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On (ϵ) - Lorentzian para-Sasakian Manifolds

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Abstract. The object of this paper is to study (ϵ)-Lorentzian para-Sasakian manifolds. Some typical identities for the curvature tensor and the Ricci tensor of (ϵ)-Lorentzian para-Sasakian manifold are investigated. Further, we study globally ϕ-Ricci symmetric and weakly ϕ-Ricci symmetric (ϵ)-Lorentzian para-Sasakian manifolds and obtain interesting results.

Keywords:(ϵ)-Lorentzian para-Sasakian manifold, Einstein manifold, globally ϕ-Ricci symmetric manifold, Weakly ϕ-Ricci symmetric manifold.

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1. Introduction

In 1969, T. Takahashi [17] introduced almost contact manifolds equipped with associated indefinite metrics. He studied Sasakian manifolds equipped with an associated indefinite metric. These indefinite almost contact metric manifolds and indefinite Sasakian manifolds are also called (ϵ)-almost contact metric manifolds and (ϵ)-Sasakian manifolds, respectively [1, 4, 12, 21]. The index of a metric plays significant roles in differential geometry since it generates variety of vector fields such as space-like, time-like, and light-like fields. In 2010, M. M. Tripathi and his co-authors [20] studied almost paracontact manifolds equipped with an indefinite metrics. In particular, they studied para-Sasakian manifolds with an indefinite metric known as (ϵ)-para-Sasakian manifolds. In [7], K. Matsumoto introduced the notion of Lorenzian para-Sasakian manifold. I. Mihai and R. Rosca [9] defined the same notion independently. Lorenzian para-Sasakian manifolds were further studied by many authors. For details we may refer to the papers ([8, 10, 14, 15]) and the references therein. Recently, Rajendra Prasad and Vibha Srivastava [11] introduced the notion of Lorentzian para-Sasakian manifolds with indefinite metric which also include usual LP-Sasakian manifold. Such a notion is called Indefinite Lorentzian para-Sasakian manifold (or, (ϵ)-Lorentzian para-Sasakian manifold). This notion was further studied by Haseeb, Prakash and Siddiqui [6].

The paper is organized as follows: Section 2 is devoted to preliminaries. In section 3, some typical identities for the curvature tensor and the Ricci tensor are presented. We prove that if a pseudo-Riemannian manifold is one of: flat, proper recurrent, or proper Ricci recurrent, then it cannot admit (ϵ)-Lorentzian para-Sasakian structure. Also, we show that, for an (ϵ)-Lorentzian para-Sasakian manifold, the conditions of being symmetric, semi-symmetric, or of constant scalar curvature are all identical. In this section,it is also proved that for an (ϵ)-Lorentzian para-Sasakian manifold, the conditions being Ricci-semi-symmetric, Ricci symmetric and Einstein are all identical. In section 4, we study globally ϕ-Ricci symmetric and weakly ϕ-Ricci symmetric (ϵ)-Lorentzian para-Sasakian manifolds. It is proved that, an (ϵ)-Lorentzian para-Sasakian manifold is globally ϕ-Ricci symmetric if and only if it is an Einstein manifold. Also, we discuss the nature of associated 1-forms of a weakly ϕ-Ricci symmetric (ϵ)-Lorentzian para-Sasakian manifold.

2. Preliminaries

A differentiable manifold of dimension n is called an (ϵ)- Lorentzian para-Sasakian manifold (briefly, (ϵ)-LP-Sasakian), if it admits a (1, 1)-tensor field ϕ, a contravariant vector field ξ, a 1-form η and a Lorentzian metric g, which

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satisfies

$$\phi^2 X = X + \eta(X)\xi, \quad \eta(\xi) = -1, \tag{2.1}$$
$$g(\xi,\xi) = -\epsilon, \quad \eta(X) = \epsilon g(X,\xi), \quad \phi\xi = 0, \quad \eta(\phi X) = 0, \tag{2.2}$$
$$g(\phi X, \phi Y) = g(X,Y) + \epsilon\eta(X)\eta(Y) \tag{2.3}$$
$$(\nabla_X \phi)(Y) = g(X,Y)\xi + \epsilon\eta(Y)X + 2\epsilon\eta(X)\eta(Y)\xi \tag{2.4}$$

$$\nabla_X \xi = \epsilon\phi X, \tag{2.5}$$
$$(\nabla_X \eta)Y = g(\phi X, Y) \tag{2.6}$$

for arbitrary vector fields X and Y; where ∇ denotes the operator of covariant differentiation with respect to the Lorentzian metric g (see, [7, 8]).

On an n-dimensional (ϵ)-Lorentzian para-Sasakian manifold with structure (ϕ, ξ, η, g), the following results hold (see [8]):

$$R(X,Y)\xi = \eta(Y)X - \eta(X)Y, \tag{2.7}$$
$$R(\xi,X)Y = \epsilon g(X,Y)\xi - \eta(Y)X, \tag{2.8}$$
$$g(R(X,Y)Z,\xi) = g(Y,Z)\eta(X) - g(X,Z)\eta(Y) \tag{2.9}$$
$$S(\phi X, \phi Y) = S(X,Y) + (n-1)\eta(X)\eta(Y) \tag{2.10}$$
$$S(X,\xi) = (n-1)\eta(X), \tag{2.11}$$
$$Q\xi = \epsilon(n-1)\xi, \tag{2.12}$$

for any vector fields X, Y, Z; where R is the Riemannian curvature tensor, S is the Ricci tensor and Q is the Ricci operator given by $g(QX, Y) = S(X, Y)$.

We note that if $\epsilon = 1$ and the structure vector filed ξ is space like, then an (ϵ)-LP- Sasakian manifold is an usual LP-Sasakian manifold.

An (ϵ)-Lorentzian para-Sasakian manifold is said to be Einstein manifold if its Ricci tensor S is of the form

$$S(X,Y) = \lambda g(X,Y),$$

where λ is a constant.

3. Some results on (ϵ)-Lorentzian para-Sasakian manifolds

In this section, we present some typical identies for curvature tensor R and Ricci tensor S of an (ϵ)-Lorentzian para-Sasakian manifold. We start discussion with the following:

Let M be an (ϵ)- Lorentzian para-Sasakian manifold which is flat. Then from (2.9) we get

$$g(Y,Z)\eta(X) - g(X,Z)\eta(Y) = 0 \tag{3.1}$$

Replacing X by ϕX and Z by ϕZ in (3.1), we obtain

$$g(\phi X, \phi Z) = 0,$$

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for all vector fields X and Z, a contradiction. Hence, M cannot be flat. Thus we state:

Proposition 3.1. *An (ϵ)-Lorentzian para-Sasakian manifold cannot be flat.*

A non-flat pseudo-Riemannian manifold M will be recurrent [16] if its curvature tensor R satisfies the condition

$$(\nabla_W R)(X,Y,Z,U) = \alpha(W)R(X,Y,Z,U), \tag{3.2}$$

where α is a 1-form. If $\alpha = 0$, then the manifold becomes symmetric in the sense of Cartan [2]. We say that M is proper recurrent if $\alpha \neq 0$.

Let M be an (ϵ)- Lorentzian para-Sasakian manifold which is recurrent. Then from (3.2), (2.9) and (2.5) we obtain

$$\begin{aligned} \epsilon R(X,Y,Z,\phi W) &= g(Y,Z)[g(X,\phi W) - \alpha(W)\eta(X)] \\ &\quad -g(X,Z)[g(Y,\phi W) - \alpha(W)\eta(Y)] \end{aligned} \tag{3.3}$$

for any vector fields X, Y, Z, W. Putting $X = \xi$ in above equation, we get

$$\alpha(W)g(\phi Y, \phi Z) = 0$$

for any vector fields Y, Z, W. This is not possible. Hence, M cannot be proper recurrent. Thus we state

Proposition 3.2. *An (ϵ)- Lorentzian para-Sasakian manifold cannot be proper recurrent.*

Next, suppose that an (ϵ)- Lorentzian para-Sasakian manifold is symmetric. Then by putting $\alpha = 0$ in (3.3), we obtain

$$\begin{aligned} &\epsilon R(X,Y,Z,\phi W) \\ = \;& g(Y,Z)g(X,\phi W) - g(X,Z)g(Y,\phi W). \end{aligned} \tag{3.4}$$

Replacing W by ϕW in (3.4) and using (2.1) and (2.9) we get

$$R(X,Y,Z,W) = \epsilon[g(Y,Z)]g(X,W) - g(X,Z)g(Y,W)], \tag{3.5}$$

which shows that M is a space of constant curvature ϵ.

Conversely, if M is a space of constant curvature then obviously M is symmetric. This leads us to state the following result:

Proposition 3.3. *An (ϵ)- Lorentzian para-Sasakian manifold is symmetric if and only if it is of constant curvature ϵ.*

Apart from recurrent manifolds, semi-symmetric manifolds are another important natural generalization of symmetric manifolds. A pseudo-Riemannian manifold M will be semi-symmetric if its curvature tensor R satisfies the condition

$$R(X,Y) \cdot R = 0 \tag{3.6}$$

for all vector fields X and Y, where $R(X,Y)$ acts as a derivation on R.

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Symmetric manifolds are obviously semi-symmetric, but the converse need not be true. In fact, there exists examples of semi-symmetric manifolds which are not symmetric in dimension greater than two.

Let M be an (ϵ)- Lorentzian para-Sasakian manifold which is semi-symmetric. That is, the relation (3.6) holds. In particular, for $X = \xi$ in (3.6), we get

$$R(\xi, Y)R(U, V)\xi - R(R(\xi, Y)U, V)\xi - R(U, R(\xi, Y)V)\xi = 0, \tag{3.7}$$

which in view of (2.8) gives

$$\begin{aligned} &-g(Y, R(U, V)\xi)\xi + \eta(R(U, V)\xi)Y \\ &+g(Y, U)R(\xi, V)\xi - \eta(U)R(Y, V)\xi + g(Y, V)R(U, \xi)\xi \\ &-\eta(V)R(U, Y)\xi + \eta(Y)R(U, V)\xi - R(U, V)Y = 0. \end{aligned} \tag{3.8}$$

Using (2.7) in (3.8), we have

$$R(U, V)Y = \epsilon[g(V, Y)U - g(U, Y)V]. \tag{3.9}$$

Therefore, M is of constant curvature ϵ. Consequently, M is symmetric. Thus we state the following result:

Proposition 3.4. *On an (ϵ)- Lorentzian para-Sasakian manifold, the condition of semi-symmetry implies the condition of symmetry .*

In view of Theorems 3.3 and 3.4, we summarize the following:

Corollary 3.5. *Let M be an (ϵ)- Lorentzian para-Sasakian manifold. Then the following statements are equivalent:*

(1) *M is symmetric*
(2) *M is of constant curvature ϵ*
(3) *M is semi-symmetric.*

Next, an (ϵ)- Lorentzian para-Sasakian manifold M will be Ricci-recurrent if its Ricci tensor S satisfies the condition

$$(\nabla_W S)(X, Y) = \alpha(W)S(X, Y) \tag{3.10}$$

for all vector fields X, Y and W, where α is a 1-form. If $\alpha = 0$, then the manifold M becomes *Ricci symmetric*. We say that M is proper Ricci-recurrent, if $\alpha \neq 0$.

Let M be an (ϵ)- Lorentzian para-Sasakian manifold which is proper Ricci-recurrent. Then from (3.10) and (2.11) we have

$$(\nabla_W S)(X, \xi) = (n-1)\alpha(W)\eta(X). \tag{3.11}$$

On the other hand, by (2.11) and (2.5) we obtain

$$(\nabla_W S)(X, \xi) = (n-1)(\nabla_W \eta)(X) - \epsilon S(X, \phi W). \tag{3.12}$$

Hence, by (2.6) we get

$$(n-1)g(X, \phi W) - \epsilon S(X, \phi W) = (n-1)\alpha(W)\eta(X). \tag{3.13}$$

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Putting $X = \xi$ in the foregoing equation, we get $\alpha(W) = 0$, a contradiction. Therefore, M cannot be proper Ricci-recurrent. Hence, we are able to state the following result:

Proposition 3.6. *An (ϵ)- Lorentzian para-Sasakian manifold cannot be proper Ricci-recurrent.*

An (ϵ)- Lorentzian para-Sasakian manifold M will be Ricci-semi-symmetric if its Ricci tensor S satisfies the condition

$$R(X,Y) \cdot S = 0, \tag{3.14}$$

for all vector fields X, Y on M, where $R(X,Y)$ acts as a derivation on S. In this section, we prove the following:

Proposition 3.7. *For an (ϵ)- Lorentzian para-Sasakian manifold M, the following statements are equivalent:*

(1) *M is an Einstein manifold.*
(2) *M is Ricci-symmetric.*
(3) *M is Ricci-semi-symmetric.*

Proof: Obviously,(1) $\Rightarrow$ (2) and (1) $\Rightarrow$ (3).
Let (2) be true. Then putting $\alpha = 0$ in (3.13), we get

$$\epsilon S(X, \xi W) = (n-1)g(X, \phi W) \tag{3.15}$$

Replacing W by ϕW in (3.15), we get

$$S(X,W) = (n-1)\epsilon g(X,W), \tag{3.16}$$

which shows that the statement (1) is true.
At last, let (3) be true. In particular, from (3.14) we have

$$(R(\xi, Y) \cdot S)(U, \xi) = 0, \tag{3.17}$$

which implies that

$$S(R(\xi, Y)U, \xi) + S(U, R(\xi, Y)\xi) = 0. \tag{3.18}$$

In view of (2.8) and (2.11), relation (3.18) gives (3.16). This completes the proof.

4. Globally $\phi-$Ricci symmetric and Weakly $\phi-$Ricci symmetric (ϵ)-Lorentzian para-Sasakian manifolds

The notion of locally ϕ-symmetric Sasakian manifolds was introduced by T. Takahashi [18] as a weaker version of locally symmetric Sasakian manifolds. In [3], U. C. De and A. Sarkar introduced the notion of ϕ-Ricci symmetric Kemotsu manifolds. From the definition it follows that every ϕ-symmetric Sasakian manifold is ϕ-Ricci symmetric, but the converse is not true in general. In [13] and [5], ϕ-Ricci symmetric Kenmotsu manifolds and ϕ-Ricci symmetric

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(κ,μ)-contact metric manifolds are studied, respectively. Considering the above facts in this section we study globally ϕ-Ricci symmetric and weakly ϕ-Ricci symmetric (ϵ)-Lorentzian para-Sasakian manifolds.

An (ϵ)-Lorentzian para-Sasakian manifold M is said to be globally $\phi-$Ricci symmetric if the Ricci operator satisfies

$$\phi^2((\nabla_X Q)(Y)) = 0$$

for all vector fields X and Y and $S(X,Y) = g(QX,Y)$. If the above condition is satisfied for X, Y orthogonal to ξ, then the manifold M^n is said to be locally $\phi-$Ricci symmetric.

Let us suppose that the manifold M is $\phi-$Ricci symmetric. Then we have

$$\phi^2(\nabla_X Q)(Y) = 0. \tag{4.1}$$

Using (2.1) in the above equation, we get

$$(\nabla_X Q)(Y) + \eta((\nabla_X Q)(Y))\xi = 0. \tag{4.2}$$

From (4.2), it follows that

$$g((\nabla_X Q)(Y), Z) + \epsilon\eta((\nabla_X Q)(Y))\eta(Z) = 0, \tag{4.3}$$

which on simplifying gives

$$g(\nabla_X Q(Y), Z) - S(\nabla_X Y, Z) + \epsilon\eta((\nabla_X Q)(Y))\eta(Z) = 0. \tag{4.4}$$

Replacing Y by ξ in (4.4), we get

$$g(\nabla_X Q(\xi), Z) - S(\nabla_X \xi, Z) + \epsilon\eta((\nabla_X Q)(\xi))\eta(Z) = 0. \tag{4.5}$$

Using (2.5), (2.11) and (2.12) in (4.5), we obtain

$$(n-1)\epsilon g(\phi X, Z) - S(\phi X, Z) + \epsilon\eta((\nabla_X Q)(\xi))\eta(Z) = 0. \tag{4.6}$$

Replacing Z by ϕZ in (4.6), we have

$$S(\phi X, \phi Z) = (n-1)\epsilon g(\phi X, \phi Z). \tag{4.7}$$

In view of (2.3) and (2.10), (4.7) becomes

$$S(X, Z) = (n-1)\epsilon g(X, Z), \tag{4.8}$$

which shows that the manifold M is an Einstein manifold. Conversely, suppose that the manifold M is an Einstein manifold, then

$$S(X,Y) = \alpha g(X,Y),$$

where $S(X,Y) = g(QX,Y)$ and α is a constant. Hence $QX = \alpha X$. So, we have

$$\phi^2(\nabla_X Q)(Y) = 0.$$

Hence, we can state the following theorem:

Theorem 4.1. *An (ϵ)-Lorentzian para-Sasakain manifold is ϕ-Ricci symmetric if and only if it is an Einstein manifold.*

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Now, since a ϕ-symmetric manifold is ϕ-Ricci symmetric, we have

Corollary 4.2. *A ϕ-symmetric (ϵ)- Lorentzian para-Sasakian manifold is an Einstein manifold.*

Now, we introduce the notion of weakly $\phi-$Ricci symmetric (ϵ)- Lorentzian para-Sasakian manifold.

Definition 4.3. An (ϵ)- Lorentzian para-Sasakian manifold $M(n>2)$ is said to be weakly ϕ-Ricci symmetric if the non-zero Ricci operator Q satisfies the condition

$$\phi^2(\nabla_X Q)(Y) = A(X)Q(Y) + B(Y)Q(X) + g(QX,Y)\rho, \tag{4.9}$$

for any vector fields X and Y; where ρ is a vector field such that $g(\rho, V) = C(V)$, A, B and C are 1-forms not being simultaneously zero.

Let us consider a weakly $\phi-$Ricci symmetric (ϵ)- Lorentzian para-Sasakian manifold $M(n>2)$. Since the manifold is weakly $\phi-$Ricci symmetric, we have (4.9) which, by virtue of (2.1) yields

$$(\nabla_X Q)(Y) + \eta((\nabla_X Q)(Y))\xi = A(X)Q(Y) + B(Y)Q(X) + S(X,Y)\rho,$$

from which it follows that

$$\begin{aligned} & g(\nabla_X Q(Y), Z) - g(Q\nabla_X Y, Z) + \epsilon\eta((\nabla_X Q)(Y))\eta(Z) \\ = \;& A(X)S(Y,Z) + B(Y)S(X,Z) + S(X,Y)C(Z). \end{aligned} \tag{4.10}$$

Putting $Y=\xi$ in (4.10) and so using (2.5), (2.11) and (2.12) we have

$$\begin{aligned} & (n-1)g(\phi X, Z) + \epsilon S(\phi X, Z) \\ = \;& (n-1)\{A(X)\eta(Z) + \eta(X)C(Z)\} + B(\xi)S(X,Z). \end{aligned} \tag{4.11}$$

Setting $X=Z=\xi$ in (4.11) we obtain

$$A(\xi) + B(\xi) + C(\xi) = 0. \tag{4.12}$$

Claim: $A+B+C=0$ holds for all vector fields on M.
In deed, setting $Z=\xi$ in (4.11) we get

$$A(X) = (B(\xi) + C(\xi))\eta(X). \tag{4.13}$$

In view of (4.12), the relation (4.13) reduces to

$$A(X) = -A(\xi)\eta(X). \tag{4.14}$$

Again, taking $Z=\xi$ in (4.10) and so using (2.1) we obtain

$$0 = A(X)S(Y,\xi) + B(Y)S(X,\xi) + C(\xi)S(X,Y). \tag{4.15}$$

Plugging $X=\xi$ in the above equation and then using (2.11), it follows that

$$B(Y) = (A(\xi) + C(\xi))\eta(Y). \tag{4.16}$$

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In view of (4.12), the foregoing relation turns to

$$B(X) = -B(\xi)\eta(X). \tag{4.17}$$

Further, by putting $X = \xi$ in (4.11) we easily obtain

$$C(X) = -C(\xi)\eta(X). \tag{4.18}$$

Adding (4.14), (4.17), (4.18) and then using (4.12) we obtain

$$A(X) + B(X) + C(X) = 0 \tag{4.19}$$

for all X. Hence, we can state the following:

Theorem 4.4. *On a weakly ϕ-Ricci symmetric (ϵ)-Lorentzian para-Sasakian manifold $M(n > 2)$ the sum of associated 1-forms A, B, C is zero everywhere.*

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