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Monotone Orbitally Nonexpansive and Cyclic Mappings in Partially Ordered Uniformly Convex Banach Spaces

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Abstract. In the setting of uniformly convex Banach spaces equipped with a partially ordered relation, we survey the existence of fixed points for monotone orbitally nonexpansive mappings. In this way, we extend and improve the main results of Alfuraidan and Khamsi [M. R. Alfuraidan, M. A. Khamsi, *Proc. Amer. Math. Soc.*, **146**, (2018), 2451-2456]. Examples are given to show the usability of our main conclusions. We also study the existence of an optimal solution for cyclic contractions in such spaces.

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1. Introduction

Nonexpansive mappings are those mappings which have Lipschitz constant equal to one. For example, contractions, isometries, and the resolvents of accretive operators are all nonexpansive. Recently, a new direction of research has been discovered dealing with the extension of the Banach contraction principle to partially ordered metric spaces. For example, Bacher and Khamsi [4] considered the case of monotone nonexpansive mappings and tried to answer the question whether the classical fixed point theorems for nonexpansive mappings still hold for monotone nonexpansive mappings. The difficulty in dealing with monotone nonexpansive mappings is that the monotone Lipschitzian mappings enjoy suitable properties only on comparable elements. In fact, they may not be even continuous, a property obviously shared by Lipschitzian mappings. In continuity several generalizations have received attention, for example those due to Goebel and Kirk [9, 11], Goebel et al. [10], Suzuki [16], Garcia-Falset et al. [8] and Aoyama and Kohsaka [3].

Now, assume that A and B are two nonempty subsets of a metric space. It might happen that a mapping T from A to B lacks any fixed point, so that it is extremely important to know whether there exists a point x in A such that the distance of x to Tx is minimum. Note that if this distance is equal to zero, then x is a fixed point of T. A point x in A is said to be a *best proximity point* of T provided that the distance of x to Tx is equal to the distance of A to B. Indeed, best proximity point theorems are connected to optimal approximate solutions of some equations. Existence of best proximity points for a class of non-self mappings, called *cyclic contractions*, was studied in [7] in the setting of uniformly convex Banach spaces. After that in [1, 2] the authors proved some fixed point theorems as well as best proximity point theorems for cyclic contractions in partially ordered metric spaces using appropriate geometric properties of uniformly convex Banach spaces, and so extended the main conclusions of [13] (see also [15] for a different approach to the same problem).

This paper is organized as follows: in Section 2, we recall some definitions and notations which will be used in our coming discussion. In Section 3, we extend the Goebel-Kirk's fixed point theorem by considering the class of monotone orbitally nonexpansive mappings in uniformly convex Banach spaces with a partially ordered relation. Finally, in Section 4, we introduce a class of T-cyclic contractions using a partially ordered relation and prove a common best proximity point theorem in uniformly convex Banach spaces.

2. Preliminaries

Throughout this paper, $(X, ||\cdot||)$ is a real Banach space endowed with a partial order "$\preceq$". We will say that $x, y \in X$ are comparable whenever $x \preceq y$

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or $y \preceq x$. As usual we adopt the convention $x \succeq y$ if and only if $y \preceq x$. The linear structure of X is assumed to be compatible with the order structure in the following sense:

(i) $x \preceq y \Longrightarrow x + z \preceq y + z$ for all $x, y, z \in X$;
(ii) $x \preceq y \Longrightarrow \alpha x \preceq \alpha y$ for all $x, y \in X$ and $\alpha > 0$.

Moreover, we assume that order intervals are closed. Recall that an order interval is any of the subsets:

(i) $[a, \rightarrow) = \{x \in X : a \preceq x\}$;
(ii) $(\leftarrow, a] = \{x \in X : x \preceq a\}$.

Definition 2.1. Let $(X, ||\cdot||, \preceq)$ be an ordered Banach space and C be a nonempty subset of X. A mapping $T : C \to X$ is said to be:

(i) monotone or order-preserving if $Tx \preceq Ty$ whenever $x \preceq y$;
(ii) monotone k-Lipschitzian, if T is monotone and $||Tx - Ty|| \leq k||x - y||$ for every $x, y \in X$ such that x and y are comparable, $k \in \mathbb{R}^+ = (0, +\infty)$.

If A is a nonempty subset of X, $conv(A)$ will denote the convex hull of the set A, that is the smallest convex set that contains A. From now on, C stands for a given nonempty, closed, convex, and bounded subset of X. A mapping $T : C \to X$ is nonexpansive if and only if

$$||Tx - Ty|| \leq ||x - y|| \quad \text{for all } x, y \in C.$$

If $T : C \to C$ is a mapping and $x_0 \in C$, the sequence $\{T^n x_0 : n = 0, 1, 2, ...\}$ is often called the orbit of T starting at x_0, which will be denoted by $orb(T, x_0)$. A point $x \in X$ is said to be a fixed point of T whenever $Tx = x$. The set of fixed points of T will be denoted by $Fix(T)$.

Recall that monotone Lipschitzian mappings are not necessarily continuous. They usually have a good topological behavior on comparable elements but not on the entire set on which they are defined.

We also recall that a sequence $\{x_n\}_{n\in\mathbb{N}}$ in a partially ordered set $(X, \preceq)$ is said to be

(i) monotone increasing if $x_n \preceq x_{n+1}$ for all $n \in \mathbb{N}$;
(ii) monotone decreasing if $x_{n+1} \preceq x_n$ for all $n \in \mathbb{N}$;
(iii) monotone if it is either monotone increasing or decreasing.

Let A and B be nonempty subsets of a metric space (X, d). A mapping $T : A \cup B \to A \cup B$ is said to be a *cyclic* mapping if $T(A) \subseteq B, T(B) \subseteq A$. A point $p \in A \cup B$ is called a best proximity point of the cyclic mapping T provided that $d(p, Tp) = \text{dist}(A, B)$, where $\text{dist}(A, B) := \inf\{d(x, y) : (x, y) \in A \times B\}$.

We finish this section by recalling the notion of uniformly convexity of Banach spaces.

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Definition 2.2. A Banach space X is said to be uniformly convex if there exists a strictly increasing function $\delta : [0,2] \to [0,1]$ such that the following implication holds for all $x, y, p \in X, R > 0$ and $r \in [0, 2R]$:

$$\begin{cases} \|x-p\| \le R, \\ \|y-p\| \le R, \\ \|x-y\| \ge r \end{cases} \Rightarrow \|\frac{x+y}{2} - p\| \le (1 - \delta(\frac{r}{R}))R.$$

It is well known that Hilbert spaces and l^p spaces $(1 < p < \infty)$ are uniformly convex Banach spaces.

3. Monotone orbitally nonexpansive mappings

Let C be a nonempty subset of a normed linear space X. A mapping $T : C \to C$ is said to be asymptotically nonexpansive if

$$||T^n x - T^n y|| \le k_n ||x - y|| \quad \text{for all } x, y \in C,$$

where $\{k_n\}$ is a sequence of real numbers such that $\lim_{n\to\infty} k_n = 1$ (see [9]). Very recently, the class of asymptotically nonexpansive mappings was extended to monotone asymptotic nonexpansive mappings as follows.

Definition 3.1. (see [14]) Let C be a nonempty subset of a normed linear space X equipped with a partially ordered relation. A mapping $T : C \to C$ is said to be monotone asymptotically nonexpansive if there exists a sequence of real numbers $\{k_n\}$ such that $\lim_{n\to\infty} k_n = 1$ and $||T^n x - T^n y|| \le k_n ||x - y||$, for every comparable elements $x, y \in C$.

At the same time an interesting generalization of nonexpansive mappings, called orbitally nonexpansive mappings, was introduced in [12].

Definition 3.2. A mapping $T : C \to C$ is said to be orbitally nonexpansive if for every nonempty, closed, convex, T-invariant subset $D \subset C$, there exists some $x_0 \in D$ such that $\limsup_{n\to\infty} ||T^n x_0 - Tx|| \le \limsup_{n\to\infty} ||T^n x_0 - x||$ for all $x \in D$.

Motivated by Definition 3.1, we can generalize the class of monotone orbitally nonexpansive mappings as follows.

Definition 3.3. A mapping $T : C \to C$ is said to be monotone orbitally nonexpansive if for every nonempty, closed, convex, T-invariant subset $D \subset C$, there exists some $x_0 \in D$ with monotone increasing (resp. decreasing) orbit $\{T^n x_0\}$ in D such that $\limsup_{n\to\infty} ||T^n x_0 - Tx|| \le \limsup_{n\to\infty} ||T^n x_0 - x||$ for all $x \in D$ such that $T^n x_0 \preceq x$ (resp. $x \preceq T^n x_0$) for all $n \in \mathbb{N}$.

Next example shows that the class of monotone orbitally nonexpansive mappings cannot be concluded from orbitally nonexpansive mappings.

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EXAMPLE 3.4. Let $X = \mathbb{R}$ be endowed with the usual metric and with the natural partially ordered relation "$\leq$". Suppose $A = [-1, 1]$ and define the self-mapping $T : A \to A$ by

$$Tx = \begin{cases} -x & \text{if } x \in (\mathbb{Q}^+ \cap A) \cup \{0\}, \\ \sqrt{-x} & \text{if } x \in \mathbb{Q}^- \cap A, \\ 0 & \text{if } x \in \mathbb{Q}^c \cap A, \end{cases}$$

where $\mathbb{Q}^+$ and $\mathbb{Q}^-$ denote the sets of positive and negative rational numbers, respectively. Note that T is not continuous. Assume that D is a closed, convex and T-invariant subset of A. Take $D = [a, b]$, where $a, b \in A$ and $a < b$. In view of the fact that D is T-invariant, we must have $D = [-1, 1]$. Put $x_0 := 0$. Thus for any $x \in D$ with $T^n x_0 \leq x$, we have $x \geq 0$. Now, if $x \in \mathbb{Q}^c \cap A$ then $Tx = 0$, and if $x \in \mathbb{Q}^+ \cap A$ then $Tx = -x$. In both cases, we obtain

$$\limsup_{n\to\infty} |T^n x_0 - Tx| \leq x = \limsup_{n\to\infty} |T^n x_0 - x|,$$

that is, T is monotone orbitally nonexpansive. We claim that T is not an orbitally nonexpansive mapping. To this end, we consider the following cases:
Case 1. If $x_0 \in (\mathbb{Q}^+ \cap D) \cup \{0\}$, then $Tx_0 = -x_0, T^2 x_0 = \sqrt{x_0}$ and $T^n x_0 = 0$ for all $n \geq 3$. Now for $x \in \mathbb{Q}^- \cap D$ with $x \neq -1$ we have

$$\limsup_{n\to\infty} |T^n x_0 - Tx| = \sqrt{-x} > -x = \limsup_{n\to\infty} |T^n x_0 - x|. \tag{3.1}$$

Case 2. If $x_0 \in \mathbb{Q}^- \cap D$, then $Tx_0 = \sqrt{-x_0}$ and $T^n x_0 = 0$ for all $n \geq 2$. For $x \in \mathbb{Q}^- \cap D$ with $x \neq -1$ the result follows from (3.1).
Case 3. If $x_0 \in \mathbb{Q}^c \cap D$, then $T^n x_0 = 0$ for all $n \in \mathbb{N}$. Again, if $x \in \mathbb{Q}^- \cap D$ with $x \neq -1$, then the conclusion follows from (3.1).

The following technical lemma will be useful to prove our main result.

Lemma 3.5. *(Lemma 3.2 of* [14]*) Let C be a nonempty closed convex subset of an uniformly convex Banach space $(X, ||\cdot||)$. Let $\tau : C \to [0, +\infty)$ be a type function, i.e., there exists a bounded sequence $\{x_n\} \in X$ such that $\tau(x) = \limsup_{n\to\infty} ||x_n - x||$, for every $x \in C$. Then τ has a unique minimum point $z \in C$ such that $\tau(z) = \inf\{\tau(x); x \in C\} = \tau_0$.*

We now state the first main result of this section.

Theorem 3.6. *Let $(X, ||\cdot||, \preceq)$ be a partially ordered Banach space for which order intervals are convex and closed. Assume that $(X, ||\cdot||)$ is uniformly convex. Let C be a nonempty convex, closed, bounded subset of X not reduced to one point. Let $T : C \to C$ be a monotone orbitally nonexpansive mapping. Then T has a fixed point iff there exists $x_0 \in C$ such that x_0 and Tx_0 are comparable.*

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Proof. Obviously if x is a fixed point of T, then x and $Tx = x$ are comparable. Let $x_0 \in C$ be such that x_0 and Tx_0 are comparable. Without loss of generality, assume that $x_0 \preceq Tx_0$. Since T is monotone, then we have $T^n x_0 \preceq T^{n+1}x_0$, for all $n \in \mathbb{N}$. In other words, the orbit $\{T^n x_0\}$ is monotone increasing. Since the order intervals are closed and convex and X is reflexive, we conclude that

$$C_\infty = \bigcap_{n\geq 0} \{x \in C; T^n x_0 \preceq x\} \neq \emptyset.$$

Consider $x \in C_\infty$. Then, $T^n x_0 \preceq x$ and since T is monotone, we get

$$T^n x_0 \preceq T(T^n x_0) = T^{n+1}x_0 \preceq Tx, \text{ for every } n \geq 0, \text{ i.e., } T(C_\infty) \subset C_\infty.$$

Consider the type function $\tau : C_\infty \to [0, +\infty)$ generated by $\{T^n x_0\}$, i.e.

$$\tau(x) = \limsup_{n\to\infty} ||T^n x_0 - x||.$$

Lemma 4.5 implies the existence of a unique $z \in C_\infty$ such that

$$\tau(z) = \inf\{\tau(x); x \in C_\infty\} = \tau_0.$$

Since $z \in C_\infty$, we have $Tz \in C_\infty$ which implies that

$$\tau(Tz) = \limsup_{n\to\infty} ||T^n x_0 - Tz|| \leq \limsup_{n\to\infty} ||T^n x_0 - z|| = \tau_0$$

and so $Tz = z$, i.e. z is a fixed point of T. □

Here, we recall a relevant geometric property of Banach spaces.

A Banach space $(X, ||\cdot||)$ is said to have normal structure if each nonempty, bounded, closed and convex subset C of X with $diam(C) := sup\{||x - y|| : x, y \in C\} > 0$ contains a point $y \in C$ such that

$$r_C(y) := \sup\{||y - x|| : x \in C\} < diam(C).$$

It is well known that every uniformly convex Banach space has normal structure, but the converse is not true (see [5] for more details).

Theorem 3.7. *Let C be a nonempty, weakly compact, convex subset of a partially ordered Banach space $(X, ||\cdot||, \preceq)$ with normal structure. Let $T : C \to C$ be a monotone orbitally nonexpansive mapping. Then T has a fixed point iff there exists $x_0 \in C$ such that x_0 and Tx_0 are comparable.*

Proof. If T has a fixed point x_0. It is clear that x_0 and $Tx_0 = x_0$ are comparable. Conversely, since C is a weakly compact set, from a standard application of Zorn's lemma, there is a nonempty, closed, convex, T-invariant subset D of C with no proper subsets joining these characteristics. From the definition of monotone orbitally nonexpansive mapping, there exists $x_0 \in D$ with monotone increasing (resp. decreasing) orbit $\{T^n x_0\}$ in D such that

$$\limsup_{n\to\infty} ||T^n x_0 - Tx|| \leq \limsup_{n\to\infty} ||T^n x_0 - x||$$

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for all $x \in D$ such that $T^n x_0 \preceq x$ (resp. $x \preceq T^n x_0$) for all $n \in \mathbb{N}$. Without loss of generality, we can assume that $\{T^n x_0\}$ is increasing. Note that $orb(T, x_0)$ is bounded and not (eventually) constant. Since the Banach space $(X, ||\cdot||)$ has normal structure, the real function $g : C \to [0, \infty)$ defined by

$$g(x) := \limsup_{n\to\infty} ||x - T^n x_0||$$

is not constant on $conv\{T^n x_0 : n = 1, 2, ...\}$, see [6]. Then g takes at least two different real values, that is, there exist $\nu_1, \nu_2 \in conv\{T^n x_0 : n = 1, 2, ...\} \subset C$ such that $r_1 := g(\nu_1) < g(\nu_2) =: r_2$. Take $r := \frac{1}{2}(r_1 + r_2)$ and consider the set $M := \{x \in C : g(x) \leq r\}$. Because of $\nu_1 \in M$ and $\nu_2 \notin M$ thus $\emptyset \neq M \subset C$. On the other hand, if $x_1, x_2 \in M$ then

$$\begin{aligned} g(\lambda x_1 + (1-\lambda)x_2) &= \limsup_{n\to\infty} ||T^n x_0 - (\lambda x_1 + (1-\lambda)x_2)|| \\ &= \limsup_{n\to\infty} ||\lambda(T^n x_0 - x_1) + (1-\lambda)(T^n x_0 - x_2)|| \\ &\leq \lambda \limsup_{n\to\infty} ||T^n x_0 - x_1|| + (1-\lambda) \limsup_{n\to\infty} ||T^n x_0 - x_2|| \\ &\leq \lambda g(x_1) + (1-\lambda) g(x_2). \end{aligned}$$

Moreover, for every $x \in M$, since T is a monotone orbitally nonexpansive mapping, we have

$$g(Tx) = \limsup_{n\to\infty} ||T^n x_0 - Tx|| \leq \limsup_{n\to\infty} ||T^n x_0 - x|| = g(x) \leq r.$$

Thus, M is a nonempty, closed, convex and T-invariant proper subset of C, a contradiction to the minimality of C. Thus there exists $z \in C$ such that $T^n x_0 = z$ for large enough n. Note that

$$||Tz - z|| = \limsup_{n\to\infty} ||T^n x_0 - Tz|| \leq \limsup_{n\to\infty} ||T^n x_0 - z|| = 0.$$

Therefore $Tz = z$, and hence T has a fixed point in C. □

In what follows, we present some sufficient conditions to obtain the class of monotone orbitally nonexpansive mappings. To this end, we need the following concepts.

Definition 3.8. Let C be a nonempty, closed and convex subset of a Banach space X. A mapping $T : C \to C$ is said to be asymptotocally regular at $x_0 \in C$ provided that $\lim_{n\to\infty} ||T^n x_0 - T^{n+1} x_0|| = 0$.

Definition 3.9. $T : C \to X$ satisfies the monotone E_μ-condition for some $\mu \geq 1$, on C, if T is monotone and $||x - Ty|| \leq \mu||x - Tx|| + ||x - y||$ for all $x, y \in C$ such that x and y are comparable. Indeed, T satisfies the monotone E-condition on C if T satisfies the monotone E_μ-condition on C for some $\mu \geq 1$.

Theorem 3.10. *Let C be a nonempty, closed and convex subset of an ordered Banach space $(X, ||\cdot||, \preceq)$ and $T : C \to C$ satisfies the monotone E-condition on C. If T is asymptotically regular on C and there is at least one element in*

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every nonempty, closed, convex and T-invariant subset of C comparable with its image under T, then T is monotone orbitally nonexpansive.

Proof. Let D be a nonempty, closed, convex and T-invariant subset of C. By assumption there is $x_0 \in D$ comparable with Tx_0. Without loss of generality, take $x_0 \preceq Tx_0$ so that $orb(T, x_0)$ is increasing. Now let $x \in D$ and $T^n x_0 \preceq x$ for all $n \in \mathbb{N}$. Since T satisfies the monotone E-condition, it satisfies the monotone E_μ-condition for some $\mu \geq 1$ and we have

$$||T^n x_0 - Tx|| \leq \mu ||T^n x_0 - T^{n+1} x_0|| + ||T^n x_0 - x||,$$

for all $n \in \mathbb{N}$. Hence

$$\limsup_{n\to\infty} ||T^n x_0 - Tx|| \leq \mu \limsup_{n\to\infty} ||T^n x_0 - T^{n+1} x_0|| + \limsup_{n\to\infty} ||T^n x_0 - x||.$$

The asymptotical regularity of T implies that

$$\limsup_{n\to\infty} ||T^n x_0 - Tx|| \leq \limsup_{n\to\infty} ||T^n x_0 - x||.$$

Therefore T is orbitally nonexpansive. □

It is remarkable to note that the inverse of Theorem 3.10 does not hold, necessarily.

Example 3.11. Consider $X = [0, 1]$ with the usual ordered relation to " $\leq$ ". Let $T : [\frac{1}{2}, 1] \to [\frac{1}{2}, 1]$ be given by $Tx = \sqrt{x}$. Clearly T is monotone and each closed, convex and T-invariant subset of $[0, 1]$ is just of the form $C = [a, 1]$ for some $\frac{1}{2} \leq a \leq 1$. Now if $x_0 \in [a, 1]$, then $T^n x_0 = x_0^{\frac{1}{2^n}} \to 1$ as $n \to \infty$. Thus the only element comparable with the orbit of x_0 is $x = 1$ and so

$$\limsup_{n\to\infty} |T^n x_0 - Tx| = 0 = \limsup_{n\to\infty} |T^n x_0 - x|,$$

that is, T is an orbitally nonexpansive mapping. We now show that T does not satisfy the monotone E-condition. Let $x_0 = 1$ and $y \in (\frac{1}{2}, 1)$ be an arbitrary element. Then

$$\sqrt{y} = |x_0 - Ty| > n|x_0 - Tx_0| + |x_0 - y| = 1 - y$$

for all $n \geq 1$.

Next corollary is a straightforward consequence of Theorems 3.10 and 4.7.

Corollary 3.12. *Let $(X, ||\cdot||, \preceq)$ be a partially ordered uniformly convex Banach space for which order intervals are convex and closed. Let C be a nonempty convex, closed, bounded subset of X not reduced to one point. If T is asymptotically regular and satisfies the monotone E-condition on C, then T has a fixed point iff there exists $x_0 \in C$ such that x_0 and Tx_0 are comparable.*

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4. T-cyclic contractions

In this section, we study some sufficient conditions to ensure the existence of common best proximity points for two cyclic mappings. We begin our discussion by introducing the following notions.

Definition 4.1. Let (A, B) be a nonempty pair in a metric space (X, d) and let "$\preceq$" be a partially ordered relation on A. The mappings $S, T : A \to B$ are said to be order preserving provided that $Sx = Ty \Rightarrow x \preceq y$, for $x, y \in A$.

Let us illustrate this notion with the following examples.

Example 4.2. Consider $X = \mathbb{R}$ with the usual metric and the natural relation "$\leq$". Assume that $A = B = [0, \infty)$ and define the mappings $S, T : A \to B$ with $Sx = x^3$ and $Ty = y^2$. Then S, T are order preserving. Indeed, if $Sx = Ty$, then $x^3 = y^2$ which implies that $x = \sqrt[3]{y^2} \leq y$.

Example 4.3. Consider the Banach space $X = \mathcal{C}([0, 1])$ of all complex-valued continuous functions defined on $[0, 1]$. Consider the partially ordered relation "$\preceq$" on X as $f \preceq g \Leftrightarrow f_1 \leq g_1$, $f_2 \leq g_2$, where $f = f_1 + if_2$ and $g = g_1 + ig_2$. Suppose

$$A = \{f = f_1 + if_2 \in X : f(0) = 0\}, \ B = \{g = g_1 + ig_2 \in X : g_1 \geq 0, g(0) = 1\}.$$

Define $S, T : A \to B$ with $Sf = (f_1 + 1) + if_2$ and $Tg = 1 + ig_2$. Now if $Sf = Tg$, then we must have $f_1 \equiv 0$ and $f_2 = g_2$ which implies that $f \preceq g$, that is, S, T are order preserving.

Definition 4.4. Let (A, B) be a nonempty pair in a metric space (X, d) and let "$\preceq$" be a partially ordered relation on X. Let $S, T : A \cup B \to A \cup B$ be two cyclic mappings. The mapping S is said to be a T-cyclic contraction provided that

$$d(S\acute{x}, S^2x) \leq \alpha d(T\acute{x}, T^2x) + (1 - \alpha)\text{dist}(A, B), \tag{4.1}$$

for some $\alpha \in (0, 1)$ and for all $(x, \acute{x}) \in A^2 \cup B^2$ with $x \preceq \acute{x}$.

Next lemma plays an important role in our results of this section.

Lemma 4.5. *Let (A, B) be a nonempty pair in a metric space (X, d) and let "$\preceq$" be a partially ordered relation on X. Assume $S, T : A \cup B \to A \cup B$ are two cyclic mappings such that S, T are commuting and order preserving on X and $S(A) \subseteq T(A) \subseteq B, S(B) \subseteq T(B) \subseteq A$ and let S be a T-cyclic contraction mapping. Then there exists an increasing sequence $\{x_n\} \in A$ for which the following implications hold:*

(a) $d(Sx_{2n}, S^2x_{2n}) \leq \alpha^n d(Sx_n, S^2x_0) + (1 - \alpha^n)\text{dist}(A, B)$;
(b) $d(Sx_{2n+1}, S^2x_{2n}) \leq \alpha^n d(Sx_{n+1}, S^2x_0) + (1 - \alpha^n)\text{dist}(A, B)$;
(c) $d(Sx_{2n-1}, S^2x_{2n-1}) \leq \alpha^{n-1} d(Sx_n, S^2x_1) + (1 - \alpha^{n-1})\text{dist}(A, B)$;
(d) $d(Sx_{2n}, S^2x_{2n-1}) \leq \alpha^{n-1} d(Sx_{n+1}, S^2x_1) + (1 - \alpha^{n-1})\text{dist}(A, B)$.

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Proof. Choose $x_0 \in A$. Since $S(A) \subseteq T(A)$, there exists an element $x_1 \in A$ such that $Sx_0 = Tx_1$. By the fact that S, T are order preserving, we have $x_0 \preceq x_1$. Again since $S(A) \subseteq T(A)$, there exists an element $x_2 \in A$ such that $Sx_1 = Tx_2$ and so $x_1 \preceq x_2$. Continuing this process we obtain a sequence $\{x_n\}$ in A such that $Sx_n = Tx_{n+1}$ and $x_0 \preceq x_1 \preceq \cdots \preceq x_n \preceq \cdots$. By the contractive condition in (4.1) we conclude that

$$\begin{aligned}
d(Sx_{2n}, S^2x_{2n}) &\leq \alpha d(Tx_{2n}, T^2x_{2n}) + (1-\alpha)\text{dist}(A,B) \quad (x_{2n} \preceq x_{2n}) \\
&= \alpha d(Sx_{2n-1}, T(Sx_{2n-1})) + (1-\alpha)\text{dist}(A,B) \\
&= \alpha d(Sx_{2n-1}, S(Tx_{2n-1})) + (1-\alpha)\text{dist}(A,B) \\
&\qquad\qquad\qquad\qquad (S, T \text{ are commuting}) \\
&= \alpha d(Sx_{2n-1}, S^2x_{2n-2}) + (1-\alpha)\text{dist}(A,B) \\
&\leq \alpha[\alpha d(Tx_{2n-1}, T^2x_{2n-2}) + (1-\alpha)\text{dist}(A,B)] \\
&\qquad\qquad + (1-\alpha)\text{dist}(A,B)(x_{2n-2} \preceq x_{2n-1}) \\
&= \alpha^2 d(Tx_{2n-1}, T^2x_{2n-2}) + (1-\alpha^2)\text{dist}(A,B) \\
&= \alpha^2 d(Sx_{2n-2}, S^2x_{2n-4}) + (1-\alpha^2)\text{dist}(A,B) \\
&\leq \cdots \leq \alpha^n d(Sx_n, S^2x_0) + (1-\alpha^n)\text{dist}(A,B),
\end{aligned}$$

which implies that (a) is satisfied. Similarly, we can see that the relations (b), (c) and (d) hold true. □

Remark 4.6. Under the assumptions of Lemma 4.5 if moreover B is bounded, then

$$d(Sx_{2n}, S^2x_{2n}) \to \text{dist}(A,B), \quad d(Sx_{2n-1}, S^2x_{2n-1}) \to \text{dist}(A,B),$$

$$d(Sx_{2n+1}, S^2x_{2n}) \to \text{dist}(A,B), \quad d(Sx_{2n}, S^2x_{2n-1}) \to \text{dist}(A,B).$$

Proof. It follows from (a) that

$$\begin{aligned}
d(Sx_{2n}, S^2x_{2n}) &\leq \alpha^n d(Sx_n, S^2x_0) + (1-\alpha^n)\text{dist}(A,B) \\
&\leq \alpha^n[d(Sx_n, Sx_0) + d(Sx_0, S^2x_0)] + (1-\alpha^n)\text{dist}(A,B) \\
&\leq \alpha^n[\text{diam}(B) + d(Sx_0, S^2x_0)] + (1-\alpha^n)\text{dist}(A,B) \\
&\to \text{dist}(A,B).
\end{aligned}$$

Equivalently, the other implications hold. □

Here, we establish the following fixed point theorem for two cyclic mappings.

Theorem 4.7. *Let (A,B) be a nonempty pair in a metric space (X,d) such that B is bounded and complete and let "$\preceq$" be a partially ordered relation on A. Assume $S, T : A \cup B \to A \cup B$ are two cyclic mappings such that S, T are*

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commuting and order preserving on A *and* $S(A) \subseteq T(A) \subseteq B, S(B) \subseteq T(B) \subseteq A$. *Suppose*

$$d(S\acute{x}, S^2x) \leq \alpha d(T\acute{x}, T^2x), \tag{4.2}$$

for some $\alpha \in (0,1)$ *and for all* $x, \acute{x} \in A$ *with* $x \preceq \acute{x}$ *and let* $T|_B$ *be continuous. Then* $A \cap B \neq \emptyset$, $\mathrm{Fix}(T)$ *is nonempty and* $S(\mathrm{Fix}(T)) \subseteq \mathrm{Fix}(S)$. *Furthermore, if* $S|_B$ *is continuous, then* $\mathrm{Fix}(T) \cap \mathrm{Fix}(S) \neq \emptyset$.

Proof. Consider the increasing sequence $\{x_n\}$ in A as in the proof of Lemma 4.5. We claim that $\{Sx_n\}$ is a Cauchy sequence in B. From the relations (a) and (b) of Lemma 4.5 we have

$$\begin{aligned} d(Sx_{2n}, Sx_{2n+1}) &\leq d(Sx_{2n}, S^2x_{2n}) + d(Sx_{2n+1}, S^2x_{2n}) \\ &\leq \alpha^n d(Sx_n, S^2x_0) + \alpha^n d(Sx_{n+1}, S^2x_0) \\ &\leq 2\alpha^n[\mathrm{diam}(B) + d(Sx_0, S^2x_0)], \end{aligned}$$

which leads to $\sum_{n=0}^{\infty} d(Sx_{2n}, Sx_{2n+1}) < \infty$. Similarly, by using the relations (c) and (d) we can see that $\sum_{n=1}^{\infty} d(Sx_{2n-1}, Sx_{2n}) < \infty$. Thereby,

$$\sum_{n=0}^{\infty} d(Sx_n, Sx_{n+1}) < \infty,$$

that is, $\{Sx_n\}$ is Cauchy. In view of the fact that B is complete, $Sx_n \to p \in B$ and so, $Tx_n \to p$. Since $T|_B$ is continuous, $T^2x_n \to Tp$ and $STx_n = TSx_n \to Tp$. We now have

$$d(Sx_n, STx_n) = d(Sx_n, S^2x_{n-1}) \leq \alpha d(Tx_n, T^2x_{n-1}) \quad (x_{n-1} \preceq x_n).$$

Letting $n \to \infty$ in the above relation, we obtain $d(p, Tp) \leq \alpha d(p, Tp)$ which implies that $p = Tp$. Besides, $d(Sp, S^2p) \leq \alpha d(Tp, T^2p)$ and so $Sp = S^2p$. Thus $Sp \in \mathrm{Fix}(S)$ which implies $S(\mathrm{Fix}(T)) \subseteq \mathrm{Fix}(S)$. Now assume that S is continuous on B. Then $S^2x_n \to Sp$ and

$$S^2x_{n-1} = STx_n = TSx_n \to Tp = p.$$

Thereby, $Sp = p$ and hence $p \in \mathrm{Fix}(T) \cap \mathrm{Fix}(S) \neq \emptyset$. □

Example 4.8. Consider $X = \mathbb{R}^2$ with the metric

$$d((x_1, y_1), (x_2, y_2)) = \max\{|x_1 - x_2|, |y_1 - y_2|\},$$

where $(x_i, y_i) \in \mathbb{R}^2$ for $i = 1, 2$. Let

$$A = \{(x, 0) : 0 \leq x \leq 1\}, \quad B = \{(0, y) : 0 \leq y \leq 1\}.$$

We define $S, T : A \cup B \to A \cup B$ by

$$S(x, 0) = (0, \frac{x}{3}), \quad S(0, y) = (\frac{y}{3}, 0), \quad T(x, 0) = (0, \frac{x}{2}), \quad T(0, y) = (\frac{y}{2}, 0).$$

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Clearly, $S(A) \subseteq T(A) \subseteq B, S(B) \subseteq T(B) \subseteq A$ and S, T are commuting. Define the partial order " $\preceq$ " on A in the following way:

$$(x, 0) \preceq (\acute{x}, 0) \Leftrightarrow x \leq \acute{x}.$$

Then S, T are order preserving on A. Moreover, if $(x, 0) \preceq (x', 0)$, then

$$\begin{aligned} d(S(x',0), S^2(x,0)) &= d((0, \frac{x'}{3}), (\frac{x}{9}, 0)) = \max\{\frac{x}{9}, \frac{x'}{3}\} = \frac{x'}{3} \\ &\leq \alpha \frac{x'}{2} = \alpha \max\{\frac{x}{4}, \frac{x'}{2}\} = \alpha d((0, \frac{x'}{2}), (\frac{x}{4}, 0)) \\ &= \alpha d(T(x',0), T^2(x,0)), \end{aligned}$$

where $\alpha \in [\frac{2}{3}, 1)$. Hence, all of the conditions of Theorem 4.7 hold and S, T have a common fixed point which is $p = (0, 0) \in A \cap B$.

Next lemmas will be used in the main result of this section.

Lemma 4.9. *(Lemma 3.7 of [7]) Let (A, B) be a nonempty and closed pair in a uniformly convex Banach space X such that A is convex. Let $\{x_n\}$ and $\{y_n\}$ be sequences in A and B, respectively, such that either of the following holds:*

$$\lim_{m\to\infty} \sup_{n\geq m} \|x_m - y_n\| = \text{dist}(A, B) \text{ or } \lim_{n\to\infty} \sup_{m\geq n} \|x_m - y_n\| = \text{dist}(A, B).$$

Then $\{x_n\}$ is a Cauchy sequence.

Lemma 4.10. *(Lemma 3.8 of [7]) Let (A, B) be a nonempty and closed pair in a uniformly convex Banach space X such that A is convex. Let $\{x_n\}$ and $\{z_n\}$ be sequences in A and $\{y_n\}$ be a sequence in B, satisfying $\|x_n - y_n\| \to \text{dist}(A, B)$, $\|z_n - y_n\| \to \text{dist}(A, B)$. Then $\|x_n - z_n\| \to 0$.*

We are now ready to state the main result of this section.

Theorem 4.11. *Let (A, B) be nonempty, closed and convex pair in a uniformly convex Banach space X such that B is bounded, and let "$\preceq$" be a partially ordered relation on $A \cup B$. Let $S, T : A \cup B \to A \cup B$ be two cyclic mappings, such that S, T are commuting and order preserving on A, $S(A) \subseteq T(A)$, $S(B) \subseteq T(B)$, and let S be a T-cyclic contraction mapping. If T is continuous, $S|_A$ is monotone then there exists an element $q \in B$ such that $q \in \text{BPP}(T)$, $Sq \in \text{BPP}(S)$. If in addition $S|_B$ is continuous, then $q \in \text{BPP}(T) \bigcap \text{BPP}(S)$.*

Proof. Consider the increasing sequence $\{x_n\}$ as in the proof of Lemma 4.5. Thus

$$Sx_n = Tx_{n+1},\ \|Sx_n - S^2x_n\| \to \text{dist}(A, B),\ \|Sx_{n+1} - S^2x_n\| \to \text{dist}(A, B).$$

It follows from Lemma 4.10 that $\|Sx_n - Sx_{n+1}\| \to 0$. Again

$$\|Sx_{n+1} - S^2x_n\| \to \text{dist}(A, B)$$

and

$$\|Sx_{n+1} - S^2x_{n+1}\| \to \text{dist}(A, B),\ \|S^2x_n - S^2x_{n+1}\| \to 0.$$

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We claim that for every $\varepsilon > 0$ there exists $N_0 \in \mathbb{N}$ such that, for all $m > n \geq N_0$, we have $\|Sx_m - S^2x_n\|^* < \varepsilon$, where $\|a-b\|^* := \|a-b\| - \text{dist}(A,B)$ for any $(a,b) \in A \times B$. Suppose the contrary. Then there exists $\varepsilon > 0$ such that for all $k \in \mathbb{N}$ there exist $m_k > n_k \geq k$ for which

$$\|Sx_{m_k} - S^2x_{n_k}\|^* \geq \varepsilon, \quad \|Sx_{m_k-1} - S^2x_{n_k}\|^* < \varepsilon.$$

Thus

$$\varepsilon \leq \|Sx_{m_k} - S^2x_{n_k}\|^* \leq \|Sx_{m_k} - Sx_{m_k-1}\| + \|Sx_{m_k-1} - S^2x_{n_k}\|^*,$$

which implies that $\lim_{k\to\infty} \|Sx_{m_k} - S^2x_{n_k}\|^* = \varepsilon$. We now have

$$\begin{aligned}
&\|Sx_{m_k} - S^2x_{n_k}\|^* \\
&\leq \|Sx_{m_k} - Sx_{m_k+1}\| + \|Sx_{m_k+1} - S^2x_{n_k+1}\|^* + \|S^2x_{n_k+1} - S^2x_{n_k}\| \\
&\leq \|Sx_{m_k} - Sx_{m_k+1}\| + \alpha\|Tx_{m_k+1} - T^2x_{n_k+1}\|^* + \|S^2x_{n_k+1} - S^2x_{n_k}\| \\
&\qquad\qquad (x_{n_k+1} \preceq x_{m_k+1}) \\
&= \|Sx_{m_k} - Sx_{m_k+1}\| + \alpha\|Sx_{m_k} - S^2x_{n_k-1}\|^* + \|S^2x_{n_k+1} - S^2x_{n_k}\| \\
&\leq \|Sx_{m_k} - Sx_{m_k+1}\| + \alpha^2\|Tx_{m_k} - T^2x_{n_k-1}\|^* + \|S^2x_{n_k+1} - S^2x_{n_k}\| \\
&\qquad\qquad (x_{n_k-1} \preceq x_{m_k}) \\
&= \|Sx_{m_k} - Sx_{m_k+1}\| + \alpha^2\|Sx_{m_k-1} - S^2x_{n_k-3}\|^* + \|S^2x_{n_k+1} - S^2x_{n_k}\| \\
&\leq \|Sx_{m_k} - Sx_{m_k+1}\| + \alpha^2[\|Sx_{m_k-1} - S^2x_{n_k}\|^* + \|S^2x_{n_k} - S^2x_{n_k-3}\|] \\
&+ \|S^2x_{n_k+1} - S^2x_{n_k}\| \\
&\leq \|Sx_{m_k} - Sx_{m_k+1}\| + \alpha^2[\|Sx_{m_k-1} - S^2x_{n_k}\|^* + \|S^2x_{n_k} - S^2x_{n_k-1}\| \\
&+ \|S^2x_{n_k-1} - S^2x_{n_k-2}\| + \|S^2x_{n_k-2} - S^2x_{n_k-3}\|] + \|S^2x_{n_k+1} - S^2x_{n_k}\|.
\end{aligned}$$

Letting $k \to \infty$ we obtain $\varepsilon \leq \alpha^2\varepsilon$ which is a contradiction and so the relation (4.2) holds. Thereby

$$\lim_{m\to\infty} \sup_{n\geq m} \|Sx_m - S^2x_n\| = \text{dist}(A,B).$$

It now follows from Lemma 4.9 that $\{Sx_n\}$ is a Cauchy sequence in B. Since B is complete, there exists an element $q \in B$ such that $Sx_n \to q$ and $Tx_n \to q$. By the fact that S, T are commuting and T is continuous, we get

$$STx_n = TSx_n \to Tq, \quad T^2x_n \to Tq.$$

Thus $\|q - Tq\| = \lim_{n\to\infty} \|Sx_n - STx_n\| = \lim_{n\to\infty} \|Sx_n - S^2x_{n-1}\| = \text{dist}(A,B)$, that is, $q \in \text{BPP}(T)$. On the other hand, we have

$$\begin{aligned}
S^3x_{n-2} &= S(S(Sx_{n-2})) = S(S(Tx_{n-1})) \\
&= S(T(Sx_{n-1})) = S(T(Tx_n)) = T^2(Sx_n) \to T^2q.
\end{aligned}$$

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Similarly, $S^2x_n = S(Sx_n) = S(Tx_{n+1}) = T(Sx_{n+1}) \to Tq$. In view of the fact that $S|_A$ is monotone, we have

$$\begin{aligned}\|Tq - T^2q\| &= \lim_{n\to\infty} \|S^2x_n - S^3x_{n-2}\| = \|S(Sx_n) - S^2(Sx_{n-2})\| \\ &\le \alpha\|T(Sx_n) - T^2(Sx_{n-2})\| + (1-\alpha)\mathrm{dist}(A,B) \quad (Sx_{n-2} \preceq Sx_n).\end{aligned}$$

Letting $n \to \infty$, we conclude that $\|Tq - T^2q\| = \mathrm{dist}(A,B)$. We now have

$$\|Sq - S^2q\| \le \alpha\|Tq - T^2q\| + (1-\alpha)\mathrm{dist}(A,B) = \mathrm{dist}(A,B),$$

and so $Sq \in \mathrm{BPP}(S)$. Note that if $S|_B$ is continuous, then

$$\|q - Sq\| = \lim_{n\to\infty} \|Sx_n - S^2x_n\| = \mathrm{dist}(A,B),$$

and so $q \in \mathrm{BPP}(T) \bigcap \mathrm{BPP}(S)$. □

Example 4.12. Consider the Hilbert space $X = l^2$ with the canonical basis $\{e_n\}$ and let

$$A = \{te_1 + e_2 : 0 \le t \le 1\}, \quad B = \{t'e_1 + 2e_2 : 0 \le t' \le 1\}.$$

Then $\mathrm{dist}(A,B) = 1$. Define $S, T : A \cup B \to A \cup B$ by

$$S(te_1 + e_2) = \frac{t}{3}e_1 + 2e_2, \quad S(t'e_1 + 2e_2) = \frac{t'}{3}e_1 + e_2,$$

$$T(te_1 + e_2) = \frac{t}{2} + 2e_2, \quad T(t'e_1 + 2e_2) = \frac{t'}{2}e_1 + e_2.$$

Clearly, $S(A) \subseteq T(A) \subseteq B, S(B) \subseteq T(B) \subseteq A$ and S, T are commuting. Define the partial order " $\preceq$ " on $A \cup B$ as follows: if $x := te_1 + ke_2 \in A \cup B$ and $y := t'e_1 + k'e_2 \in A \cup B$, where $t, t' \in [0,1]$ and $k, k' \in \{1,2\}$, then $x \preceq y \Leftrightarrow t \le t',\ k \le k'$. It is easy to see that S, T are order preserving on A, and S is a T-cyclic contraction. It now follows from Theorem 4.11 that $\mathrm{BPP}(T) \bigcap \mathrm{BPP}(S) \neq \emptyset$.

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