

## $(C, C')$ -Controlled g-Fusion Frames in Hilbert Spaces

Habib Shakoory<sup>a</sup>, Reza Ahmadi<sup>b\*</sup>, Gholamreza Rahimlou<sup>c</sup>, Vahid Sadri<sup>c</sup>

<sup>a</sup>Department of Mathematics, Shabestar Branch, Islamic Azad University  
Shabestar, Iran

<sup>b</sup>Research Institute for Fundamental Sciences, University of Tabriz, Tabriz,  
Iran

<sup>c</sup>Department of Mathematics, Faculty of Tabriz Branch, Technical and  
Vocational University (TVU), East Azarbaijan, Iran

E-mail: [habibshakoory@yahoo.com](mailto:habibshakoory@yahoo.com)

E-mail: [rahmadi@tabrizu.ac.ir](mailto:rahmadi@tabrizu.ac.ir)

E-mail: [grahimlou@gmail.com](mailto:grahimlou@gmail.com)

E-mail: [vahidsadri57@gmail.com](mailto:vahidsadri57@gmail.com)

ABSTRACT. Controlled frames in Hilbert spaces have been recently introduced by P. Balazs and etc. for improving the numerical efficiency of interactive algorithms for inverting the frame operator. In this paper we develop a theory based on g-fusion frames on Hilbert spaces, which provides exactly the frameworks not only to model new frames on Hilbert spaces but also for deriving robust operators. In particular, we can define analysis, synthesis and frame operators with representation space compatible for  $(C, C')$ -Controlled g-fusion frames, which even yield a reconstruction formula. Also, some useful concepts such as Q-dual and perturbation are introduced and investigated.

**Keywords:** G-fusion frame, Controlled fusion frame, Controlled g-fusion frame, Q-dual.

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\*Corresponding Author

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## 1. INTRODUCTION

Frames, as a generalization of the bases in Hilbert spaces, were first introduced by Duffin and Schaeffer [5] during their study of nonharmonic Fourier series in 1952. Controlled frames for spherical wavelets were introduced in [2] for obtaining a numerically more efficient approximation algorithm and the related theory for general frames were developed in [1]. Also, Controlled frames as generalization of frames, have been introduced for getting an improved solution of a linear system of the equation  $Ax = B$ , which can be solved by the equation  $PAx = PB$ , where  $P$  is a suitable matrix for getting a better duplicate algorithm [2]. Recent developments in this direction can be found in [8, 9, 10, 11, 12].

Throughout this paper  $H$  and  $K$  are separable Hilbert spaces,  $\{H_j\}_{j \in \mathbb{I}}$  is a sequence of Hilbert spaces and  $I \subseteq \mathbb{Z}$ . We denote by  $\mathcal{B}(H, K)$  the set of all the bounded and linear operators from  $H$  to  $K$ . If  $H = K$ , then  $\mathcal{B}(H, H)$  will be denoted as  $\mathcal{B}(H)$ . Also,  $GL(H)$  is called the set of all bounded linear operators which have bounded inverses on  $H$ . It is easy to check that if  $C, C' \in GL(H)$ , then  $C^*, C^{-1}$  and  $CC'$  are in  $GL(H)$ . Assume that  $Id_H$  is the identity operator on  $H$  and  $\pi_W$  is the orthogonal projection from  $H$  onto a closed subspace  $V \subseteq H$ .

## 2. PRELIMINARIES

In this section, some necessary definitions and lemmas are introduced.

**Definition 2.1.** A sequence  $\{f_i\}_{i \in \mathbb{I}}$  in  $H$  is a frame if there exist constants  $0 < A \leq B < \infty$  such that for all  $f \in H$

$$A \|f\|^2 \leq \sum_{i \in \mathbb{I}} |\langle f, f_i \rangle|^2 \leq B \|f\|^2.$$

The constants  $A, B$  are frame bounds;  $A$  is the lower bound and  $B$  is the upper bound. The frame is tight if  $A = B$ , it is called a Parseval frame if  $A = B = 1$ . If we only have the upper bound, We call  $\{f_i\}_{i \in \mathbb{I}}$  a Bessel sequence. If  $\{f_i\}_{i \in \mathbb{I}}$  is a Bessel sequence then the following operators are bounded:

$$T : l^2(I) \mapsto H,$$

$$T(c_i) = \sum_{i \in \mathbb{I}} c_i f_i$$

$$T^* : H \mapsto l^2(I),$$

$$T^* f = \{\langle f, f_i \rangle\}_{i \in \mathbb{I}}$$

$$S : H \mapsto H,$$

$$Sf = TT^* f = \sum_{i \in \mathbb{I}} \langle f, f_i \rangle f_i.$$

These operators are called synthesis operator; analysis operator and frame operator, respectively. The representation space employed in this setting is

$$l^2(I) = \left\{ \{c_i\}_{i \in \mathbb{I}} : c_i \in \mathbb{C}, \sum_{i \in \mathbb{I}} \|c_i\|^2 < \infty \right\}.$$

**Definition 2.2.** Let  $W := \{W_i\}_{i \in \mathbb{I}}$  be a family of closed subspaces of  $H$ ,  $\{v_i\}_{i \in \mathbb{I}}$  be a family of weights i.e.  $v_i > 0$  for all  $i \in \mathbb{I}$  and  $\Lambda_i \in \mathcal{B}(H, H_i)$ . We say  $\Lambda := (W_i, \Lambda_i, v_i)$  is a g-fusion frame for  $H$  if there exist constants  $0 < A \leq B < \infty$  such that for all  $f \in H$

$$A \|f\|^2 \leq \sum_{i \in \mathbb{I}} v_i^2 \|\Lambda_i \pi_{W_i} f\|^2 \leq B \|f\|^2.$$

We call  $\Lambda$  a Parseval g-fusion frame if  $A = B = 1$ . When the right hand of above inequality holds,  $\Lambda$  is called a g-fusion Bessel sequence for  $H$  with bound  $B$ . We define the space

$$\mathcal{H}_2 := \left( \sum_{j \in \mathbb{J}} \bigoplus H_j \right)_{l_2}$$

by:

$$\mathcal{H}_2 = \left\{ \{f_j\}_{j \in \mathbb{J}} : f_j \in H_j, \sum_{j \in \mathbb{J}} \|f_j\|^2 < \infty \right\}.$$

with the inner product defined by

$$\langle \{f_j\}, \{g_j\} \rangle = \sum_{j \in \mathbb{J}} \langle f_j, g_j \rangle.$$

It is clear that  $\mathcal{H}_2$  is a Hilbert space with pointwise operations. Suppose that  $\Lambda$  be a g-fusion frame for  $H$ . Then The synthesis and analysis operator are denoted by (for more details refer to [13])

$$\begin{aligned} T_{\Lambda'} : \mathcal{H}_2 &\longrightarrow H, \\ T_{\Lambda'}(\{f_j\}_{j \in \mathbb{J}}) &= \sum_{j \in \mathbb{J}} v_j \pi_{W_j} \Lambda_j^* f_j \end{aligned}$$

and

$$\begin{aligned} T_{\Lambda'}^* : H &\longrightarrow \mathcal{H}_2, \\ T_{\Lambda'}^* f &: \{v_j \Lambda_j \pi_{W_j} f\}_{j \in \mathbb{J}}. \end{aligned}$$

Now, the g-fusion frame operator is defined by

$$\begin{aligned} S_{\Lambda'} : H &\longrightarrow H, \\ S_{\Lambda'} f &= \sum_{j \in \mathbb{J}} v_j^2 \pi_{W_j} \Lambda_j^* \Lambda_j \pi_{W_j} f. \end{aligned}$$

and

$$\langle S_{\Lambda'} f, f \rangle = \sum_{j \in \mathbb{J}} v_j^2 \|\Lambda_j \pi_{W_j} f\|^2.$$

Therefore

$$AI \leq S_{\Lambda'} \leq BI.$$

This means that  $S_{\Lambda'}$  is a bounded, positive and invertible operator. So, we have the reconstruction formula for any  $f \in H$ :

$$f = \sum_{j \in \mathbb{J}} v_j^2 \pi_{W_j} \Lambda_j^* \Lambda_j \pi_{W_j} S_{\Lambda'}^{-1} f = \sum_{j \in \mathbb{J}} v_j^2 S_{\Lambda'}^{-1} \pi_{W_j} \Lambda_j^* \Lambda_j \pi_{W_j} f.$$

**Lemma 2.3.** ([13])  $\Lambda$  is a  $g$ -fusion frame for  $H$  if and only if the operator

$$S_{\Lambda'} : f \longrightarrow \sum_{j \in \mathbb{J}} v_j^2 \pi_{W_j} \Lambda_j^* \Lambda_j \pi_{W_j} f$$

is a well-defined, bounded and surjective.

**Lemma 2.4.** ([1]) Let  $u : H \rightarrow H$  be a linear operator. Then the following statements are equivalent:

- (1) for some  $0 < A \leq B < \infty$  we get  $AI \leq u \leq BI$ , ;
- (2)  $u$  is positive and  $A \|f\|^2 \leq \|u^{\frac{1}{2}} f\|^2 \leq B \|f\|^2$ , for some  $0 < A \leq B < \infty$ ;
- (3)  $u \in GL^+(H)$ .

**Definition 2.5.** A sequence  $\Lambda = \{\Lambda_i \in \mathcal{B}(H, H_i) : i \in \mathbb{I}\}$  is called generalized frame, or simply a  $g$ -frame, for  $H$  with respect to  $\{H_i : i \in \mathbb{I}\}$  if there exist constants  $A > 0$  and  $B < \infty$  such that for all  $f \in H$

$$A \|f\|^2 \leq \sum_{i \in \mathbb{I}} \|\Lambda_i f\|^2 \leq B \|f\|^2.$$

The numbers  $A$  and  $B$  are called  $g$ -frame bounds.  $\Lambda = \{\Lambda_i \in \mathcal{B}(H, H_i) : i \in \mathbb{I}\}$  is called tight  $g$ -frame if  $A = B$  and Parseval  $g$ -frame if  $A = B = 1$ . If the second part of the above inequality holds, the sequence is called  $g$ -Bessel sequence.

**Definition 2.6.** Let  $C, C' \in GL^+(H)$ . The family  $\Lambda = \{\Lambda_i \in \mathcal{B}(H, H_i) : i \in \mathbb{I}\}$  will be called a  $(C, C')$ -controlled  $g$ -frame for  $H$ , if  $\Lambda$  is a  $g$ -Bessel sequence and there exist constants  $A > 0$  and  $B < \infty$  such that for all  $f \in H$

$$A \|f\|^2 \leq \sum_{i \in \mathbb{I}} \langle \Lambda_i C' f, \Lambda_i C f \rangle \leq B \|f\|^2.$$

$A$  and  $B$  will be called controlled frame bounds. If the second part of the above inequality holds, it will be called  $(C, C')$ -controlled  $g$ -Bessel sequence with bound  $B$ .

3. (C, C′)-CONTROLLED G-FUSION FRAMES

**Definition 3.1.** Let  $W := \{W_i\}_{i \in \mathbb{I}}$  be a family of closed subspaces of  $H$  and  $\{v_i\}_{i \in \mathbb{I}}$  be a family of weights i.e.  $v_i > 0$  for all  $i \in \mathbb{I}$ . Let  $\{H_i\}_{i \in \mathbb{I}}$  be a sequence of Hilbert spaces,  $C, C' \in GL(H)$  and  $\Lambda_i \in \mathcal{B}(H, H_i)$ .  $\Lambda_{CC'} := (W_i, \Lambda_i, v_i)$  is a  $(C, C')$ -controlled g-fusion frame (briefly  $CC'$ -GF) for  $H$  if there exist constants  $0 < A \leq B < \infty$  such that for all  $f \in H$

$$A \|f\|^2 \leq \sum_{i \in \mathbb{I}} v_i^2 \langle \Lambda_i \pi_{W_i} C' f, \Lambda_i \pi_{W_i} C f \rangle \leq B \|f\|^2.$$

We call  $\Lambda_{CC'}$  is a Parseval  $CC'$ -GF if  $A = B = 1$ . If only the second Inequality is required, We call  $\Lambda_{CC'}$  is a  $(C, C')$ -Controlled Bessel g-fusion sequence (briefly  $CC'$ -GBS) with bound  $B$ .

If we assume  $\Lambda_{CC'}$  is a  $CC'$ -GF for  $H$  and  $C^* \pi_{W_i} \Lambda_i^* \Lambda_i \pi_{W_i} C'$  is a positive operator for each  $i \in \mathbb{I}$ , then  $C^* \pi_{W_i} \Lambda_i^* \Lambda_i \pi_{W_i} C' = C'^* \pi_{W_i} \Lambda_i^* \Lambda_i \pi_{W_i} C$  and therefore

$$A \|f\|^2 \leq \sum_{i \in \mathbb{I}} v_i^2 \left\| (C^* \pi_{W_i} \Lambda_i^* \Lambda_i \pi_{W_i} C')^{1/2} f \right\|^2 \leq B \|f\|^2.$$

Let

$$\mathcal{K}_{\Lambda_j}^2 := \{v_i (C^* \pi_{W_i} \Lambda_i^* \Lambda_i \pi_{W_i} C')^{1/2} f : f \in H\} \subset \left(\bigoplus_{i \in \mathbb{I}} H\right)_{l^2}.$$

It is easy to check that  $\mathcal{K}_{\Lambda_j}^2$  is a closed subspace. We can define the *controlled analysis operator*  $T_{\Lambda}^*$  by

$$T_{\Lambda}^* : H \rightarrow \mathcal{K}_{\Lambda_j}^2, \\ T_{\Lambda}^* f = \{v_i (C^* \pi_{W_i} \Lambda_i^* \Lambda_i \pi_{W_i} C')^{1/2} f\}_{i \in \mathbb{I}}.$$

It is easy to check that the controlled analysis operator is bounded linear operator. Thus,  $T_{\Lambda} := (T_{\Lambda}^*)^*$  is well-defined and bounded and the *controlled synthesis operator*  $T_{\Lambda}$  can be defined by

$$T_{\Lambda} : \mathcal{K}_{\Lambda_j}^2 \rightarrow H, \\ T_{\Lambda} \left( v_i (C^* \pi_{W_i} \Lambda_i^* \Lambda_i \pi_{W_i} C')^{1/2} f \right) = \sum_{i \in \mathbb{I}} v_i^2 C^* \pi_{W_i} \Lambda_i^* \Lambda_i \pi_{W_i} C' f.$$

Now, we can define the  $CC'$ -GF operator  $S_{CC'}$  on  $H$  by

$$S_{CC'} f := T_{\Lambda} T_{\Lambda}^* f = \sum_{i \in \mathbb{I}} v_i^2 C^* \pi_{W_i} \Lambda_i^* \Lambda_i \pi_{W_i} C' f.$$

We can write for each  $f \in H$

$$\langle S_{CC'} f, f \rangle = \sum_{i \in \mathbb{I}} v_i^2 \langle C^* \pi_{W_i} \Lambda_i^* \Lambda_i \pi_{W_i} C' f, f \rangle \\ = \sum_{i \in \mathbb{I}} v_i^2 \langle \Lambda_i \pi_{W_i} C' f, \Lambda_i \pi_{W_i} C f \rangle,$$

therefore, we get

$$AId_H \leq S_{CC'} \leq BId_H.$$

**Theorem 3.2.**  $\Lambda_{CC'}$  be a  $CC'$ -GBS for  $H$  with bound  $B$  if and only if the operator

$$\begin{aligned} T_\Lambda : \mathcal{K}_{\Lambda_j}^2 &\rightarrow H, \\ T_\Lambda(v_i(C^* \pi_{W_i} \Lambda_i^* \Lambda_i \pi_{W_i} C')^{\frac{1}{2}} f) &= \sum_{i \in \mathbb{I}} v_i^2 C^* \pi_{W_i} \Lambda_i^* \Lambda_i \pi_{W_i} C' f. \end{aligned}$$

is well -defined and bounded operator with  $\|T_\Lambda\| \leq \sqrt{B}$ .

*Proof.* The necessary condition follows from the definition of  $CC'$ -GBS. We only need to prove that the sufficient condition holds. Let  $T_\Lambda$  be well-defined and bounded operator with  $\|T_\Lambda\| \leq \sqrt{B}$ . For any  $f \in H$ , we have

$$\begin{aligned} \sum_{i \in \mathbb{I}} v_i^2 \langle \Lambda_i \pi_{W_i} C' f, \Lambda_i \pi_{W_i} C' f \rangle &= \sum_{i \in \mathbb{I}} v_i^2 \langle C^* \pi_{W_i} \Lambda_i^* \Lambda_i \pi_{W_i} C' f, f \rangle \\ &= \langle T_\Lambda(v_i(C^* \pi_{W_i} \Lambda_i^* \Lambda_i \pi_{W_i} C')^{\frac{1}{2}} f), f \rangle \\ &\leq \|T_\Lambda\| \left\| (v_i(C^* \pi_{W_i} \Lambda_i^* \Lambda_i \pi_{W_i} C')^{\frac{1}{2}} f) \right\| \|f\|. \end{aligned}$$

But

$$\left\| (v_i(C^* \pi_{W_i} \Lambda_i^* \Lambda_i \pi_{W_i} C')^{\frac{1}{2}} f) \right\|^2 = \sum_{i \in \mathbb{I}} v_i^2 \langle \Lambda_i \pi_{W_i} C' f, \Lambda_i \pi_{W_i} C' f \rangle.$$

It follows that

$$\sum_{i \in \mathbb{I}} v_i^2 \langle \Lambda_i \pi_{W_i} C' f, \Lambda_i \pi_{W_i} C' f \rangle \leq B \|f\|^2$$

and this means that  $\Lambda_{CC'}$  is a  $CC'$ -GBS for  $H$ .  $\square$

**Theorem 3.3.** Let  $C \in GL^+(H)$ .  $\Lambda := (W_i, \Lambda_i, v_i)$  is a  $g$ -fusion frame for  $H$  if and only if  $\Lambda$  is a  $CC$ -GF.

*Proof.* Suppose that  $\Lambda$  is a  $CC$ -GF with Bounds  $A$  and  $B$  for  $H$ . for each  $f \in \mathbb{H}$ , we obtain

$$\begin{aligned} A \|f\|^2 &= A \|CC^{-1} f\|^2 \\ &\leq A \|C\|^2 \cdot \|C^{-1} f\|^2 \\ &\leq \|C\|^2 \sum_{i \in \mathbb{I}} v_i^2 \|\Lambda_i \pi_{W_i} CC^{-1} f\|^2 \\ &= \|C\|^2 \sum_{i \in \mathbb{I}} v_i^2 \|\Lambda_i \pi_{W_i} f\|^2 \end{aligned}$$

Hence

$$A \|C\|^{-2} \|f\|^2 \leq \sum_{i \in \mathbb{I}} v_i^2 \|\Lambda_i \pi_{W_i} f\|^2.$$

On the other hand, for any  $f \in H$ , we have

$$\begin{aligned} \sum_{i \in \mathbb{I}} v_i^2 \|\Lambda_i \pi_{W_i} f\|^2 &= \sum_{i \in \mathbb{I}} v_i^2 \|\Lambda_i \pi_{W_i} C C^{-1} f\|^2 \\ &\leq B \|C^{-1} f\|^2 \\ &\leq B \|C^{-1}\|^2 \cdot \|f\|^2 \end{aligned}$$

Thus,  $\Lambda$  is a g-fusion frame for  $H$  with bounds  $A \|C\|^{-2}, B \|C^{-1}\|^2$ .

Conversely, assume that  $\Lambda$  is a g-fusion frame for  $H$  with bounds  $A, B$ . Then, for each  $f \in H$  we get

$$\sum_{i \in \mathbb{I}} v_i^2 \langle \Lambda_i \pi_{W_i} C f, \Lambda_i \pi_{W_i} C f \rangle = \sum_{i \in \mathbb{I}} v_i^2 \|\Lambda_i \pi_{W_i} C f\|^2 \leq B \|C\|^2 \|f\|^2.$$

For the lower bound, we can write for any  $f \in H$ ,

$$\begin{aligned} A \|f\|^2 &= A \|C^{-1} C f\|^2 \\ &\leq A \|C^{-1}\|^2 \|C f\|^2 \\ &\leq \|C^{-1}\|^2 \sum_{i \in \mathbb{I}} v_i^2 \|\Lambda_i \pi_{W_i} C f\|^2. \end{aligned}$$

Therefore,  $\Lambda$  is a  $CC$ -GF for  $H$  with bounds  $A \|C^{-1}\|^{-2}, B \|C^{-1}\|^{-2}$ . □

**Theorem 3.4.** *Let  $\Lambda_{CC'}$  be a  $CC'$ -GF for  $H$  with bounds  $A, B$ . Then,  $\Lambda_{CC'}$  is a g-fusion frame for  $H$ . Furthermore, if  $S_{\Lambda'}$  is its g-fusion frame operator, then  $C^{-1} S_{\Lambda'} C' = C'^* S_{\Lambda'} C$ .*

*Proof.* We define

$$\begin{aligned} S : H &\longrightarrow H, \\ S f &:= C^{*-1} S_{CC'} C'^{-1} (f) = \sum_{i \in \mathbb{I}} v_i^2 \pi_{W_i} \Lambda_i^* \Lambda_i \pi_{W_i} f. \end{aligned}$$

It is easy to check that  $S$  is well-defined. Let  $f \in H$ , since  $GL(H)$  is a  $C^*$ -subalgebra in  $\mathcal{B}(H)$ , so  $(C^*)^{-1} = (C^{-1})^*$ , we have

$$\|S\| = \sup_{\|g\|=1} \|Sg\| \leq B \|C^{-1}\| \|C'\|,$$

so,  $S$  is bounded and

$$\sum_{i \in \mathbb{I}} v_i^2 \|\Lambda_i \pi_{W_i} f\|^2 = \|\langle S f, f \rangle\| \leq B \|C^{-1}\| \|C'\| \|f\|^2.$$

Let  $g \in H$  and define  $f := C'S_{CC'}^{-1}C^{-1}g$ . Then  $S$  is surjective and by Lemma 2.3,  $\Lambda_{CC'}$  is a  $g$ -fusion frame in  $H$  and its  $g$ -fusion frame operator is  $S_{\Lambda'} := S$ . So, we get

$$C^{-1}S_{\Lambda'}C' = S_{CC'} = S_{CC'}^* = C'^*S_{\Lambda'}C.$$

□

**Theorem 3.5.** Let  $\Lambda_{CC'} = (W_i, \Lambda_i, v_i)$  and  $\Theta_{CC'} = (W_i, \Theta_i, v_i)$  be two  $CC'$ -BGF for  $H$  with bounds  $B_1$  and  $B_2$ , respectively. Suppose that  $T_{\Lambda}$  and  $T_{\Theta}$  be their controlled analysis operators such that  $T_{\Theta}T_{\Lambda}^* = Id_H$ . Then, both  $\Lambda_{CC'}$  and  $\Theta_{CC'}$  are  $CC'$ -GF for  $H$ .

*Proof.* For each  $f \in H$ , we have

$$\begin{aligned} \|f\|^4 &= \langle f, f \rangle^2 \\ &= \langle T_{\Lambda}^*f, T_{\Theta}^*f \rangle^2 \\ &\leq \|T_{\Lambda}^*f\|^2 \cdot \|T_{\Theta}^*f\|^2 \\ &= \left( \sum_{i \in \mathbb{I}} v_i^2 \langle \Lambda_i \pi_{W_i} C'f, \Lambda_i \pi_{W_i} Cf \rangle \right) \left( \sum_{i \in \mathbb{I}} v_i^2 \langle \Theta_i \pi_{W_i} C'f, \Theta_i \pi_{W_i} Cf \rangle \right) \\ &\leq \left( \sum_{i \in \mathbb{I}} v_i^2 \langle \Lambda_i \pi_{W_i} C'f, \Lambda_i \pi_{W_i} Cf \rangle \right) B_2 \|f\|^2. \end{aligned}$$

Thus

$$B_2^{-1} \|f\|^2 \leq \sum_{i \in \mathbb{I}} v_i^2 \langle \Lambda_i \pi_{W_i} C'f, \Lambda_i \pi_{W_i} Cf \rangle,$$

and  $\Lambda_{CC'}$  is a  $CC'$ -GF for  $H$ . Similarly,  $\Theta_{CC'}$  is a  $CC'$ -GF with the lower bound  $B_1^{-1}$ . □

**Theorem 3.6.** Let  $\Lambda = (W_i, \Lambda_i, v_i)$  be a  $g$ -fusion frame for  $H$ . Then  $\Lambda_{CC'}$  is a Parseval  $CC'$ -GF for  $H$  if and only if There exist a operator  $L : H \rightarrow H$  such that  $C = UL^{-q}$  and  $C' = VL^{-p}$ , where  $U, V$  are two operators on  $H$  such that  $VU^* = Id_H$  and  $p, q \in \mathbb{R}$ ,  $p + q = 1$ .

*Proof.* Let  $\Lambda_{CC'}$  be a Parseval  $CC'$ -GF for  $H$ . So  $S_{CC'} = Id_H$ . Therefore, for each pairs of real numbers  $p, q$  such that  $p + q = 1$ , we obtain

$$Id_H = S_{CC'} = C'LC^* = C'L^pL^qC^*.$$

We define  $V := C'L^p$  and  $U := CL^q$ . So

$$VU^* = C'L^pL^qC^* = C'LC^* = S_{CC'} = Id_H.$$

Conversely, Let  $V, U$  be two operators on  $H$  such that  $VU^* = Id_H$ . We define  $C := UL^{-q}$  and  $C' := VL^{-p}$  be two operators on  $H$  where  $p, q \in \mathbb{R}$  and  $p + q = 1$ . So for any  $f \in H$ :

$$f = C'LC^*(f) = \sum_{i \in \mathbb{I}} v_i^2 C^* \pi_{W_i} \Lambda_i^* \Lambda_i \pi_{W_i} C'f$$

Therefore ,  $\Lambda_{CC'}$  is a parseval  $CC'$ -GF for  $H$ . □

As a special case of this theorem we have the following well-known result which is the most basic result for generating a Parseval frame. See Lemma 1.7 in [3].

4. Q-DUALITY AND PERTURBATION OF (C, C′)-CONTROLLED G-FUSION FRAME

This section is devoted to studying the behavior of the canonical dual of a g-fusion frame under perturbations. We consider perturbations of  $CC'$ -GF in analogy to the perturbations in [7] for frames, Also Q-duals are useful tools for establishing reconstruction formula. For more information we refer the reader to [6, 7].

**Definition 4.1.** Assume that  $\Lambda_{CC'} = (W_i, \Lambda_i, v_i)$  be a  $CC'$ -GF for  $H$ . We call a  $CC'$ -GBS as  $\Theta_{CC'} := (W_i, \Theta_i, v_i)$  the Q-dual  $CC'$ -GF of  $\Lambda_{CC'}$ , if there exist a bounded linear operator  $Q : \mathcal{K}_{\Lambda_j}^2 \rightarrow \mathcal{K}_{\Theta_j}^2$  such that:

$$T_\Lambda Q^* T_\Theta^* = C.$$

**Lemma 4.2.** Let  $\Lambda_{cc'} = (W_i, \Lambda_i, v_i)$  and  $\Theta_{cc'} = (W_i, \Theta_i, v_i)$  be  $CC'$ -GBS for  $H$ , and Let  $Q : \mathcal{K}_{\Lambda_j}^2 \rightarrow \mathcal{K}_{\Theta_j}^2$ , Then the following conditions are equivalent.

- (1)  $T_\Theta Q T_\Lambda^* = C$ ;
- (2)  $T_\Lambda Q^* T_\Theta^* = C^*$ ;
- (3)  $\langle Cf, g \rangle = \langle QT_\Lambda^* f, T_\Theta^* g \rangle = \langle Q^* T_\Theta^* f, T_\Lambda^* g \rangle$ .

*Proof.* Straightforward. □

**Theorem 4.3.** If  $\Theta_{CC'}$  be a Q-dual for  $\Lambda_{CC'}$ , Then  $\Theta_{CC'}$  is a  $CC'$ -GF for  $H$ .

*Proof.* Let  $f \in H$  and  $B$  an upper bound for  $\Lambda_{CC'}$ . Therefore

$$\begin{aligned}
\|f\|^4 &= \|\langle f, f \rangle\|^2 \\
&= \|\langle f, C^*(C^*)^{-1} f \rangle\|^2 \\
&= \|\langle T_\Lambda Q^* T_\Theta^* f, (C^*)^{-1} f \rangle\|^2 \\
&= \|\langle T_\Theta^* f, QT_\Lambda^*(C^*)^{-1} f \rangle\|^2 \\
&\leq \|T_\Theta^* f\|^2 \|Q\|^2 \|T_\Lambda^*(C^*)^{-1} f\|^2 \\
&\leq \|T_\Theta^* f\|^2 \|Q\|^2 B \|C^{-1}\|^2 \|f\|^2 \\
&= \|Q\|^2 B \|C^{-1}\|^2 \|f\|^2 \sum_{i \in \mathbb{I}} v_i^2 \langle \Theta_i \pi_{W_i} C' f, \Theta_i \pi_{W_i} C f \rangle.
\end{aligned}$$

Hence

$$B^{-1} \|Q\|^{-2} \|C^{-1}\|^{-2} \|f\|^2 \leq \sum_{i \in \mathbb{I}} v_i^2 \langle \Theta_i \pi_{W_i} C' f, \Theta_i \pi_{W_i} C f \rangle$$

and this completes the proof.  $\square$

**Corollary 4.4.** *Assume  $C_{op}$  and  $D_{op}$  are the optimal bounds of  $\Theta_{CC'}$ , respectively. Then*

$$C_{op} \geq B_{op}^{-1} \|Q\|^{-2} \|C^{-1}\|^{-2} \quad \text{and} \quad D_{op} \geq A_{op}^{-1} \|Q\|^{-2} \|C^{-1}\|^{-2},$$

which  $A_{op}$  and  $B_{op}$  are the optimal bounds of  $\Lambda_{CC'K}$ , respectively.

Suppose that  $\Lambda_C$  be a  $C^2$ -GF for  $H$ . Since  $S_C \geq A_C$ , then by Douglas theorem, [4], there exists an operator  $U \in \mathcal{B}(H, \mathcal{K}_{\Lambda_j}^2)$  such that

$$T_C U = I_H. \quad (4.1)$$

Now, we define the  $j$ -th component of  $Uf$  by  $U_j f = (Uf)_j$  for each  $f \in H$ . By this operator, we may construct some  $Q$ -duals  $C^2$ -GF for  $\Lambda_C$ .

**Theorem 4.5.** *Let  $\Lambda_C$  be a  $C^2$ -GF for  $H$ . If  $U$  be an operator as in (4.1) and  $\widetilde{W}_i := U_i^* C^* W_i$  such that  $\Theta_C := (\widetilde{W}_i, \Lambda_i, v_i)$  is a  $C^2$ -GF for  $H$ . Then  $\Theta$  is a  $Q$ -dual  $C^2$ -GF for  $\Lambda_C$ .*

*Proof.* Define the mapping

$$\begin{aligned} \Psi_0 : \mathcal{R}(T_\Theta^*) &\rightarrow \mathcal{K}_{\Lambda_j}^2, \\ \Psi_0(T_\Theta^* f) &= U C f. \end{aligned}$$

Then  $\Psi_0$  is well-defined. indeed, if  $f_1, f_2 \in H$  and  $T_\Theta^* f_1 = T_\Theta^* f_2$ , then  $\pi_{\widetilde{W}_i} \Lambda_i^* \Lambda_i \pi_{\widetilde{W}_i} C(f_1 - f_2) = 0$ . Therefore, for any  $i \in \mathbb{I}$ ,

$$\Lambda_i^* \Lambda_i \pi_{\widetilde{W}_i} C(f_1 - f_2) \in (\widetilde{W}_i)^\perp = \mathcal{R}(U_i^*)^\perp = \ker U_i.$$

Thus,

$$U_i \Lambda_i^* \Lambda_i \pi_{\widetilde{W}_i} C(f_1 - f_2) = 0,$$

and so,  $C(f_1 - f_2) \in \pi_{\widetilde{W}_i}$ . Hence  $U_i C f_1 = U_i C f_2$ , for all  $i \in \mathbb{I}$ . Moreover

$$\begin{aligned} \|\Psi_0\| &= \sup_{f \neq 0} \frac{\|\Psi_0 f\|}{\|T_\Theta^* f\|} \\ &\leq \sup_{f \neq 0} \frac{\|U C f\|}{\sqrt{A_\Theta} \|f\|} \\ &= \frac{\|U\| \|C\|}{\sqrt{A_\Theta}} < \infty, \end{aligned}$$

where,  $A_\Theta$  is a lower frame bound of  $\Theta_C$ . Therefore,  $\Psi_0$  is a bounded operator. So, it has a unique linear extension (also denoted  $\Psi_0$ ) to  $\overline{\mathcal{R}(T_\Theta^*)}$ . Define

$$\Psi = \begin{cases} \Psi_0, & \text{on } \overline{\mathcal{R}(T_\Theta^*)}, \\ 0, & \text{on } \overline{\mathcal{R}(T_\Theta^*)}^\perp \end{cases}$$

and let  $Q := \Psi^*$ . This implies that  $Q^* \in \mathcal{B}(\mathcal{K}_\Theta^2, \mathcal{K}_{\Lambda_j}^2)$  and

$$T_C Q^* T_\Theta^* = T_C \Psi T_\Theta^* = T_C U C = C.$$

□

**Definition 4.6.** Let  $\Lambda_{CC'} = (W_i, \Lambda_i, v_i)$  and  $\Theta_{CC'} = (W_i, \Theta_i, v_i)$  be two  $CC'$ -GBS for  $H$  and  $0 \leq \lambda_1, \lambda_2 < 1$ . Let a sequence of positive numbers  $\{c_i\}_{i \in \mathbb{I}}$  such that  $\beta := \{c_i\}_{i \in \mathbb{I}} \in l^2(I)$ . If

$$\begin{aligned} \left\| v_i(C^* \pi_{W_i} \Lambda_i^* \Lambda_i \pi_{W_i} C' - C^* \pi_{W_i} \Theta_i^* \Theta_i \pi_{W_i} C')^{\frac{1}{2}} f \right\|_2 &\leq \lambda_1 \left\| v_i(C^* \pi_{W_i} \Lambda_i^* \Lambda_i \pi_{W_i} C')^{\frac{1}{2}} f \right\|_2 \\ &+ \lambda_2 \left\| v_i(C^* \pi_{W_i} \Theta_i^* \Theta_i \pi_{W_i} C')^{\frac{1}{2}} f \right\|_2 + \|\beta\|_2 \|f\| \end{aligned}$$

Then we say that  $\Theta_{CC'}$  is a  $(\lambda_1, \lambda_2, \beta, C, C')$ -perturbation of  $\Lambda_{CC'}$ .

**Theorem 4.7.** Let  $\Lambda_{CC'}$  be a  $CC'$ -GF for  $H$  with frame bounds  $A, B$  and  $\Theta_{CC'}$  be a  $(\lambda_1, \lambda_2, \beta, C, C')$ -perturbation of  $\Lambda_{CC'}$ . Then  $\Theta_{CC'}$  is a  $CC'$ -GF for  $H$  with bounds:

$$\left( \frac{(1 - \lambda_1)\sqrt{A} - \|\beta\|_2}{1 + \lambda_2} \right)^2, \quad \left( \frac{(1 + \lambda_1)\sqrt{B} + \|\beta\|_2}{1 - \lambda_2} \right)^2$$

*Proof.* Let  $f \in H$ , We have

$$\begin{aligned} &\left\| v_i(C^* \pi_{W_i} \Theta_i^* \Theta_i \pi_{W_i} C')^{\frac{1}{2}} f \right\|_2 \\ &= \left\| v_i(C^* \pi_{W_i} \Theta_i^* \Theta_i \pi_{W_i} C' - C^* \pi_{W_i} \Lambda_i^* \Lambda_i \pi_{W_i} C')^{\frac{1}{2}} f + v_i(C^* \pi_{W_i} \Lambda_i^* \Lambda_i \pi_{W_i} C')^{\frac{1}{2}} f \right\|_2 \\ &\leq \left\| v_i(C^* \pi_{W_i} \Theta_i^* \Theta_i \pi_{W_i} C' - C^* \pi_{W_i} \Lambda_i^* \Lambda_i \pi_{W_i} C')^{\frac{1}{2}} f \right\|_2 + \left\| v_i(C^* \pi_{W_i} \Lambda_i^* \Lambda_i \pi_{W_i} C')^{\frac{1}{2}} f \right\|_2 \\ &\leq \lambda_1 \left\| v_i(C^* \pi_{W_i} \Lambda_i^* \Lambda_i \pi_{W_i} C')^{\frac{1}{2}} f \right\|_2 + \lambda_2 \left\| v_i(C^* \pi_{W_i} \Theta_i^* \Theta_i \pi_{W_i} C')^{\frac{1}{2}} f \right\|_2 \\ &\quad + \|\beta\|_2 \|f\| + \left\| v_i(C^* \pi_{W_i} \Lambda_i^* \Lambda_i \pi_{W_i} C')^{\frac{1}{2}} f \right\|_2. \end{aligned}$$

Hence,

$$\begin{aligned} (1 - \lambda_2) &\left\| v_i(C^* \pi_{W_i} \Theta_i^* \Theta_i \pi_{W_i} C')^{\frac{1}{2}} f \right\|_2 \\ &\leq (1 + \lambda_1) \left\| v_i(C^* \pi_{W_i} \Lambda_i^* \Lambda_i \pi_{W_i} C')^{\frac{1}{2}} f \right\|_2 + \|\beta\|_2 \|f\|. \end{aligned}$$

Since  $\Lambda_{CC'}$  is a  $CC'$ -GF with bounds  $A, B$  and analysis operator  $T_\Lambda^*$  and synthesis operator  $T_\Lambda$ , we have

$$\left\langle \sum_{i \in \mathbb{I}} v_i^2 C^* \pi_{W_i} \Lambda_i^* \Lambda_i \pi_{W_i} C' f, f \right\rangle = \langle S_{CC'} f, f \rangle = \langle T_\Lambda T_\Lambda^* f, f \rangle = \langle T_\Lambda^* f, T_\Lambda^* f \rangle$$

thus,

$$\|T_\Lambda^* f\|^2 = \sum_{i \in \mathbb{I}} v_i^2 \langle \Lambda_i \pi_{W_i} C' f, \Lambda_i \pi_{W_i} C' f \rangle \leq B \|f\|^2.$$

So

$$\begin{aligned} \left\| v_i(C^* \pi_{W_i} \Theta_i^* \Theta_i \pi_{W_i} C')^{\frac{1}{2}} f \right\|_2 &\leq \frac{(1 + \lambda_1) \left\| v_i(C^* \pi_{W_i} \Lambda_i^* \Lambda_i \pi_{W_i} C')^{\frac{1}{2}} f \right\| + \|\beta\|_2 \|f\|}{1 - \lambda_2} \\ &\leq \frac{((1 + \lambda_1)\sqrt{B} \|f\| + \|\beta\|_2 \|f\|)}{1 - \lambda_2}. \end{aligned}$$

Now, for the lower bound, we have

$$\begin{aligned} &\left\| v_i(C^* \pi_{W_i} \Theta_i^* \Theta_i \pi_{W_i} C')^{\frac{1}{2}} f \right\|_2 \\ &= \left\| v_i(C^* \pi_{W_i} \Lambda_i^* \Lambda_i \pi_{W_i} C')^{\frac{1}{2}} f - v_i(C^* \pi_{W_i} \Lambda_i^* \Lambda_i \pi_{W_i} C' - C^* \pi_{W_i} \Theta_i^* \Theta_i \pi_{W_i} C')^{\frac{1}{2}} f \right\|_2 \\ &\geq \left\| v_i(C^* \pi_{W_i} \Lambda_i^* \Lambda_i \pi_{W_i} C')^{\frac{1}{2}} f \right\|_2 - \left\| v_i(C^* \pi_{W_i} \Theta_i^* \Theta_i \pi_{W_i} C' - C^* \pi_{W_i} \Theta_i^* \Theta_i \pi_{W_i} C')^{\frac{1}{2}} f \right\|_2 \\ &\geq \left\| v_i(C^* \pi_{W_i} \Lambda_i^* \Lambda_i \pi_{W_i} C')^{\frac{1}{2}} f \right\|_2 - \lambda_1 \left\| v_i(C^* \pi_{W_i} \Lambda_i^* \Lambda_i \pi_{W_i} C')^{\frac{1}{2}} f \right\|_2 \\ &\quad - \lambda_2 \left\| v_i(C^* \pi_{W_i} \Theta_i^* \Theta_i \pi_{W_i} C')^{\frac{1}{2}} f \right\|_2 - \|\beta\|_2 \|f\|. \end{aligned}$$

Therefore

$$\begin{aligned} &(1 + \lambda_2) \left\| v_i(C^* \pi_{W_i} \Theta_i^* \Theta_i \pi_{W_i} C')^{\frac{1}{2}} f \right\|_2 \\ &\geq (1 - \lambda_1) \left\| v_i(C^* \pi_{W_i} \Lambda_i^* \Lambda_i \pi_{W_i} C')^{\frac{1}{2}} f \right\|_2 - \|\beta\|_2 \|f\|, \end{aligned}$$

or

$$\left\| v_i(C^* \pi_{W_i} \Theta_i^* \Theta_i \pi_{W_i} C')^{\frac{1}{2}} f \right\|_2 \geq \frac{(1 - \lambda_1) \left\| v_i(C^* \pi_{W_i} \Lambda_i^* \Lambda_i \pi_{W_i} C') f \right\| - \|\beta\|_2 \|f\|}{1 + \lambda_2}.$$

Since  $\Lambda_{C'}$  is a  $CC'$ -GF with bounds  $A, B$  and analysis operator  $T_\Lambda^*$ , we have

$$\|T_\Lambda^* f\|^2 = \left\| v_i(C^* \pi_{W_i} \Lambda_i^* \Lambda_i \pi_{W_i} C')^{\frac{1}{2}} f \right\|^2 = \sum_{i \in I} v_i^2 \langle \Lambda_i \pi_{W_i} C' f, \Lambda_i \pi_{W_i} C' f \rangle \geq A \|f\|^2.$$

Thus

$$\left\| v_i(C^* \pi_{W_i} \Theta_i^* \Theta_i \pi_{W_i} C')^{\frac{1}{2}} f \right\|_2 \geq \frac{(1 - \lambda_1)\sqrt{A} \|f\| - \|\beta\|_2 \|f\|}{1 + \lambda_2}$$

and the proof is completed.  $\square$

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#### REFERENCES

1. P. Balazs, J. P. Antoine, A. Grybos, Weighted and Controlled Frames: Mutual Relationship and First Numerical Properties, *Int. J. Wavelets Multiresolut. Inf. Process.*, **8**(1), (2010), 109-132.
2. I. Bogdanova, P. Vandergheynst, J. P. Antoine, L. Jacques, M. Morvidone, Stereographic Wavelet Frames on the Sphere, *Appl. Comput. Harmon. Anal.*, **19**, (2005), 223-252.
3. P. G. Casazza, G. Kutyniok, *Finite Frames, Theory and Applications*, Applied and Numerical Harmonic Analysis, Boston, Birkhauser, 2013.
4. R. G. Douglas, On Majorization, Factorization and Range Inclusion of Operators on Hilbert Spaces, *Proc Amer. Math. Soc.*, **17**(2), (1996), 413-415.

5. R. J. Duffin, A. C. Schaeffer, A Class of Nonharmonic Fourier Series, *Trans. Amer. Math. Soc.*, **72**(1), (1952), 341-366.
6. P. Găvruta, On the Duality of Fusion Frames, *J. Math. Anal. Appl.*, **333**, (2007), 871–879.
7. S. Heineken, E. Matusiak, V. Paternostro, Perturbed Frame Sequences: Canonical Dual Systems, Approximate Reconstructions and Applications, *Int. J. Wavelets Multiresolut. Inf. Process.*, **12**(2), (2014).
8. D. Hua, Y. Huang, Controlled  $K$ - $g$ -frames in Hilbert Spaces, *Results. Math.*, **72**(3), (2016), 1227-1238.
9. A. Khosravi, K. Musazadeh, Controlled Fusion Frames, *Methods Funct. Anal. Top.*, **18**(3), (2012), 256-265.
10. K. Musazadeh, H. Khandani, Some Results on Controlled Frames in Hilbert Spaces, *Acta Math. Sci.*, **36B**(3), (2016), 655-665.
11. A. Rahimi, A. Fereydooni, Controlled  $G$ -frames and Their  $G$ -multipliers in Hilbert Spaces, *An. St. Univ. Ovidius Constanta.*, **21**(2), (2013), 223-236.
12. A. Rahimi, S. Najafzadeh, M. Nouri, Controlled  $K$ -frames and Their Invariance under Compact Perturbation, arXiv:1602.03982, (2016).
13. V. Sadri, Gh. Rahimlou, R. Ahmadi, R. Zarghami Farfar, Generalized Fusion Frames in Hilbert Spaces, Submitted, (2018).