

Bi-concave Functions Defined by Al-Oboudi Differential Operator

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ABSTRACT. The purpose of the present paper is to introduce a class $D_{\Sigma, \delta}^n C_0(\alpha)$ of bi-concave functions defined by Al-Oboudi differential operator. We find estimates on the Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ for functions in this class. Several consequences of these results are also pointed out in the form of corollaries.

Keywords: Bi-concave functions, Al-Oboudi differential operator, Coefficient estimates.

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1. INTRODUCTION

Let A indicate an analytic function family, which is normalized under the condition of $f(0) = f'(0) - 1 = 0$ in the open unit disk $\Delta = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$ and given by the following Taylor-Maclaurin series:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \quad (1.1)$$

Further, by S we shall denote the class of all functions in A which are univalent in Δ .

It is well known that every function $f \in S$ has an inverse f^{-1} , satisfying $f^{-1}(f(z)) = z$, ($z \in \Delta$) and $f(f^{-1}(w)) = w$, ($|w| < r_0(f)$; $r_0(f) \geq \frac{1}{4}$),

where

$$f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3) w^3 - (5a_2^3 - 5a_2 a_3 + a_4) w^4 + \dots$$

(for details, see Duren [13]). A function $f \in A$ is said to be bi-univalent in Δ if both f and f^{-1} are univalent in Δ . Let Σ stand for the class of bi-univalent functions defined in the unit disk Δ . For a brief history of functions in the class Σ , see [25] (see also [10, 11, 14, 17, 20, 26, 27]). More recently, Srivastava *et al.* [25], Altınkaya and Yalcın [3] made an effort to introduce various subclasses of the bi-univalent function class Σ and found non-sharp coefficient estimates on the initial coefficients $|a_2|$ and $|a_3|$ (see also [21, 15]). But determination of the bounds for the coefficients

$$|a_n|, \quad n \in \mathbb{N} \setminus \{1, 2\}; \quad \mathbb{N} = \{1, 2, 3, \dots\}$$

is still an open problem. In the literature, there are only a few works determining the general coefficient bounds $|a_n|$ for the analytic bi-univalent functions (see, for example [4, 16, 28]).

The study of operators plays an important role in Geometric Function Theory in Complex Analysis and its related fields (see, for example [2, 18, 19]). Recently, the interest in this area has been increasing because it permits detailed investigations of problems with physical applications. For $f \in A$, we consider the following differential operator introduced by Al-Oboudi [1],

$$D_\delta^0 f(z) = f(z),$$

$$D_\delta^1 f(z) = (1 - \delta)f(z) + \delta f'(z) \quad (\delta \geq 0),$$

$$\vdots$$

$$D_\delta^k f(z) = D_\delta(D_\delta^{k-1} f(z)) \quad (k \in \mathbb{N}).$$

Additionally, in view of (1.1), we deduce that

$$D_\delta^k f(z) = z + \sum_{n=2}^{\infty} [1 + (n-1)\delta]^k a_n z^n \quad (k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\})$$

with $D_\delta^k f(0) = 0$.

It is of interest to note that D_1^k is the Salagean's differential operator [23].

2. PRELIMINARIES

Conformal maps of the unit disk onto convex domains are a classical topic. Recently, Avkhadiev and Wirths [6] discovered that conformal maps onto concave domains (the complements of convex closed sets) have some novel properties.

A function $f : \Delta \rightarrow \mathbb{C}$ is said to belong to the family $C_0(\alpha)$ if f satisfies the following conditions:

- f is analytic in Δ with the standard normalization $f(0) = f'(0) - 1 = 0$. In addition it satisfies $f(1) = \infty$.
- f maps Δ conformally onto a set whose complement with respect to \mathbb{C} is convex.
- The opening angle of $f(\Delta)$ at ∞ is less than or equal to $\pi\alpha$, $\alpha \in (1, 2]$.

The class $C_0(\alpha)$ is referred to as the class of concave univalent functions and for a detailed discussion about concave functions, we refer to Avkhadiev et al. [7], Cruz and Pommerenke [12] and references there in.

In particular, the inequality

$$\Re \left(1 + \frac{zf''(z)}{f'(z)} \right) < 0 \quad (z \in \Delta)$$

is used - sometimes also as a definition - for concave functions $f \in C_0$ (see e.g. [22] and others).

Bhowmik et al. [9] showed that an analytic function f maps Δ onto a concave domain of angle $\pi\alpha$, if and only if $\Re(P_f(z)) > 0$, where

$$P_f(z) = \frac{2}{\alpha - 1} \left[\frac{\alpha + 1}{2} \frac{1+z}{1-z} - 1 - z \frac{f''(z)}{f'(z)} \right].$$

There has been a number of investigations on basic subclasses of concave univalent functions (see, for example [5], [8] and [24]).

Let us recall now the following definition required in sequel.

Definition 2.1. Let the functions $h, p : \Delta \rightarrow \mathbb{C}$ be so constrained that

$$\min \{ \Re(h(z)), \Re(p(z)) \} > 0$$

and

$$h(0) = p(0) = 1.$$

Motivated by each of the above definitions, we now define a new subclass of bi-concave analytic functions involving Al-Oboudi differential operator D_δ^k .

Definition 2.2. A function $f \in \Sigma$ given by (1.1) is said to be in the class

$$D_{\Sigma; \delta}^k C_0(\alpha) \quad (k \in \mathbb{N}_0, \delta \geq 0, \alpha \in (1, 2], z, w \in \Delta)$$

if the following conditions are satisfied:

$$\frac{2}{\alpha - 1} \left[\frac{\alpha + 1}{2} \frac{1+z}{1-z} - 1 - z \frac{[D_{\Sigma; \delta}^k f(z)]''}{[D_{\Sigma; \delta}^k f(z)]'} \right] \in h(\Delta) \quad (2.1)$$

and

$$\frac{2}{\alpha - 1} \left[\frac{\alpha + 1}{2} \frac{1-w}{1+w} - 1 - w \frac{[D_{\Sigma; \delta}^k g(w)]''}{[D_{\Sigma; \delta}^k g(w)]'} \right] \in p(\Delta), \quad (2.2)$$

where $g = f^{-1}$.

Remark 2.3. There are several choices of k and δ which would provide interesting subclasses of the class $D_{\Sigma, \delta}^k C_0(\alpha)$. For example,

(i) For $k = 0$, it can be directly verified that the functions h and p satisfy the hypotheses of Definition 2.1. Now if $f \in C_{\Sigma; 0}(\alpha)$ then

$$f \in \Sigma, \quad \frac{2}{\alpha - 1} \left[\frac{\alpha + 1}{2} \frac{1 + z}{1 - z} - 1 - z \frac{f''(z)}{f'(z)} \right] \in h(\Delta) \quad (z \in \Delta)$$

and

$$\frac{2}{\alpha - 1} \left[\frac{\alpha + 1}{2} \frac{1 + w}{1 - w} - 1 - w \frac{g''(w)}{g'(w)} \right] \in p(\Delta) \quad (w \in \Delta),$$

where $g = f^{-1}$.

(ii) For $\delta = 1$, it can be directly verified that the functions h and p satisfy the hypotheses of Definition 2.1. Now if $f \in D_{\Sigma}^k C_0(\alpha)$ then

$$f \in \Sigma, \quad \frac{2}{\alpha - 1} \left[\frac{\alpha + 1}{2} \frac{1 + z}{1 - z} - 1 - z \frac{[D_{\Sigma}^k f(z)]''}{[D_{\Sigma}^k f(z)]'} \right] \in h(\Delta) \quad (k \in \mathbb{N}_0, z \in \Delta)$$

and

$$\frac{2}{\alpha - 1} \left[\frac{\alpha + 1}{2} \frac{1 - w}{1 + w} - 1 - w \frac{[D_{\Sigma}^k g(w)]''}{[D_{\Sigma}^k g(w)]'} \right] \in p(\Delta) \quad (k \in \mathbb{N}_0, w \in \Delta),$$

where $g = f^{-1}$.

3. MAIN RESULTS AND THEIR CONSEQUENCES

We begin by finding the estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in the class $D_{\Sigma, \delta}^k C_0(\alpha)$.

Theorem 3.1. *Let f given by (1.1) be in the class $D_{\Sigma, \delta}^k C_0(\alpha)$. Then*

$$|a_2| \leq \min \left\{ \sqrt{\frac{(\alpha+1)^2}{4(1+\delta)^{2k}} + \frac{(\alpha-1)^2(|h'(0)|^2 + |p'(0)|^2)}{32(1+\delta)^{2k}} + \frac{(\alpha^2-1)(|h'(0)| + |p'(0)|)}{8(1+\delta)^{2k}}}, \right. \\ \left. \sqrt{\frac{(\alpha-1)(|h''(0)| + |p''(0)|)}{16|2(1+\delta)^{2k} - 3(1+2\delta)^k|} + \frac{(\alpha+1)}{2|2(1+\delta)^{2k} - 3(1+2\delta)^k|}} \right\} \quad (3.1)$$

and

$$|a_3| \leq \min \left\{ \frac{8(\alpha+1)^2 + (\alpha-1)^2(|h'(0)|^2 + |p'(0)|^2)}{32(1+\delta)^{2k}} + \frac{(\alpha^2-1)(|h'(0)| + |p'(0)|)}{8(1+\delta)^{2k}} + \frac{(\alpha-1)(|h''(0)| + |p''(0)|)}{48(1+2\delta)^k}, \right. \\ \left. \frac{|3(\alpha-1)(1+2\delta)^k - (\alpha-1)(1+\delta)^{2k}||h''(0)| + (\alpha-1)(1+\delta)^{2k}|p''(0)|}{24(1+\delta)^{2k}|2(1+\delta)^{2k} - 3(1+2\delta)^k|} + \frac{\alpha+1}{2|2(1+\delta)^{2k} - 3(1+2\delta)^k|} \right\}. \quad (3.2)$$

Proof. Let $f \in D_{\Sigma; \delta}^k C_0(\alpha)$ and g be the analytic extension of f^{-1} to Δ . It follows from (2.1) and (2.2) that

$$\frac{2}{\alpha - 1} \left[\frac{\alpha + 1}{2} \frac{1 + z}{1 - z} - 1 - z \frac{[D_{\Sigma; \delta}^k f(z)]''}{[D_{\Sigma; \delta}^k f(z)]'} \right] = h(z) \quad (3.3)$$

and

$$\frac{2}{\alpha - 1} \left[\frac{\alpha + 1}{2} \frac{1 - w}{1 + w} - 1 - w \frac{[D_{\Sigma; \delta}^k g(w)]''}{[D_{\Sigma; \delta}^k g(w)]'} \right] = p(w), \quad (3.4)$$

where h and p satisfy the conditions of Definition 2.1. Furthermore, the functions $h(z)$ and $p(w)$ have the following Taylor-Maclaurin series expansions:

$$h(z) = 1 + h_1 z + h_2 z^2 + \dots$$

and

$$p(w) = 1 + p_1 w + p_2 w^2 + \dots,$$

respectively. Now, equating the coefficients in (3.3) and (3.4), we get

$$\frac{2 [(\alpha + 1) - 2(1 + \delta)^k a_2]}{\alpha - 1} = h_1, \quad (3.5)$$

$$\frac{2 [(\alpha + 1) + 4(1 + \delta)^{2k} a_2^2 - 6(1 + 2\delta)^k a_3]}{\alpha - 1} = h_2, \quad (3.6)$$

$$-\frac{2 [(\alpha + 1) - 2(1 + \delta)^k a_2]}{\alpha - 1} = p_1, \quad (3.7)$$

$$\frac{2 [(\alpha + 1) + 4(1 + \delta)^{2k} a_2^2 - 6(1 + 2\delta)^k (2a_2^2 - a_3)]}{\alpha - 1} = p_2. \quad (3.8)$$

From (3.5) and (3.7), we find that

$$h_1 = -p_1. \quad (3.9)$$

Also, from (3.5), we can write

$$a_2 = \frac{\alpha + 1}{2(1 + \delta)^k} - \frac{h_1(\alpha - 1)}{4(1 + \delta)^k}. \quad (3.10)$$

Next, by using (3.5), (3.7), (3.9) and (3.10), we get

$$a_2^2 = \frac{(\alpha + 1)^2}{4(1 + \delta)^{2k}} + \frac{(\alpha - 1)^2 (h_1^2 + p_1^2)}{32(1 + \delta)^{2k}} - \frac{(\alpha^2 - 1)(h_1 - p_1)}{8(1 + \delta)^{2k}}. \quad (3.11)$$

By adding (3.6) to (3.8), we get

$$a_2^2 = \frac{(\alpha - 1)(h_2 + p_2)}{8[2(1 + \delta)^{2k} - 3(1 + 2\delta)^k]} - \frac{\alpha + 1}{2[2(1 + \delta)^{2k} - 3(1 + 2\delta)^k]}. \quad (3.12)$$

Therefore, we find from the equations (3.11) and (3.12) that

$$|a_2|^2 \leq \frac{(\alpha + 1)^2}{4(1 + \delta)^{2k}} + \frac{(\alpha - 1)^2 (|h'(0)|^2 + |p'(0)|^2)}{32(1 + \delta)^{2k}} + \frac{(\alpha^2 - 1)(|h'(0)| + |p'(0)|)}{8(1 + \delta)^{2k}}$$

and

$$|a_2|^2 \leq \frac{(\alpha-1)(|h''(0)|+|p''(0)|)}{16|2(1+\delta)^{2k}-3(1+2\delta)^k|} + \frac{(\alpha+1)}{2|2(1+\delta)^{2k}-3(1+2\delta)^k|}.$$

Similarly, subtracting (3.8) from (3.6), we have

$$a_3 = a_2^2 - \frac{(\alpha-1)(h_2-p_2)}{24(1+2\delta)^k}. \quad (3.13)$$

Then, upon substituting the value of in view of a_2^2 from (3.11) and (3.12) into (3.13), it follows that

$$a_3 = \frac{(\alpha+1)^2}{4(1+\delta)^{2k}} + \frac{(\alpha-1)^2(h_1^2+p_1^2)}{32(1+\delta)^{2k}} - \frac{(\alpha^2-1)(h_1-p_1)}{8(1+\delta)^{2k}} - \frac{(\alpha-1)(h_2-p_2)}{24(1+2\delta)^k}$$

and

$$a_3 = \frac{(\alpha-1)(h_2+p_2)}{8[2(1+\delta)^{2k}-3(1+2\delta)^k]} - \frac{\alpha+1}{2[2(1+\delta)^{2k}-3(1+2\delta)^k]} - \frac{(\alpha-1)(h_2-p_2)}{24(1+2\delta)^k}.$$

Consequently, we have

$$|a_3| \leq \frac{8(\alpha+1)^2+(\alpha-1)^2(|h'(0)|^2+|p'(0)|^2)}{32(1+\delta)^{2k}} + \frac{(\alpha^2-1)(|h'(0)|+|p'(0)|)}{8(1+\delta)^{2k}} + \frac{(\alpha-1)(|h''(0)|+|p''(0)|)}{48(1+2\delta)^k}$$

and

$$|a_3| \leq \frac{|3(\alpha-1)(1+2\delta)^k-(\alpha-1)(1+\delta)^{2k}||h''(0)|+(\alpha-1)(1+\delta)^{2k}|p''(0)|}{24(1+\delta)^{2k}|2(1+\delta)^{2k}-3(1+2\delta)^k|} + \frac{\alpha+1}{2|2(1+\delta)^{2k}-3(1+2\delta)^k|}.$$

This completes the proof of the theorem. \square

It is easily seen that, by specializing the functions h and p involved in the Theorem, several coefficient estimates can be obtained as special cases.

Corollary 3.2. *If we set*

$$h(z) = \left(\frac{1+z}{1-z}\right)^\gamma = 1 + 2\gamma z + 2\gamma^2 z^2 + \dots \quad (0 < \gamma \leq 1),$$

$$p(z) = \left(\frac{1-z}{1+z}\right)^\gamma = 1 - 2\gamma z + 2\gamma^2 z^2 + \dots \quad (0 < \gamma \leq 1),$$

then inequalities (3.1) and (3.2) become

$$|a_2| \leq \min \left\{ \sqrt{\frac{(\alpha+1)^2+(\alpha-1)^2\gamma^2+2(\alpha^2-1)\gamma}{4(1+\delta)^{2k}}}, \sqrt{\frac{(\alpha+1)+(\alpha-1)\gamma^2}{2|2(1+\delta)^{2k}-3(1+2\delta)^k|}} \right\}$$

and

$$|a_3| \leq \min \left\{ \frac{(\alpha+1)^2+(\alpha-1)^2\gamma^2+2(\alpha^2-1)\gamma}{4(1+\delta)^{2k}} + \frac{(\alpha-1)\gamma^2}{6(1+2\delta)^k}, \right. \\ \left. \frac{|3(\alpha-1)(1+2\delta)^k-(\alpha-1)(1+\delta)^{2k}|\gamma^2+(\alpha-1)(1+\delta)^{2k}\gamma^2}{6(1+\delta)^{2k}|2(1+\delta)^{2k}-3(1+2\delta)^k|} + \frac{\alpha+1}{2|2(1+\delta)^{2k}-3(1+2\delta)^k|} \right\}.$$

Corollary 3.3. *If we let*

$$h(z) = \frac{1 + (1 - 2\beta)z}{1 - z} = 1 + 2(1 - \beta)z + 2(1 - \beta)z^2 + \dots \quad (0 \leq \beta < 1),$$

$$p(z) = \frac{1 - (1 - 2\beta)z}{1 + z} = 1 - 2(1 - \beta)z + 2(1 - \beta)z^2 + \dots \quad (0 \leq \beta < 1),$$

then inequalities (3.1) and (3.2) become

$$|a_2| \leq \min \left\{ \sqrt{\frac{(\alpha+1)^2 + (\alpha-1)^2(1-\beta)^2 + 2(\alpha^2-1)(1-\beta)}{4(1+\delta)^{2k}}}, \sqrt{\frac{(\alpha+1) + (\alpha-1)(1-\beta)}{2|2(1+\delta)^{2k} - 3(1+2\delta)^k|}} \right\}$$

and

$$|a_3| \leq \min \left\{ \frac{(\alpha+1)^2 + (\alpha-1)^2(1-\beta)^2 + 2(\alpha^2-1)(1-\beta)}{4(1+\delta)^{2k}} + \frac{(\alpha-1)(1-\beta)}{6(1+2\delta)^k}, \right. \\ \left. \frac{|3(\alpha-1)(1+2\delta)^k - (\alpha-1)(1+\delta)^{2k}|(1-\beta) + (\alpha-1)(1+\delta)^{2k}(1-\beta)}{6(1+\delta)^{2k}|2(1+\delta)^{2k} - 3(1+2\delta)^k|} + \frac{\alpha+1}{2|2(1+\delta)^{2k} - 3(1+2\delta)^k|} \right\}.$$

Theorem 3.4. *Let f given by (1.1) be in the class $C_{\Sigma;0}(\alpha)$. Then*

$$|a_2| \leq \min \left\{ \sqrt{\frac{(\alpha+1)^2}{4} + \frac{(\alpha-1)^2(|h'(0)|^2 + |p'(0)|^2)}{32} + \frac{(\alpha^2-1)(|h'(0)| + |p'(0)|)}{8}}, \right. \\ \left. \sqrt{\frac{(\alpha-1)(|h''(0)| + |p''(0)|)}{16} + \frac{(\alpha+1)}{2}} \right\} \quad (3.14)$$

and

$$|a_3| \leq \min \left\{ \frac{8(\alpha+1)^2 + (\alpha-1)^2(|h'(0)|^2 + |p'(0)|^2)}{32} + \frac{(\alpha^2-1)(|h'(0)| + |p'(0)|)}{8} + \frac{(\alpha-1)(|h''(0)| + |p''(0)|)}{48}, \right. \\ \left. \frac{|3(\alpha-1)(1+2\delta)^n - (\alpha-1)(1+\delta)^{2n}||h''(0)| + (\alpha-1)(1+\delta)^{2n}|p''(0)|}{24} + \frac{\alpha+1}{2} \right\}. \quad (3.15)$$

Corollary 3.5. *If we set*

$$h(z) = \left(\frac{1+z}{1-z} \right)^\gamma = 1 + 2\gamma z + 2\gamma^2 z^2 + \dots \quad (0 < \gamma \leq 1),$$

$$p(z) = \left(\frac{1-z}{1+z} \right)^\gamma = 1 - 2\gamma z + 2\gamma^2 z^2 + \dots \quad (0 < \gamma \leq 1),$$

then inequalities (3.14) and (3.15) become

$$|a_2| \leq \min \left\{ \sqrt{\frac{(\alpha+1)^2 + (\alpha-1)^2\gamma^2 + 2(\alpha^2-1)\gamma}{4}}, \sqrt{\frac{(\alpha+1) + (\alpha-1)\gamma^2}{2}} \right\}$$

and

$$|a_3| \leq \min \left\{ \frac{(\alpha+1)^2 + (\alpha-1)^2 \gamma^2 + 2(\alpha^2-1)\gamma}{4} + \frac{(\alpha-1)\gamma^2}{6}, \right. \\ \left. \frac{\gamma^2(\alpha-1) + (\alpha+1)}{2} \right\}.$$

Corollary 3.6. *If we let*

$$h(z) = \frac{1 + (1-2\beta)z}{1-z} = 1 + 2(1-\beta)z + 2(1-\beta)z^2 + \dots \quad (0 \leq \beta < 1),$$

$$p(z) = \frac{1 - (1-2\beta)z}{1+z} = 1 - 2(1-\beta)z + 2(1-\beta)z^2 + \dots \quad (0 \leq \beta < 1),$$

then inequalities (3.14) and (3.15) become

$$|a_2| \leq \min \left\{ \sqrt{\frac{(\alpha+1)^2 + (\alpha-1)^2(1-\beta)^2 + 2(\alpha^2-1)(1-\beta)}{4}}, \sqrt{\frac{(\alpha+1) + (\alpha-1)(1-\beta)}{2}} \right\}$$

and

$$|a_3| \leq \min \left\{ \frac{(\alpha+1)^2 + (\alpha-1)^2(1-\beta)^2 + 2(\alpha^2-1)(1-\beta)}{4} + \frac{(\alpha-1)(1-\beta)}{6}, \right. \\ \left. \frac{(1-\beta)(\alpha-1) + (\alpha+1)}{2} \right\}.$$

Theorem 3.7. *Let f given by (1.1) be in the class $D_{\Sigma}^{\alpha}C_0(\alpha)$. Then*

$$|a_2| \leq \min \left\{ \sqrt{\frac{(\alpha+1)^2}{2^{2k+2}} + \frac{(\alpha-1)^2(|h'(0)|^2 + |p'(0)|^2)}{2^{2k+5}} + \frac{(\alpha^2-1)(|h'(0)| + |p'(0)|)}{2^{2k+3}}}, \right. \\ \left. \sqrt{\frac{(\alpha-1)(|h''(0)| + |p''(0)|)}{16(3^{k+1} - 2^{2k+1})} + \frac{(\alpha+1)}{2(3^{k+1} - 2^{2k+1})}} \right\} \quad (3.16)$$

and

$$|a_3| \leq \min \left\{ \frac{8(\alpha+1)^2 + (\alpha-1)^2(|h'(0)|^2 + |p'(0)|^2)}{2^{2k+5}} + \frac{(\alpha^2-1)(|h'(0)| + |p'(0)|)}{2^{2k+3}} + \frac{(\alpha-1)(|h''(0)| + |p''(0)|)}{16 \cdot 3^{k+1}}, \right. \\ \left. \frac{(\alpha-1)(3^{k+1} - 2^{2k})|h''(0)| + (\alpha-1)2^{2k}|p''(0)|}{3 \cdot 2^{2k+3}(3^{k+1} - 2^{2k+1})} + \frac{\alpha+1}{2(3^{k+1} - 2^{2k+1})} \right\}. \quad (3.17)$$

Corollary 3.8. *If we set*

$$h(z) = \left(\frac{1+z}{1-z} \right)^{\gamma} = 1 + 2\gamma z + 2\gamma^2 z^2 + \dots \quad (0 < \gamma \leq 1),$$

$$p(z) = \left(\frac{1-z}{1+z} \right)^{\gamma} = 1 - 2\gamma z + 2\gamma^2 z^2 + \dots \quad (0 < \gamma \leq 1),$$

then inequalities (3.16) and (3.17) become

$$|a_2| \leq \min \left\{ \sqrt{\frac{(\alpha+1)^2 + (\alpha-1)^2 \gamma^2 + 2(\alpha^2-1)\gamma}{2^{2k+2}}}, \sqrt{\frac{(\alpha+1) + (\alpha-1)\gamma^2}{2(3^{k+1}-2^{2k+1})}} \right\}$$

and

$$|a_3| \leq \min \left\{ \frac{(\alpha+1)^2 + (\alpha-1)^2 \gamma^2 + 2(\alpha^2-1)\gamma}{2^{2k+2}} + \frac{(\alpha-1)\gamma^2}{2 \cdot 3^{k+1}}, \right. \\ \left. \frac{(\alpha-1)(3^{k+1}-2^{2k})\gamma^2 + (\alpha-1)2^{2k}\gamma^2}{3 \cdot 2^{2k+1}(3^{k+1}-2^{2k+1})} + \frac{\alpha+1}{2(3^{k+1}-2^{2k+1})} \right\}.$$

Corollary 3.9. If we let

$$h(z) = \frac{1 + (1-2\beta)z}{1-z} = 1 + 2(1-\beta)z + 2(1-\beta)z^2 + \dots \quad (0 \leq \beta < 1),$$

$$p(z) = \frac{1 - (1-2\beta)z}{1+z} = 1 - 2(1-\beta)z + 2(1-\beta)z^2 + \dots \quad (0 \leq \beta < 1),$$

then inequalities (3.16) and (3.17) become

$$|a_2| \leq \min \left\{ \sqrt{\frac{(\alpha+1)^2 + (\alpha-1)^2(1-\beta)^2 + 2(\alpha^2-1)(1-\beta)}{2^{2k+2}}}, \sqrt{\frac{(\alpha+1) + (\alpha-1)(1-\beta)}{2(3^{k+1}-2^{2k+1})}} \right\}$$

and

$$|a_3| \leq \min \left\{ \frac{(\alpha+1)^2 + (\alpha-1)^2(1-\beta)^2 + 2(\alpha^2-1)(1-\beta)}{2^{2k+2}} + \frac{(\alpha-1)(1-\beta)}{2 \cdot 3^{k+1}}, \right. \\ \left. \frac{(\alpha-1)(3^{k+1}-2^{2k})(1-\beta) + (\alpha-1)2^{2k}(1-\beta)}{3 \cdot 2^{2k+1}(3^{k+1}-2^{2k+1})} + \frac{\alpha+1}{2(3^{k+1}-2^{2k+1})} \right\}.$$

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