

## Horadam Polynomials Estimates for $\lambda$ -Pseudo-Starlike Bi-Univalent Functions

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**ABSTRACT.** In the present investigation, we use the Horadam Polynomials to establish upper bounds for the second and third coefficients of functions belongs to a new subclass of analytic and  $\lambda$ -pseudo-starlike bi-univalent functions defined in the open unit disk  $U$ . Also, we discuss Fekete-Szegö problem for functions belongs to this subclass.

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### 1. INTRODUCTION

Let  $\mathcal{A}$  stand for the family of functions  $f$  which are analytic in the open unit disk  $U = \{z \in \mathbb{C} : |z| < 1\}$  that have the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \quad (1.1)$$

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Further, let  $S$  indicate the subclass of  $\mathcal{A}$  consisting of the form (1.1) which are univalent in  $U$ . It is well known (see [3]) that every function  $f \in S$  has an inverse  $f^{-1}$  defined by  $f^{-1}(f(z)) = z$ , ( $z \in U$ ) and  $f(f^{-1}(w)) = w$ , ( $|w| < r_0(f), r_0(f) \geq \frac{1}{4}$ ), where

$$g(w) = f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3) w^3 - (5a_2^3 - 5a_2 a_3 + a_4) w^4 + \dots \quad (1.2)$$

A function  $f \in \mathcal{A}$  is said to be bi-univalent in  $U$  if both  $f$  and  $f^{-1}$  are univalent in  $U$ . Let  $\Sigma$  stands for the class of bi-univalent functions in  $U$  given by (1.1). In fact, Srivastava et al. [13] has apparently revived the study of analytic and bi-univalent functions in recent years, it was followed by such works as those by Frasin and Aouf [4], Goyal and Goswami [5], Caglar et al. [2] and others (see, for example [9, 10, 11, 12, 14]).

A function  $f \in S$  is said to be  $\lambda$ -pseudo-starlike function of order  $\beta$   $0 \leq \beta < 1$  in  $U$ , if it satisfies (see [1]): This class of functions was denoted by  $\mathcal{L}_\lambda(\beta)$ . It is observed that for  $\lambda = 1$ , we have the class of starlike functions.

With a view to recalling the principal of subordination between analytic functions, let the functions  $f$  and  $g$  be analytic in  $U$ . We say that the function  $f$  is said to be subordinate to  $g$ , if there exists a Schwarz function  $w$  analytic in  $U$  with  $w(0) = 0$  and  $|w(z)| < 1$  ( $z \in U$ ) such that  $f(z) = g(w(z))$ . This subordination is denoted by  $f \prec g$  or  $f(z) \prec g(z)$  ( $z \in U$ ). It is well known that, if the function  $g$  is univalent in  $U$ , then  $f \prec g$  if and only if  $f(0) = g(0)$  and  $f(U) \subset g(U)$ .

The Horadam polynomials  $h_n(r)$  are defined by the following repetition relation (see [6]):

$$h_n(r) = prh_{n-1}(r) + qh_{n-2}(r) \quad (r \in \mathbb{R}, n \in \mathbb{N}),$$

with

$$h_1(r) = a \quad \text{and} \quad h_2(r) = br, \quad (1.3)$$

for some real constant  $a, b, p$  and  $q$ .

The generating function of the Horadam polynomials  $h_n(r)$  (see [7]) is given by

$$\Pi(r, z) = \sum_{n=1}^{\infty} h_n(r) z^{n-1} = \frac{a + (b - ap)rz}{1 - prz - qz^2}. \quad (1.4)$$

## 2. MAIN RESULTS

We begin this section by defining the subclass  $\mathcal{L}_\Sigma(\delta, \lambda, r)$  as follows:

**Definition 2.1.** For  $\delta \in \mathbb{C} \setminus \{0\}$ ,  $\lambda \geq 1$  and  $r \in \mathbb{R}$ , a function  $f \in \Sigma$  is said to be in the class  $\mathcal{L}_\Sigma(\delta, \lambda, r)$  if it satisfies the subordinations:

$$1 + \frac{1}{\delta} \left( \frac{z (f'(z))^\lambda}{f(z)} - 1 \right) \prec \Pi(r, z) + 1 - a$$

and

$$1 + \frac{1}{\delta} \left( \frac{w(f'(w))^\lambda}{f(w)} - 1 \right) \prec \Pi(r, w) + 1 - a,$$

where  $a$  is real constant and the function  $g = f^{-1}$  is given by (1.2).

**Theorem 2.2.** For  $\delta \in \mathbb{C} \setminus \{0\}$ ,  $\lambda \geq 1$  and  $r \in \mathbb{R}$ , let  $f \in \mathcal{A}$  be in the class  $\mathcal{L}_\Sigma(\delta, \lambda, r)$ . Then

$$|a_2| \leq \frac{\sqrt{2\delta} |br| \sqrt{|br|}}{\sqrt{\left| \left[ \delta(4\lambda^2 - 5\lambda + 1)b - 2p(\lambda - 1)^2 \right] br^2 - 2qa(\lambda - 1)^2 \right|}}$$

and

$$|a_3| \leq \frac{\delta |br|}{3\lambda - 1} + \frac{\delta^2 b^2 r^2}{(\lambda - 1)^2}.$$

*Proof.* Let  $f \in \mathcal{L}_\Sigma(\delta, \lambda, r)$ . Then there are two analytic functions  $u, v : U \rightarrow U$  given by

$$u(z) = u_1 z + u_2 z^2 + u_3 z^3 + \cdots \quad (z \in U) \quad (2.1)$$

and

$$v(w) = v_1 w + v_2 w^2 + v_3 w^3 + \cdots \quad (w \in U), \quad (2.2)$$

with  $u(0) = v(0) = 0$ ,  $|u(z)| < 1$ ,  $|v(w)| < 1$ ,  $z, w \in U$  such that

$$1 + \frac{1}{\delta} \left( \frac{z(f'(z))^\lambda}{f(z)} - 1 \right) = \Pi(r, u(z)) + 1 - a$$

and

$$1 + \frac{1}{\delta} \left( \frac{w(f'(w))^\lambda}{f(w)} - 1 \right) = \Pi(r, v(w)) + 1 - a.$$

Or, equivalently

$$1 + \frac{1}{\delta} \left( \frac{z(f'(z))^\lambda}{f(z)} - 1 \right) = 1 + h_1(r) + h_2(r)u(z) + h_3(r)u^2(z) + \cdots \quad (2.3)$$

and

$$1 + \frac{1}{\delta} \left( \frac{w(f'(w))^\lambda}{f(w)} - 1 \right) = 1 + h_1(r) + h_2(r)v(w) + h_3(r)v^2(w) + \cdots. \quad (2.4)$$

Combining (2.1), (2.2), (2.3) and (2.4) yields

$$1 + \frac{1}{\delta} \left( \frac{z(f'(z))^\lambda}{f(z)} - 1 \right) = 1 + h_2(r)u_1 z + [h_2(r)u_2 + h_3(r)u_1^2] z^2 + \cdots \quad (2.5)$$

and

$$1 + \frac{1}{\delta} \left( \frac{w(f'(w))^\lambda}{f(w)} - 1 \right) = 1 + h_2(r)v_1 w + [h_2(r)v_2 + h_3(r)v_1^2] w^2 + \cdots. \quad (2.6)$$

It is quite well-known that if  $|u(z)| < 1$  and  $|v(w)| < 1$ ,  $z, w \in U$ , then

$$|u_i| \leq 1 \quad \text{and} \quad |v_i| \leq 1 \quad \text{for all } i \in \mathbb{N}. \quad (2.7)$$

Comparing the corresponding coefficients in (2.5) and (2.6), after simplifying, we have

$$\frac{2\lambda - 1}{\delta} a_2 = h_2(r)u_1, \quad (2.8)$$

$$\frac{3\lambda - 1}{\delta} a_3 + \frac{2\lambda(\lambda - 2) + 1}{\delta} a_2^2 = h_2(r)u_2 + h_3(r)u_1^2, \quad (2.9)$$

$$-\frac{2\lambda - 1}{\delta} a_2 = h_2(r)v_1 \quad (2.10)$$

and

$$\frac{3\lambda - 1}{\delta} (2a_2^2 - a_3) + \frac{2\lambda(\lambda - 2) + 1}{\delta} a_2^2 = h_2(r)v_2 + h_3(r)v_1^2. \quad (2.11)$$

It follows from (2.8) and (2.10) that

$$u_1 = -v_1 \quad (2.12)$$

and

$$\frac{2(\lambda - 1)^2}{\delta^2} a_2^2 = h_2^2(r)(u_1^2 + v_1^2). \quad (2.13)$$

If we add (2.9) to (2.11), we find that

$$\frac{4\lambda^2 - 5\lambda + 1}{\delta} a_2^2 = h_2(r)(u_2 + v_2) + h_3(r)(u_1^2 + v_1^2). \quad (2.14)$$

Substituting the value of  $u_1^2 + v_1^2$  from (2.13) in the right hand side of (2.14), we deduce that

$$a_2^2 = \frac{\delta^2 h_2^3(r)(u_2 + v_2)}{\delta (4\lambda^2 - 5\lambda + 1) h_2^2(r) - 2h_3(r)(\lambda - 1)^2}. \quad (2.15)$$

Further computations using (1.3), (2.7) and (2.15), we obtain

$$|a_2| \leq \frac{\sqrt{2}\delta |br| \sqrt{|br|}}{\sqrt{\left| \left[ \delta (4\lambda^2 - 5\lambda + 1) b - 2p(\lambda - 1)^2 \right] br^2 - 2qa(\lambda - 1)^2 \right|}}.$$

Next, if we subtract (2.11) from (2.9), we can easily see that

$$\frac{2(3\lambda - 1)}{\delta} (a_3 - a_2^2) = h_2(r)(u_2 - v_2) + h_3(r)(u_1^2 - v_1^2). \quad (2.16)$$

In view of (2.12) and (2.13), we get from (2.16)

$$a_3 = \frac{\delta h_2(r)(u_2 - v_2)}{2(3\lambda - 1)} + \frac{\delta^2 h_2^2(r)(u_1^2 + v_1^2)}{2(\lambda - 1)^2}.$$

Thus applying (1.3), we obtain

$$|a_3| \leq \frac{\delta |br|}{3\lambda - 1} + \frac{\delta^2 b^2 r^2}{(\lambda - 1)^2}.$$

This completes the proof of Theorem 2.2  $\square$

In the next theorem, we discuss the Fekete-Szegö problem for the subclass  $\mathcal{L}_\Sigma(\delta, \lambda, r)$ .

**Theorem 2.3.** *For  $\delta \in \mathbb{C} \setminus \{0\}$ ,  $\lambda \geq 1$  and  $r, \mu \in \mathbb{R}$ , let  $f \in \mathcal{A}$  be in the class  $\mathcal{L}_\Sigma(\delta, \lambda, r)$ . Then*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{\delta |br|}{3\lambda-1}; \\ \text{for } |\mu - 1| \leq \frac{|[\delta(4\lambda^2-5\lambda+1)b-2p(\lambda-1)^2]br^2-2qa(\lambda-1)^2|}{2\delta b^2 r^2 (3\lambda-1)}, \\ \frac{2\delta^2 |br|^3 |\mu-1|}{|[\delta(4\lambda^2-5\lambda+1)b-2p(\lambda-1)^2]br^2-2qa(\lambda-1)^2|}; \\ \text{for } |\mu - 1| \geq \frac{|[\delta(4\lambda^2-5\lambda+1)b-2p(\lambda-1)^2]br^2-2qa(\lambda-1)^2|}{2\delta b^2 r^2 (3\lambda-1)}. \end{cases}$$

*Proof.* It follows from (2.15) and (2.16) that

$$\begin{aligned} a_3 - \mu a_2^2 &= \frac{\delta h_2(r)(u_2 - v_2)}{2(3\lambda - 1)} + (1 - \mu)a_2^2 \\ &= \frac{\delta h_2(r)(u_2 - v_2)}{2(3\lambda - 1)} + \frac{\delta^2 h_2^3(r)(u_2 + v_2)(1 - \mu)}{\delta(4\lambda^2 - 5\lambda + 1)h_2^2(r) - 2h_3(r)(\lambda - 1)^2} \\ &= h_2(r) \left[ \left( \psi(\mu, r) + \frac{\delta}{2(3\lambda - 1)} \right) u_2 + \left( \psi(\mu, r) - \frac{\delta}{2(3\lambda - 1)} \right) v_2 \right], \end{aligned}$$

where

$$\psi(\mu, r) = \frac{\delta^2 h_2^2(r)(1 - \mu)}{\delta(4\lambda^2 - 5\lambda + 1)h_2^2(r) - 2h_3(r)(\lambda - 1)^2}.$$

According to (1.3), we find that

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{\delta |br|}{3\lambda-1}, & 0 \leq |\psi(\mu, r)| \leq \frac{\delta}{2(3\lambda-1)}, \\ 2|br||\psi(\mu, r)|, & |\psi(\mu, r)| \geq \frac{\delta}{2(3\lambda-1)}. \end{cases}$$

After some computations, we obtain

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{\delta |br|}{3\lambda-1}; \\ \text{for } |\mu - 1| \leq \frac{|[\delta(4\lambda^2-5\lambda+1)b-2p(\lambda-1)^2]br^2-2qa(\lambda-1)^2|}{2\delta b^2 r^2 (3\lambda-1)}, \\ \frac{2\delta^2 |br|^3 |\mu-1|}{|[\delta(4\lambda^2-5\lambda+1)b-2p(\lambda-1)^2]br^2-2qa(\lambda-1)^2|}; \\ \text{for } |\mu - 1| \geq \frac{|[\delta(4\lambda^2-5\lambda+1)b-2p(\lambda-1)^2]br^2-2qa(\lambda-1)^2|}{2\delta b^2 r^2 (3\lambda-1)}. \end{cases}$$

□

Putting  $\mu = 1$  in Theorem 2.3, we obtain the following result:

**Corollary 2.4.** For  $\delta \in \mathbb{C} \setminus \{0\}$ ,  $\lambda \geq 1$  and  $r \in \mathbb{R}$ , let  $f \in \mathcal{A}$  be in the class  $\mathcal{L}_\Sigma(\delta, \lambda, r)$ . Then

$$|a_3 - a_2^2| \leq \frac{\delta |br|}{3\lambda - 1}.$$

*Remark 2.5.* If we put  $\lambda = 1$  in our Theorems, we have the result for well-known class  $S_\Sigma^*(r)$  of bi-starlike functions which was considered recently by Srivastava et al. [8].

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