{G_j}(\overline{u})$, with $j \in J$, for all $A_s^H \in A_{H_s}(\overline{u})$, $A_s^K \in A_{K_s}(\overline{u})$, with $s \in S$, for all $A^h \in A_h(\overline{u})$ for all $A^V \in A_V(\overline{x}) \times \{0\}$, for all $d \in \mathbb{R}^n \times \mathbb{R}^m$ such that $dist(d, K(E, \overline{u})) < \sigma$ we have*

$$\begin{aligned}&\sum_{i=1}^{p} \tfrac{\alpha_i}{g_i(\overline{u})} \left(A_i^f(d)\right) + \sum_{j=1}^{q} \tfrac{\beta_j}{G_j(\overline{u})} \left(A_j^F(d)\right) \\ &+\sum_{s=1}^{r} \tfrac{\gamma_s}{K_s(\overline{u})} \left(A_s^H(d)\right) + \delta\left(A^h(d) - A^V(d)\right) \quad \geq 0.\end{aligned} \tag{4.7}$$

If for all $d \in K(E, \overline{u})$, $B_i^f \in clB_{f_i}(\overline{u})$, $B_i^g \in clB_{g_i}(\overline{u})$ with $i \in I$, $B_j^F \in clB_{F_j}(\overline{u})$, $B_j^G \in clB_{G_j}(\overline{u})$ with $j \in J$, $B_s^H \in clB_{H_s}(\overline{u})$, $B_s^K \in clB_{K_s}(\overline{u})$, with $s \in S$, $B^h \in clB_h(\overline{u})$, $B^V \in clB_V(\overline{x}) \times \{0\}$ one has

$$\begin{aligned}&\sum_{i=1}^{p} \tfrac{\alpha_i}{g_i(\overline{u})} \left(B_i^f - \overline{p}_i B_i^g\right)(d,d) + \sum_{j=1}^{q} \tfrac{\beta_j}{G_j(\overline{u})} \left(B_j^F - \overline{q}_j B_j^G\right)(d,d) + \\ &+\sum_{s=1}^{r} \tfrac{\gamma_s}{K_s(\overline{u})} \left(B_s^H - \overline{r}_s B_s^K\right)(d,d) + \delta\left(B^h - B^V\right)(d,d) \qquad > 0.\end{aligned} \tag{4.8}$$

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Then, $\overline{u} = (\overline{x}, \overline{y})$ is a local weak efficient solution of (P).

Proof. It follows the path of that of Theorem 4.1. Obviously the difference lies in the definition of the second order approximation. To proceed, first assume that $\overline{u}$ is not a local weak efficient solution of (P). Then, there exists a sequence $(u_k)_k \in E$ converging to $\overline{u}$ such that

$$u_k = \overline{u} + t_k d_k,\ t_k = \parallel u_k - \overline{u} \parallel,\ d_k = \frac{u_k - \overline{u}}{\parallel u_k - \overline{u} \parallel}$$

and

$$\Upsilon(u_k) - \Upsilon(\overline{u}) \in -int\ \left(\mathbb{R}^p_+\right),\quad \forall\ k. \tag{4.9}$$

Hence,

$$\sum_{i=1}^{p} \alpha_i \left(\Upsilon_i(u_k) - \Upsilon_i(\overline{u})\right) \leq 0. \tag{4.10}$$

while taking into account that $\alpha_i \geq 0$ for $i \in I$. Without loss of genarality, we can assume the existence of $d \in K(E, \overline{u})$ with $\parallel d \parallel = 1$ such that $d_k \to d$. Since, u_k is feasible and $w \in \Delta(\overline{u})$, we obtain

$$\sum_{j=1}^{q} \beta_j \left(\Gamma_j(u_k) - \Gamma_j(\overline{u})\right) + \sum_{s=1}^{r} \gamma_s \left(\Gamma_s(u_k) - \Gamma_s(\overline{u})\right) + \delta\left(h(u_k) - V(x_k)\right) = 0. \tag{4.11}$$

Combining this inequality with (4.10), it holds that

$$\begin{aligned} &\sum_{i=1}^{p} \alpha_i \left(\Upsilon_i(u_k) - \Upsilon_i(\overline{u})\right) + \sum_{j=1}^{q} \beta_j \left(\Gamma_j(u_k) - \Gamma_j(\overline{u})\right) \\ &+ \sum_{s=1}^{r} \gamma_s \left(\Gamma_s(u_k) - \Gamma_s(\overline{u})\right) + \delta\left(h(u_k) - V(x_k)\right) \qquad \leq 0. \end{aligned} \tag{4.12}$$

Let $\epsilon > 0$ be arbitrarily chosen. Then, using the approximation definition, it follows that there exist $A^f_{i,k} \in A_{f_i}(\overline{u})$, $B^f_{i,k} \in B_{f_i}(\overline{u})$, $B^g_{i,k} \in B_{g_i}(\overline{u})$ with $i \in I$, $A^F_{j,k} \in A_{F_j}(\overline{u})$, $B^F_{j,k} \in B_{F_j}(\overline{u})$, $B^G_{j,k} \in B_{G_j}(\overline{u})$ with $j \in J$, $A^H_{s,k} \in A_{H_s}(\overline{u})$, $B^H_{s,k} \in B_{H_s}(\overline{u})$, $B^K_{s,k} \in B_{K_s}(\overline{u})$, with $s \in S$, $A^h_k \in A_h(\overline{u})$, $A^V_k \in A_V(\overline{x}) \times \{0\}$, $B^h_k \in B_h(\overline{u})$ and $B^V_k \in B_V(\overline{x}) \times \{0\}$ such that

$$\begin{aligned} &\sum_{i=1}^{p} \tfrac{\alpha_i}{g_i(\overline{u})} \left(A^f_{i,k}(d_k)\right) + \sum_{j=1}^{q} \tfrac{\beta_j}{G_j(\overline{u})} \left(A^F_{j,k}(d_k)\right) + \sum_{s=1}^{r} \tfrac{\gamma_s}{K_s(\overline{u})} \left(A^H_{s,k}(d_k)\right) \\ &+ t_k \sum_{i=1}^{p} \tfrac{\alpha_i}{g_i(\overline{u})} \left(B^f_{i,k} - \overline{p}_i B^g_{i,k}\right)(d_k, d_k) + t_k \sum_{j=1}^{q} \tfrac{\beta_j}{G_j(\overline{u})} \left(B^F_{j,k} - \overline{q}_j B^G_{j,k}\right)(d_k, d_k) \\ &+ t_k \sum_{s=1}^{r} \tfrac{\gamma_s}{K_s(\overline{u})} \left(B^H_{s,k} - \overline{r}_s B^K_{s,k}\right)(d_k, d_k) + \delta\left(A^h_k - A^V_k\right)(d_k) \\ &+ t_k \delta\left(B^h_k - B^V_k\right)(d_k, d_k) + t_k \epsilon \parallel d_k \parallel^2 \left(\sum_{i=1}^{p} b^{f,g}_{i,k} + \sum_{j=1}^{q} b^{F,G}_{j,k} + \sum_{s=1}^{r} b^{H,K}_{s,k} + b^{h,V}_k\right) \quad \leq 0, \end{aligned}$$

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where, $b^{f,g}_{i,k}$, $i \in I$, $b^{F,G}_{j,k}$, $j \in J$, $b^{H,K}_{s,k}$, $s \in S$ and $b^{h,V}_{k}$ are elements of the closed unit ball $\mathbb{B}_{\mathbb{R}}$.
Hence, by assumption of the theorem, due to $d \in K(E, \overline{u})$, one has

$$\begin{aligned}
&\sum_{i=1}^{p} \tfrac{\alpha_i}{g_i(\overline{u})} \left(B^f_{i,k} - \overline{p}_i B^g_{i,k}\right)(d_k, d_k) + \sum_{j=1}^{q} \tfrac{\beta_j}{G_j(\overline{u})} \left(B^F_{j,k} - \overline{q}_j B^G_{j,k}\right)(d_k, d_k) \\
&+\sum_{s=1}^{r} \tfrac{\gamma_s}{K_s(\overline{u})} \left(B^H_{s,k} - \overline{r}_s B^K_{s,k}\right)(d_k, d_k) + \delta\left(B^h_k - B^V_k\right)(d_k, d_k) \\
&+\epsilon \parallel d_k \parallel^2 \left(\sum_{i=1}^{p} b^{f,g}_{i,k} + \sum_{j=1}^{q} b^{F,G}_{j,k} + \sum_{s=1}^{r} b^{H,K}_{s,k} + b^{h,V}_k\right) \leq 0,
\end{aligned} \tag{4.13}$$

Due to the boundedness of the second order approximations, extracting a subsequence if necessary and passing to the limit in (4.13) for k tending to infinity, we derive the existence of $B^f_i \in clB_{f_i}(\overline{u})$, $B^g_i \in clB_{g_i}(\overline{u})$ with $i \in I$, $B^F_j \in clB_{F_j}(\overline{u})$, $B^G_j \in clB_{G_j}(\overline{u})$ with $j \in J$, $B^H_s \in clB_{H_s}(\overline{u})$, $B^K_s \in clB_{K_s}(\overline{u})$, with $s \in S$, $B^h \in clB_h(\overline{u})$, $B^V \in clB_V(\overline{x}) \times \{0\}$ and $b^{f,g}_i$, $i \in I$, $b^{F,G}_j$, $j \in J$, $b^{H,K}_s$, $s \in S$ and $b^{h,V}$ elements of $\mathbb{B}_{\mathbb{R}}$ such that

$$\begin{aligned}
&\sum_{i=1}^{p} \tfrac{\alpha_i}{g_i(\overline{u})} \left(B^f_i - \overline{p}_i B^g_i\right)(d, d) + \sum_{j=1}^{q} \tfrac{\beta_j}{G_j(\overline{u})} \left(B^F_j - \overline{q}_j B^G_j\right)(d, d) \\
&+\sum_{s=1}^{r} \tfrac{\gamma_s}{K_s(\overline{u})} \left(B^H_s - \overline{r}_s B^K_s\right)(d, d) + \delta\left(B^h - B^V\right)(d, d) \\
&+\epsilon \parallel d \parallel^2 \left(\sum_{i=1}^{p} b^{f,g}_i + \sum_{j=1}^{q} b^{F,G}_j + \sum_{s=1}^{r} b^{H,K}_s + b^{h,V}\right) \leq 0,
\end{aligned}$$

Letting $\epsilon \to 0$, we derive

$$\begin{aligned}
&\sum_{i=1}^{p} \tfrac{\alpha_i}{g_i(\overline{u})} \left(B^f_i - \overline{p}_i B^g_i\right)(d, d) + \sum_{j=1}^{q} \tfrac{\beta_j}{G_j(\overline{u})} \left(B^F_j - \overline{q}_j B^G_j\right)(d, d) \\
&+\sum_{s=1}^{r} \tfrac{\gamma_s}{K_s(\overline{u})} \left(B^H_s - \overline{r}_s B^K_s\right)(d, d) + \delta\left(B^h - B^V(d, d)\right) \leq 0.
\end{aligned} \tag{4.14}$$

which is a contradiction. □

5. Special case

If $\Upsilon(x, y) = f(x, y)$, $\psi(x, y) = (F_1(x, y), \cdots, F_q(x, y))$, $\phi(x, y) = 0_{\mathbb{R}^r}$ and $h(x, y) = 0$, then the problem

$$(P^{\diamond}) : \begin{cases} \text{Minimize } f(x, y) \\ \text{subject to} : \psi(x, y) \in \mathbb{R}^q_+ \end{cases} \tag{5.1}$$

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is obtained, where $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ and $\psi_j : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ are given functions; $n \geq 1$ and $m \geq 1$ are integers. In this case,

$$E = \left\{ u = (x, y) \in \mathbb{R}^n \times \mathbb{R}^m : F_j\left(x, y\right) \leq 0, \ \forall \ j \in J \right\}.$$

For $\beta \in \mathbb{R}^q_+$, consider the set

$$E_\beta = \{u = (x, y) \in E : \sum_{j=1}^{q} \beta_j F_j\left(x, y\right) = 0\}.$$

For $\overline{u} \in E$ and $\alpha \in \mathbb{R}_+$, the sets $\Delta\left(\overline{u}\right)$ and $\Pi_{\alpha,\beta}\left(\overline{u}\right)$ become

$$\Delta\left(\overline{u}\right) = \left\{ \beta \in \mathbb{R}^q_+ : \parallel \beta \parallel \leq 1 : \ \sum_{j=1}^{q} \beta_j F_j\left(x, y\right) = 0 \right\}$$

and

$$\Pi_{\alpha,\beta}\left(\overline{u}\right) = \left\{ u \in \mathbb{R}^n \times \mathbb{R}^m : \alpha A\left(u\right) + \sum_{j=1}^{q} \beta_j A_j\left(u\right) = 0, \ \forall \ A \in A_f\left(u\right), \forall \ A_j \in A_{F_j}\left(u\right), \ j \in J \right\}.$$

The next results give second order necessary and sufficient optimality conditions for $(P^\diamond)$.

Corollary 5.1. [2] *Let $\alpha \in \mathbb{R}_+$, $\beta \in \mathbb{R}^q_+$ and $\overline{u} = (\overline{x}, \overline{y})$ be a local optimal solution of problem $(P^\diamond)$. Assume that f and F_j, with $j \in J$ admit bounded second order approximations $\left(A_f\left(\overline{u}\right), B_f\left(\overline{u}\right)\right)$, $\left(A_{F_j}\left(\overline{u}\right), B_{F_j}\left(\overline{u}\right)\right)$, with $j \in J$ at $\overline{u}$. Then,*

(1) *for all $u \in \mathbb{R}^n \times \mathbb{R}^m$, there exist $(\mu, \nu) \in \mathbb{R}^{q+1}_+ \setminus \{0\}$, $A^f \in clA_f\left(\overline{u}\right)$ and $A^F_j \in clA_{F_j}\left(\overline{u}\right)$, with $j \in J$ such that*

$$\mu A^f\left(u\right) + \sum_{j=1}^{q} \nu_j A^F_j\left(u\right) \geq 0.$$

(2) *for all $d \in \Pi_{\alpha,\beta}\left(\overline{u}\right)$, $e \in K^2\left(E_\beta, \overline{u}, d\right)$, there exist $A^f \in clA_f\left(\overline{u}\right)$, $B^f \in clB_f\left(\overline{u}\right)$ and $A^F_j \in clA_{F_j}\left(\overline{u}\right)$, $B^F_j \in clB_{F_j}\left(\overline{u}\right)$, with $j \in J$ such that*

$$\alpha A^f\left(e\right) + \sum_{j=1}^{q} \beta_j A^F_j\left(e\right) + \alpha B^f\left(d, d\right) + \sum_{j=1}^{q} \beta_j B^F_j\left(d, d\right) \geq 0.$$

Corollary 5.2. *Let $\overline{u} = (\overline{x}, \overline{y})$ be a feasible point of problem $(P^\diamond)$. Suppose that f and F_j, with $j \in J$ admit bounded second order approximations $\left(A_f\left(\overline{u}\right), B_f\left(\overline{u}\right)\right)$, $\left(A_{F_j}\left(\overline{u}\right), B_{F_j}\left(\overline{u}\right)\right)$, with $j \in J$ at $\overline{u}$ such that $A_f\left(\overline{u}\right)$ and $A_{F_j}\left(\overline{u}\right)$, with $j \in J$ are compact sets. Then, $\overline{u}$ is local optimal solution of $(P^\diamond)$ if one of the following conditions is satisfied :*

(1) *There exist* $\beta \in \Delta(\overline{u})$ *and* $\alpha \in \mathbb{R}_+$ *such that for all* $d \in K(E, \overline{u}) \setminus \{0\}$, *for all* $A^f \in A_f(\overline{u})$ *and* $A_j^F \in A_{F_j}(\overline{u})$, *with* $j \in J$ *one has*

$$\alpha A^f(d) + \sum_{j=1}^{q} \beta_j A_j^F(d) > 0.$$

(2) *There exist* $\beta \in \Delta(\overline{u})$, $\alpha \in \mathbb{R}_+$ *and* $\sigma > 0$ *such that for all* $d \in \mathbb{R}^n \times \mathbb{R}^m$, *for all* $A^f \in A_f(\overline{u})$ *and* $A_j^F \in A_{F_j}(\overline{u})$, *with* $j \in J$ *such that* $dist(d, K(E, \overline{u})) < \sigma$ *one has*

$$\alpha A^f(d) + \sum_{j=1}^{q} \beta_j A_j^F(d) \geq 0$$

and for all $d \in K(E, \overline{u})$ *and for all* $B^f \in clB_f(\overline{u})$, $B_j^F \in clB_{F_j}(\overline{u})$ *with* $j \in J$ *one has*

$$\alpha B^f(d, d) + \sum_{j=1}^{q} \beta_j B_j^F(d, d) > 0.$$

6. Conclusion

In this work, we consider a multiobjective fractional bilevel programming on finite dimensional spaces with nonsmooth data. We establish first and second order optimality conditions in terms of generalized derivatives called approximations. Moreover, we assume that all data may not be Lipshitz. We used an intermediate set-valued problem to detect optimality conditions for local weak efficient solutions. The obtained conditions of orders 1 and 2 are expressed in terms of approximations of orders 1 and 2, respectively.

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