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Approximating Fixed Points of Operators Satisfying the $(B_{\gamma,\mu})$ Condition

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Abstract. Suppose C is any nonempty subset of a Banach space X. A mapping $T : C \to C$ is said to satisfy condition $(B_{\gamma,\mu})$ if there exists $\gamma \in [0,1]$ and $\mu \in [0, \frac{1}{2}]$ with $2\mu \leq \gamma$ such that for each two elements $x, y \in C$,

$$\gamma||x - Tx|| \leq ||x - y|| + \mu||y - Ty||$$

$$\text{implies } ||Tx - Ty|| \leq (1 - \gamma)||x - y|| + \mu(||x - Ty|| + ||y - Tx||).$$

In this research, we suggest some convergence results for these mappings under a up-to-date iterative process in a Banach space setting. Our results are new and improve some known results of the literature.

Keywords: Condition $(B_{\gamma,\mu})$, Condition (I), Convergence result, K^* iteration, Banach space.

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1. Introduction and preliminaries

Suppose we have a subset C of a Banach space and T possibly a selfmap of C. Then the selfmap T will be called a nonexpansive on C (or simply nonexpansive) if one has the following

$$||Tx - Ty|| \leq ||x - y||,$$

for any two elements $x, y \in C$. While a fixed point of T is some point, namely, p in the domain C such that it satisfies the relation $p = Tp$. As usual, we shall write $F(T)$, to denote the set of all such fixed points of T. In 1965, Browder [5], Gohde [9] ann Kirk [14] were the first who proved a basic existence result for nonexpansive mappings on a Banach space setting. Since fixed point theory about nonexpansive mappings have crucial applications in fixed point problems related to applied sciences. Thus it is very natural to consider some generalizations of these mappings. In 2008, Suzuki [30] suggested a weaker notion of these mappings: the selfmap T is said to satisfy a (C) condition if

$$\frac{1}{2}||x - Tx|| \leq ||x - y|| \;\Rightarrow\; ||Tx - Ty|| \leq ||x - y||,$$

for any two elements $x, y \in C$. Suzuki [30] first proved that the class of mappings with (C) condition contains properly the class of nonexpansive mappings. Moreover, he proved that the Browder-Gohde-Kirk result is still valid for mappings with (C) condition.

Inspired by Suzuki [30], Patir et al. [20] suggested a two parametric condition for mappings: the selfmap T is said to satisfy a condition $(B_{\gamma,\mu})$ if one can find a $\gamma \in [0, 1]$ and some $\mu \in [0, \frac{1}{2}]$ with $2\mu \leq \gamma$ such that

$$\gamma||x - Tx|| \leq ||x - y|| + \mu||y - Ty||$$
$$\text{implies } ||Tx - Ty|| \leq (1 - \gamma)||x - y|| + \mu(||x - Ty|| + ||y - Tx||),$$

for any two elements $x, y \in C$. Patir et al. [20] obtained same conclusions for these mappings as Suzuki [30]. They suggested the following example of mappings satisfying a condition $(B_{\gamma,\mu})$ that does not satisfy the condition (C) of Suzuki.

Example 1.1. [20] Define a mapping $T : [0, 2] \to \mathbb{R}$ by

$$T(x) = \begin{cases} 0 & \text{if } x \neq 2 \\ 1 & \text{if } x = 2. \end{cases}$$

It is easy to see that T satisfies $(B_{\gamma,\mu})$ condition, but not the (C) condition.

Patir et al. [20] proved the existence of fixed point for mappings with the condition $B_{\gamma,\mu}$ on a Banach space setting. However, once the existence of fixed point for a certain mappings is established then an iterative scheme to approximate such fixed points is always desirable (see, e.g., [8, 17, 16, 27, 28, 29] and others). Among the other things, iterative schemes for nonexpansive and

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mappings with (C) condition are widely studied (see, e.g., Mann [15], Ishikawa [11], S [3], Noor [18], Abbas [1], SP [21], S* [12], CR [6], Normal-S [22], Picard-Mann hybrid [13], Picard-S [10], Thakur et al. [32], M iteration of Ullah and Arshad [35] and so on). We present some of these iterations here.

The iteration process of Mann [15] is defined by the following formula:

$$\begin{cases} x_1 \in C, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n, n \in \mathbb{N}, \end{cases} \tag{1.1}$$

where $\alpha_n \in (0,1)$.

The iteration process of Ishikawa [11] is defined by the following formula:

$$\begin{cases} x_1 \in C, \\ y_n = (1-\beta_n)x_n + \beta_n T x_n, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n, n \in \mathbb{N}, \end{cases} \tag{1.2}$$

where $\alpha_n, \beta_n \in (0,1)$.

The iteration process of Agarwal et al. [3] (also called S-iteration) is defined by the following formula:

$$\begin{cases} x_1 \in C, \\ y_n = (1-\beta_n)x_n + \beta_n T x_n, \\ x_{n+1} = (1-\alpha_n)T x_n + \alpha_n T y_n, n \in \mathbb{N}, \end{cases} \tag{1.3}$$

where $\alpha_n, \beta_n \in (0,1)$.

It is known that the iteration process (1.3) is better than the Picard iteration $x_{n+1} = Tx_n$, Mann iteration (1.1) and Ishikawa iteration (1.2) under some restrictions for nonexpansive mappings and mappings with condition (C).

The iteration process of Gursoy and Karakaya [10] (also called Picard-S iteration) is defined by the following formula:

$$\begin{cases} x_1 \in C, \\ z_n = (1-\beta_n)x_n + \alpha_n T x_n, \\ y_n = (1-\alpha_n)T x_n + \alpha_n T z_n, \\ x_{n+1} = T y_n, n \in \mathbb{N}, \end{cases} \tag{1.4}$$

where $\alpha_n, \beta_n \in (0,1)$.

It is proved by the authors in [10] that the iteration (1.4) is essentially better than the Picard, Mann, Ishikawa, Noor, SP, CR, S, S*, Abbas, and Normal-S iterative processes.

The iteration process of Thakur et al. [32] is defined by the following formula:

$$\begin{cases} x_1 \in C, \\ z_n = (1-\beta_n)x_n + \beta_n T x_n, \\ y_n = T\left((1-\alpha_n)x_n + \alpha_n z_n\right), \\ x_{n+1} = T y_n, n \in \mathbb{N}, \end{cases} \tag{1.5}$$

where $\alpha_n, \beta_n \in (0,1)$.

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Using a numerical example, the authors [32] noted that the iteration (1.5) is still very effective than the Picard, Mann, Ishikawa, Agarwal, Noor and Abbas iterative processes for mappings with (C) condition. But it is known that the iteration (1.4) and (1.5) suggest same speed of convergence almost for all classes of mappings.

The iteration process M of Ullah and Arshad [35] is defined by the following formula:

$$\begin{cases} x_1 \in C, \\ z_n = (1-\alpha_n)x_n + \alpha_n T x_n, \\ y_n = T z_n, \\ x_{n+1} = T y_n, n \in \mathbb{N}, \end{cases} \tag{1.6}$$

where $\alpha_n \in (0,1)$.

Ullah and Arshad [35] noted that the iteration process (1.6) is more effective than all of the above mentioned iterative process in the setting of mappings with (C) condition.

Inspired by above, Ullah and Arshad [34] suggested a new iteration called K^* iteration that is defined by the following formula:

$$\begin{cases} x_1 \in C, \\ z_n = (1-\beta_n)x_n + \beta_n T x_n, \\ y_n = T((1-\alpha_n)z_n + \alpha_n T z_n), \\ x_{n+1} = T y_n, n \in \mathbb{N}, \end{cases} \tag{1.7}$$

where $\alpha_n, \beta_n \in (0,1)$.

They proved that K^* iteration is more effective than many other iterations in the setting of mappings with condition (C). In [33], Ullah and Ahmad used M iteration (1.6) to approximate fixed point of a mapping with $(B_{\gamma,\mu})$ condition. The purpose of this paper is to prove some fixed point convergence results for a mapping with $(B_{\gamma,\mu})$ condition, using the K^* iteration process (1.7). Our results improve and extend some main results of Ullah and Arshad [34] and Ullah and Ahmad [33].

Now we collect some concepts which are needed in the sequel.

Definition 1.2. [7] Suppose X is a Banach space. Then X is called uniformly convex if and only if for all $\xi \in (0,2]$, some real number $\nu > 0$ exists such that if $x, y \in X$ any elements with $||x|| \leq 1, ||y|| \leq 1, ||x-y|| > \xi$ then $||\frac{x+y}{2}|| \leq (1-\nu)$.

Definition 1.3. [2, 31] If C denotes any bounded closed convex subset of a uniformly convex Banach space X, $\{x_n\}$ and x are in X. If $r(x, \{x_n\}) := \limsup_{n\to\infty} ||x - x_n||$, then we can define the sets $r(C, \{x_n\}) = \inf\{r(x, \{x_n\}) : x \in C\}$ and $A(C, \{x_n\}) = \{x \in C : r(x, \{x_n\}) = r(C, \{x_n\})\}$. The set $A(C, \{x_n\})$ consists of exactly one point. In this case, the set $A(C, \{x_n\})$ is singleton.

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Definition 1.4. [19] Suppose X is a Banach space. Then X is called a Banach space with Opial's property provided that every $\{x_n\} \subseteq X$ whenever converges weakly to some point w of X, one has

$$\limsup_{n\to\infty} ||x_n - w|| < \limsup_{n\to\infty} ||x_n - s||,$$

for all $s \in X - \{w\}$. The known examples of Banach spaces with Opial's property are Hilbert spaces and l^p spaces $(1 < p < \infty)$.

Definition 1.5. [24] A selfmap T of a subset C of a Banach space is said to satisfy the condition (I) in the case when there is a function μ such $\mu(0) = 0$ and $\mu(s) > 0$ for any point $s > 0$ and also $||x - Tx|| \geq \mu(d(x, F(T)))$ for each point $x \in C$.

Definition 1.6. A sequence $\{x_n\}$ in X is called Fejer-monotone with respect to C if

$$||x_{n+1} - c|| \leq ||x_n - c||$$

for each $c \in C$ and $n \in \mathbb{N}$.

Lemma 1.7. [20] *Let C be a nonempty subset of a Banach space X having Opial property and $T : C \to C$ satisfies $(B_{\gamma,\mu})$ condition. If q is a fixed point of $T : C \to C$, then for each $x \in C$*

$$||q - Tx|| \leq ||q - x||.$$

Theorem 1.8. [20] *Let C be a nonempty subset of a Banach space X having Opial property. Let $T : C \to C$ satisfy condition $(B_{\gamma,\mu})$. If $\{x_n\}$ is sequence in C such that*

(i) *$\{x_n\}$ converges weakly to s,*
(ii) $\lim_{n\to\infty} ||Tx_n - x_n|| = 0$,
then $Ts = s$.

Proposition 1.9. [20] *Let C be a nonempty subset of a Banach space X. If $T : C \to C$ satisfies the $B_{\gamma,\mu}$ condition on C. Then, for all $x, y \in C$ and $c \in [0, 1]$,*

(i) $||Tx - T^2x|| \leq ||x - y||$,
(ii) *at least one of the following ((a) and (b)) holds:*
(a) $\frac{c}{2}||x - Tx|| \leq ||x - y||$
(b) $\frac{c}{2}||Tx - T^2x|| \leq ||Tx - Ty||$.
The condition (a) implies $||Tx - Ty|| \leq (1 - \frac{c}{2})||x - y|| + \mu(||x - Ty|| + ||y - Tx||)$ and condition (b) implies $||T^2x - Ty|| \leq (1 - \frac{c}{2})||Tx - y|| + \mu(||Tx - Ty|| + ||y - T^2x||)$.
(iii) $||x - Ty|| \leq (3 - c)||x - Tx|| + \left(1 - \frac{c}{2}\right) ||x - y|| + \mu(2||x - Tx|| + ||x - Ty|| + ||y - Tx|| + 2||Tx - T^2x||)$.

The following facts can be found in [4].

Proposition 1.10. *Suppose C is any nonempty closed subset of a Banach space and $\{x_n\}$ any Fejer-monotone sequence in the set C. Then $\{x_n\}$ converges to the point of C in the strong sense if and only if* $\lim_{n\to\infty} d(x_n, C) = 0$.

The following lemma is an important properly of unifromly convex Banach space that can be found in [23].

Lemma 1.11. *Let $\theta_n \in [r, v] \in (0, 1)$ and consider any two sequences, namely, $\{x_n\}$ and $\{y_n\}$ in a uniformly convex Banach space X with* $\limsup_{n\to\infty} ||x_n|| \leq e$, $\limsup_{n\to\infty} ||y_n|| \leq e$*. If one has* $\lim_{n\to\infty} ||\theta_n x_n + (1-\theta_n)y_n|| = e$ *for some real constant $e \geq 0$, then the equation* $\lim_{n\to\infty} ||x_n - y_n|| = 0$ *holds.*

2. Main Results

The following elementry lemma is essential to prove our main outcome.

Lemma 2.1. *Let C be a nonempty closed convex subset of a Banach space X and $T : C \to C$ satisfies the $(B_{\gamma,\mu})$ condition with $F(T) \neq \emptyset$. If $\{x_n\}$ is a sequence generated by (1.7), then* $\lim_{n\to\infty} ||x_n - q||$ *exists for each $q \in F(T)$.*

Proof. To establish the proof, we select any point, namely, $q \in F(T)$. Hence applying Lemma 1.7, one has

$$\begin{aligned}
||x_{n+1} - q|| &= ||Ty_n - q|| \leq ||y_n - q|| \\
&\leq ||T((1-\alpha_n)z_n + \alpha_n T z_n) - q|| \\
&\leq ||(1-\alpha_n)z_n + \alpha_n T z_n - q|| \\
&\leq (1-\alpha_n)||z_n - q|| + \alpha_n ||Tz_n - q|| \\
&\leq (1-\alpha_n)||z_n - q|| + \alpha_n ||z_n - q|| \\
&= ||z_n - q|| \\
&= ||(1-\beta_n)x_n + \beta_n T x_n - q|| \\
&\leq (1-\beta_n)||x_n - q|| + \beta_n ||Tx_n - q|| \\
&\leq (1-\beta_n)||x_n - q|| + \beta_n ||x_n - q|| \\
&\leq ||x_n - q||.
\end{aligned}$$

Subsequently, we obtained $||x_{n+1} - q|| \leq ||x_n - q||$ and hence it follows that $\{||x_n - q||\}$ is bounded and nonincreasing. Thus, we conclude that for all $q \in F(T)$, $\lim_{n\to\infty} ||x_n - q||$ exists. □

We also need the following result.

Theorem 2.2. *Let C be a nonempty closed convex subset of a uniformly convex Banach space X and $T : C \to C$ a mapping satisfying the $(B_{\gamma,\mu})$ condition. If $\{x_n\}$ is a sequence generated by (1.7). Then, $F(T) \neq \emptyset$ if and only if $\{x_n\}$ is bounded and* $\lim_{n\to\infty} ||Tx_n - x_n|| = 0$.

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Proof. Suppose that $F(T) \neq \emptyset$ and $q \in F(T)$. Then, by Lemma 2.1, $\lim_{n\to\infty} ||x_n - q||$ exists and $\{x_n\}$ is bounded. Put

$$\lim_{n\to\infty} ||x_n - q|| = c. \tag{2.1}$$

By the proof of Lemma 2.1 together with (2.1), we have

$$\limsup_{n\to\infty} ||z_n - q|| \leq \limsup_{n\to\infty} ||x_n - q|| = c. \tag{2.2}$$

By Lemma 1.7, we have

$$\limsup_{n\to\infty} ||Tx_n - q|| \leq \limsup_{n\to\infty} ||x_n - q|| = c. \tag{2.3}$$

Again by the proof of Lemma 2.1, together with (2.1), we have

$$c = \liminf_{n\to\infty} ||x_{n+1} - q|| \leq \liminf_{n\to\infty} ||z_n - q||. \tag{2.4}$$

Accordingly from the (2.2) and (2.4), one has

$$c = \lim_{n\to\infty} ||z_n - q||. \tag{2.5}$$

Also, from the (2.5), one has

$$c = \lim_{n\to\infty} ||z_n - q|| = \lim_{n\to\infty} ||(1-\beta_n)(x_n - q) + \beta_n(Tx_n - q)||.$$

Hence,

$$c = \lim_{n\to\infty} ||(1-\beta_n)(x_n - q) + \beta_n(Tx_n - q)||. \tag{2.6}$$

Now keeping (2.1), (2.3) and (2.6) in mind and so applying Lemma 1.11, one has

$$\lim_{n\to\infty} ||Tx_n - x_n|| = 0.$$

Conversely, let $q \in A(C, \{x_n\})$. Now applying the Proposition 1.9 (iii), for $\gamma = \frac{c}{2}$, $c \in [0,1]$,

$$\begin{aligned} ||x_n - Tq|| &\leq (3-c)||x_n - Tx_n|| + \left(\frac{1-c}{2}\right)||x_n - q|| + \mu(2||x_n - Tx_n|| \\ &+ ||x_n - Tq|| + ||q - Tx_n|| + 2||Tx_n - T^2x_n||) \\ &\leq (3-c)||x_n - Tx_n|| + \left(\frac{1-c}{2}\right)||x_n - q|| + \mu(2||x_n - Tx_n|| \\ &+ ||x_n - Tq|| + ||x_n - q|| + ||x_n - Tx_n|| + 2||x_n - Tx_n||) \end{aligned}$$

(by Proposition 1.9 (ii))

$$\Rightarrow (1-\mu)\limsup_{n\to\infty} ||x_n - Tq|| \leq (1 - \frac{c}{2} + \mu)\limsup_{n\to\infty} ||x_n - q||$$

$$\begin{aligned} \Rightarrow \limsup_{n\to\infty} ||x_n - Tq|| &\leq \left(\frac{1 - \frac{c}{2} + \mu}{1-\mu}\right)\limsup_{n\to\infty} ||x_n - q|| \\ &\leq \limsup_{n\to\infty} ||x_n - q|| \end{aligned}$$

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$$\left(\text{as } \frac{1-\frac{c}{2}+\mu}{1-\mu} \leq 1, \text{ for } 2\mu \leq \gamma = \frac{c}{2}\right)$$
$$\Rightarrow\ r\left(Tq, \{x_n\}\right) \leq r\left(q, \{x_n\}\right).$$

So $Tq \in A(C, \{x_n\})$. As the set $A(C, \{x_n\})$ has only one element, it follows that $Tq = q$. □

Now we are essentially in the position to establish our desired convergence results.

Theorem 2.3. *Let C a nonempty closed convex subset of a uniformly convex Banach space X having Opial property. If $T : C \to C$ satisfies the $(B_{\gamma,\mu})$ condition with $F(T) \neq \emptyset$. Then $\{x_n\}$ generated by (1.7) converges weakly to an element of $F(T)$.*

Proof. Since X is uniformly convex so it must be reflexive. Now according to Theorem 2.2, $\{x_n\}$ is bounded. Thus it has a weakly convergent subsequence which we may denote by $\{x_{n_i}\}$ of $\{x_n\}$ to some point $p_1 \in C$. In the view of Theorem 2.2, and $\lim_{i\to\infty} ||Tx_{n_i} - x_{n_i}|| = 0$. Hence applying Theorem 1.8, we obtain $p_1 \in F(T)$. We claim that p_1 is being the only weak limit of $\{x_n\}$. If one assumes that this claim is not valid then he must a subsequence which we may denote by $\{x_{n_j}\}$ of $\{x_n\}$ such that it will converge weakly to a point $p_2 \in C$ and $p_2 \neq p_1$. Same as above, it follows that, $p_2 \in F(T)$. By Lemma 2.1 and also using Opial property of the space, we have

$$\begin{aligned}
\lim_{n\to\infty} ||x_n - p_1|| &= \lim_{i\to\infty} ||x_{n_i} - p_1|| \\
&< \lim_{i\to\infty} ||x_{n_i} - p_2|| \\
&= \lim_{n\to\infty} ||x_n - p_2|| \\
&= \lim_{j\to\infty} ||x_{n_j} - p_2|| \\
&< \lim_{j\to\infty} ||x_{n_j} - p_1|| \\
&= \lim_{n\to\infty} ||x_n - p_1||.
\end{aligned}$$

Subsequently, we obtained $\lim_{n\to\infty} ||x_n - p_1|| < \lim_{n\to\infty} ||x_n - p_1||$ which is clearly a contradiction. This completed the required proof. □

Theorem 2.4. *Let C be a nonempty closed and convex subset of a uniformly convex Banach space X and $T : C \to C$ satisfies the $B_{\gamma,\mu}$ condition with $F(T) \neq \emptyset$ and $q \in F(T)$. If $\{x_n\}$ is a sequence generated by (1.7). Then $\{x_n\}$ converges to an element of $F(T)$ if and only if $\liminf_{n\to\infty} d(x_n, F(T)) = 0$ or $\limsup_{n\to\infty} d(x_n, F(T)) = 0$.*

Proof. The necessity part is obvious and hence omitted.

Conversely, we want to prove that $\{x_n\}$ is convergent in $F(T)$ whenever $\liminf_{n\to\infty} d(x_n, F(T)) = 0$. Let $q \in F(T)$ be any point. According to Lemma

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2.1, $\lim_{n\to\infty} ||x_n - q||$ exists. Hence it follows that $\lim_{n\to\infty} d(x_n, F(T)) = 0$. We prove that $\{x_n\}$ is a Cauchy sequence in C. As $\lim_{n\to\infty} d(x_n, F(T)) = 0$, for a given $\varepsilon > 0$, there exists $k_0 \in \mathbb{N}$ such that for each $n \geq k_0$,

$$d(x_n, F(T)) < \frac{\varepsilon}{2}.$$

$$\Rightarrow \ \inf\{||x_n - q|| : q \in F(T)\} < \frac{\varepsilon}{2}.$$

In particular $\inf\{||x_{k_0} - q|| : q \in F(T)\} < \frac{\varepsilon}{2}$. Therefore there exists $q \in F(T)$ such that

$$||x_{k_0} - q|| < \frac{\varepsilon}{2}.$$

Now for $k, n \geq k_0$,

$$\begin{aligned} ||x_{n+k} - x_n|| &\leq ||x_{n+k} - q|| + ||x_n - q|| \\ &\leq ||x_{k_0} - q|| + ||x_{k_0} - q|| \\ &= 2||x_{k_0} - q|| < \varepsilon. \end{aligned}$$

This shows that $\{x_n\}$ is a Cauchy sequence in C. As C is closed subset of a Banach space X, so there exists a point $p \in C$ such that $\lim_{n\to\infty} x_n = p$. Now $\lim_{n\to\infty} d(x_n, F(T)) = 0$ gives that $d(p, F(T)) = 0$. This shows that $p \in F(T)$. □

We now prove the following theorem using condition (I).

Theorem 2.5. *Let C be a nonempty closed and convex subset of a uniformly convex Banach space X and $T : C \to C$ satisfies the $(B_{\gamma,\mu})$ condition with $F(T) \neq \emptyset$. If $\{x_n\}$ is a sequence generated by (1.7). Then $\{x_n\}$ converges strongly to an element of $F(T)$ provided that T satisfies the condition (I).*

Proof. Since T satisfies the condition (I), we have $\lim_{n\to\infty} d(x_n, F(T)) = 0$. We prove that $F(T)$ is closed. Let $\{q_n\}$ be any sequence in $F(T)$ converges to some $q \in C$. Since $\gamma||q_n - Tq_n|| = 0 \leq ||q_n - q|| + \mu||q - Tq||$, by $(B_{\gamma,\mu})$ condition, we have

$$\begin{aligned} ||q_n - Tq|| &= ||Tq_n - Tq|| \\ &\leq (1-\gamma)||q_n - q|| + \mu(||q_n - Tq|| + ||q - Tq_n||) \\ &= (1-\gamma)||q_n - q|| + \mu||q_n - Tq|| + \mu||q - q_n||. \end{aligned}$$

It follows that

$$||q_n - Tq|| \leq \left(\frac{1-\gamma+\mu}{1-\mu}\right) ||q_n - q|| \leq ||q_n - q|| \text{ (as } 2\mu \leq \gamma).$$

Therefore, $q_n \to Tq$. This implies $Tq = q$ and so $q \in F(T)$. Hence $F(T)$ is closed. In the view of Lemma 2.1, $\{x_n\}$ is Fejer-monotone with respect to $F(T)$. By Proposition 1.10, $\{x_n\}$ converges strongly to an element of $F(T)$. □

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3. Example

For numerical interpretation of our results, we first construct an example of mapping which satisfies $(B_{\gamma,\mu})$ condition but not the condition (C). We then use this example to compare the quality of K^* iteration process with the leading M, Picard-S and S iterations.

Example 3.1. Consider $C = [5,7]$ be endowed with absolute valued norm. Define a mapping $T : C \to C$ by

$$Tx = \begin{cases} \frac{5+x}{2} & \text{if } x \neq 7 \\ 5 & \text{if } x = 7. \end{cases}$$

It is easy to see that T does not satisfy the condition (C). Choose $\gamma = 1$ and $\mu = \frac{1}{2}$, we prove that T satisfies the $(B_{1,\frac{1}{2}})$ condition.

Case I: For $x, y \in [5,7)$, we have

$$\begin{aligned} (1-\gamma)|x-y| + \mu(|x-Ty| + |y-Tx|) &= \frac{1}{2}\left(|x-Ty| + |y-Tx|\right) \\ &= \frac{1}{2}\left(\left|x - \left(\frac{5+y}{2}\right)\right| + \left|y - \left(\frac{5+x}{2}\right)\right|\right) \\ &\geq \frac{1}{2}\left|\frac{3x}{2} - \frac{3y}{2}\right| \\ &= \frac{3}{4}|x-y| \\ &\geq \frac{1}{2}|x-y| \\ &= |Tx - Ty|. \end{aligned}$$

Case II: For $x \in [5,7)$ and $y = 7$, we have

$$\begin{aligned} (1-\gamma)|x-y| + \mu(|x-Ty| + |y-Tx|) &= \frac{1}{2}(|x-Ty| + |y-Tx|) \\ &= \frac{1}{2}\left(|x-5| + \left|y - \left(\frac{5+x}{2}\right)\right|\right) \\ &= \frac{1}{2}|x-5| + \frac{1}{2}\left|y - \left(\frac{5+x}{2}\right)\right| \\ &\geq \frac{1}{2}|x-5| \\ &= |Tx - Ty|. \end{aligned}$$

Case III: For $x = y = 7$, we have

$$(1-\gamma)|x-y| + \mu(|x-Ty| + |y-Tx|) \geq 0 = |Tx - Ty|.$$

Hence, T satisfies the $(B_{1,\frac{1}{2}})$ condition. Note that $F(T) = \{5\}$.

Take $\alpha_n = 0.70$ and $\beta_n = 0.50$. The iterative values for $x_1 = 5.9$ are given in Table 1 and Figure 1 shows the convergence graph. Clearly the K^* iteration process converges faster to the fixed point of T in comparison with other iteration processes.

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n	K^*	M	$Picard - S$	S
1	5.9000000000	5.9000000000	5.9000000000	5.9000000000
2	5.12656250000	5.1462500000	5.1856250000	5.3712000000
3	5.0177978516	5.0237656250	5.0382851563	5.1531406250
4	5.0025028229	5.0038619141	5.0078963135	5.0631705078
5	5.0003519595	5.0006275610	5.0016286147	5.0260578345
6	5.0000494943	5.0001019787	5.0003359018	5.0107488567
7	5.0000069601	5.0000165715	5.0000692797	5.0044339034
8	5.0000009788	5.0000026929	5.0000142889	5.0018289852
9	5.0000001376	5.0000004376	5.0000029471	5.0007544564
10	5.0000000194	5.0000000711	5.0000006078	5.0003112133
11	5.0000000027	5.0000000116	5.0000001254	5.0001283755
12	5.0000000004	5.0000000019	5.0000000259	5.0000529549
13	**5.000000000**	5.0000000003	5.0000000053	5.0000218439
14	5.0000000000	**5.000000000**	5.0000000002	5.0000090106
15	5.0000000000	5.0000000000	**5.000000000**	5.0000037169
16	5.0000000000	5.0000000000	5.0000000000	5.0000015332
17	5.0000000000	5.0000000000	5.0000000000	5.0000006324
18	5.0000000000	5.0000000000	5.0000000000	5.0000002609

Table 1. Comparison of K^* iteration with some other leading iterations.

4. Conclusion and Future Plan

We proved a weak convergence and also some strong convergence results for mappings with $(B_{\gamma,\mu})$ condition under the K^* iteration process. These results are the extension of the previous results of Ullah and Arshad [34] from the setting of mappings with (C) condition to the setting of $(B_{\gamma,\mu})$ condition. We proved that in the setting of mappings with $(B_{\gamma,\mu})$ condition, the K^* iteration process is more effective under certain assumptions than the M, Picard-S and S iterative processes. Thus, our results improve the main results of Ullah and Ahmad [33] from the setting of M iteration to the general setting of K^* itration

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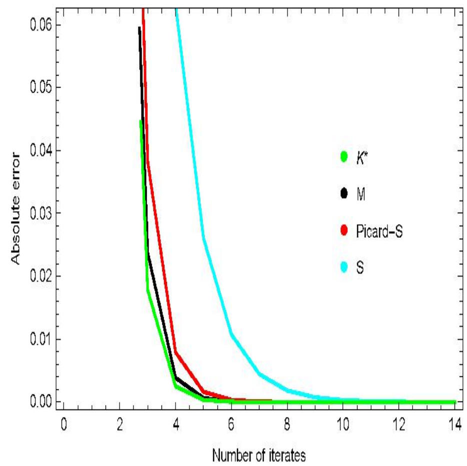

FIGURE 1. Convergence behaviors of K^*, M, Picard-S and S iterations towards the fixed point 5 of the mapping T.

process. The future plan of the authors is to prove the results of this paper in the setting of common fixed points.

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