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Coefficient Estimates for a New Subclasses of m-fold Symmetric Bi-Univalent Functions

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Abstract. The purpose of the present paper is to introduce two new subclasses of the function class Σ_m of bi-univalent functions which both f and f^{-1} are m-fold symmetric analytic functions. Furthermore, we obtain estimates on the initial coefficients for functions in each of these new subclasses. Also we explain the relation between our results with earlier known results.

Keywords: Analytic function, Univalent function, m-fold symmetric bi-univalent functions.

2000 Mathematics subject classification: 30C45, 30C50.

1. Introduction

Let A denote the class of the functions f of the form

$$f(z) = \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the open unit disk $U = \{z \in C : |z| < 1\}$ and satisfy the normalization condition $f(0) = f^{'}(0) - 1 = 0$. Let S be the subclass of A consisting of functions of the form (1.1) which are also univalent in U.

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It is well known that every $f \in S$ has an inverse f^{-1}, defined by

$$f^{-1}(f(z)) = z, (z \in U)$$

and

$$f^{-1}(f(\omega)) = \omega, (|\omega| < r_0(f), r_0(f) \geq \frac{1}{4}),$$

where

$$f^{-1}(\omega) = \omega - a_2\omega^2 + (2a_2^2 - a_3)\omega^3 - (5a_2^3 - 5a_2a_3 + a_4)\omega^4 + \cdots \qquad (1.2)$$

A function $f \in A$ is said to be bi-univalent in U if both f and f^{-1} are univalent in U . Let Σ denote the class of bi-univalent in U given by the Taylor-Maclaurin series expansion (1.1).

For each function $f \in S$, the function

$$h(z) = (f(z^m))^{\frac{1}{m}}, (z \in U; m \in N) \qquad (1.3)$$

is univalent and maps the unit disk U into a region with m-fold symmetric. A function is said to be m-fold symmetric (see [6,8]) if it has the following normalized form:

$$f(z) = z + \sum_{k=1}^{\infty} a_{mk+1} z^{mk+1}, (z \in U; m \in N) \qquad (1.4)$$

Let S_m denote the class of m-fold symmetric univalent functions in U,which are normalized by the series expansion (1.4), It may be worth nothing that the functions in the class S are said to be one-fold symmetric and each bi-univalent function generates an m-fold symmetric univalent function for each integer $m \in N$.

In 2014 Srivastava et al. [12] proved the normalized from of f is given as in (1.4) and the series expansion for f^{-1} is given as follows:

$$g(\omega) = \omega - a_{m+1}\omega^{m+1} + [(m+1)a_{m+1}^2 - a_{2m+1}]\omega^{2m+1}$$
$$-[\frac{1}{2}(m+1)(3m+2)a_{m+1}^3 - (3m+2)a_{m+1}a_{2m+1} + a_{3m+1}]\omega^{3m+1} + \cdots , \qquad (1.5)$$

where $f^{-1} = g$.

Let Σ_m denote the class of m-fold symmetric univalent functions in U , we note that for $m = 1$, formula (1.5) coincide with formula (1.2).

Brannan and Taha [5] introduced certain subclasses of the bi-univalent function class Σ similar to the familiar subclasses $S^*(\beta)$ and $K(\beta)$ of starlike and convex function of order $\beta (0 \leq \beta < 1)$ respectively. The classes of $S^*_\Sigma(\alpha)$ and $K_\Sigma(\alpha)$ of bi-starlike functions of order α, corresponding to the function classes $S^*(\alpha)$ and $K(\alpha)$, were also introduced analogously. For each of the function classes $S^*_\Sigma(\alpha)$ and $K_\Sigma(\alpha)$, they found non-sharp estimates on the initial coefficients. In recent years, the work of Srivastava et al. [11] essentially revived the investigation of various subclasses of the bi-univalent function class Σ. Recently, many authors investigated bounds for various subclasses of bi-univalent functions like, (see

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[1, 2, 3, 7, 10]).
In this paper, we introduced two new subclasses of the function class Σ of bi-univalent functions which both f and f^{-1} are m-fold symmetric analytic functions. Furthermore, we obtain estimates on the initial coefficients $|a_{m+1}|$ and $|a_{2m+1}|$ for functions in each of these new subclasses. Also we explain the relation between our results with earlier known results.
In order to prove our main results, we need to following lemma due to [9]

Lemma 1.1. *If $h \in P$, then $|t_k| \leq 2$ for each k, where P is the family of all functions h analytic in U for which $Re\{h(z)\} > 0$,*

$$h(z) = 1 + t_1 z + t_2 z^2 + t_3 z^3 + \cdots for\ z \in U.$$

2. MEAN RESULTS

2.1. Coefficient estimates of the function class $K_{\Sigma_m}(\lambda, \alpha)$.

Definition 2.1. A function f given by (1.4) is said to be in the class $K_{\Sigma_m}(\lambda, \alpha)$ if the following conditions are satisfied:

$$f \in \Sigma_m \ and \ |arg((1-\lambda)f^{'}(z) + \lambda(1 + (\frac{zf^{''}(z)}{f^{'}(z)}))| < \frac{\alpha\pi}{2}, \tag{2.1}$$
$$(0 \leq \lambda < 2, 0 < \alpha \leq 1, z \in U)$$

.

$$f \in \Sigma_m \ and \ |arg((1-\lambda)g^{'}(\omega) + \lambda(1 + (\frac{zg^{''}(\omega)}{g^{'}(\omega)}))| < \frac{\alpha\pi}{2}, \tag{2.2}$$
$$(0 \leq \lambda < 2, 0 < \alpha \leq 1, \omega \in U),$$

where the function g is given by (1.5).

We note that for $m = 1$, $\lambda = 0$ the class $K_{\Sigma_m}(\lambda, \alpha)$ reduces to the class $H_\Sigma(\alpha)$ introduced and study by Srivastsva et al. [12] and for $m = 1$, $\lambda = 1$ the class $K_{\Sigma_m}(\lambda, \alpha)$ reduces to the class $K_\Sigma(\alpha)$ introduced and study by Brannan and Taha [7].
In our first theorem, we finding the estimates on the initial coefficient $|a_{m+1}|$ and $|a_{2m+1}|$ for function in the class $K_{\Sigma_m}(\lambda, \alpha)$.

Theorem 2.2. *Let the function f given by (1.4) be in the class $K_{\Sigma_m}(\lambda, \alpha), 0 \leq \lambda < 2,$ and $0 < \alpha \leq 1$. Then*

$$|a_{m+1}| \leq \frac{2\sqrt{\alpha}}{\sqrt{(m+1)}} \times \frac{1}{\sqrt{[[1+\lambda(2m-1)](2m+1) - 2\lambda m(m+1)] - (\alpha-1)(m+1)[1+\lambda(m-1)]^2}}, \tag{2.3}$$

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and

$$|a_{2m+1}| \leq \frac{2(m+1)\alpha^2}{([1+\lambda(m-1)](m+1))^2} + \frac{2\alpha}{[1+(2m-1)](2m+1)}. \tag{2.4}$$

Proof. It follows from (2.1) and (2.2) that

$$((1-\lambda)f^{'}(z) + \lambda(1 + (\frac{zf^{''}(z)}{f^{'}(z)})) = [p(z)]^{\alpha}, \tag{2.5}$$

and

$$((1-\lambda)g^{'}(w) + \lambda(1 + (\frac{wg^{''}(w)}{g^{'}(w)})) = [q(w)]^{\alpha}, \tag{2.6}$$

where $p(z)$ and $q()$ in P and have the forms

$$p(z) = 1 + p_m z^m + p_{2m}z^{2m} + p_{3m}z^{3m} + p_{4m}z^{4m} + \cdots, \tag{2.7}$$

and

$$q(w) = 1 + q_m w^m + q_{2m}w^{2m} + q_{3m}w^{3m} + q_{4m}w^{4m} + \cdots. \tag{2.8}$$

Now, equating the coefficient in (2.5) and (2.6), we obtain

$$[1+\lambda(m-1)](m+1)a_{m+1} = \alpha p_m, \tag{2.9}$$

$$[1+\lambda(2m-1)](2m+1)a_{2m+1} - \lambda m(m+1)^2 a_{m+1}^2 = \alpha p_{2m} + 1/2\alpha(\alpha-1)p_m^2, \tag{2.10}$$

$$-[1+\lambda(m-1)](m+1)a_{m+1} = \alpha q_m, \tag{2.11}$$

and

$$[1+\lambda(2m-1)](2m+1)[(m+1)a_(^2m+1) - a_{2m+1}] - \lambda m(m+1)^2 a_{m+1}^2$$
$$= \alpha q_{2m} + 1/2\alpha(\alpha-1)q_m^2. \tag{2.12}$$

From (2.9) and (2.11), we obtain

$$p_m = -q_m, \tag{2.13}$$

and

$$2([1+\lambda(m-1)](m+1))^2 a_{m+1}^2 = \alpha^2(p_m^2 + q_m^2). \tag{2.14}$$

Now from (2.10), (2.12) and (2.14), we obtain

$$[\alpha(m+1)[[1+\lambda(2m-1)](2m+1) - 2\lambda m(m+1)] - \alpha(\alpha-1)([1+\lambda(m-1)](m+1))^2]$$
$$a_{m+1}^2 = \alpha^2(p_m^2 + q_m^2).$$

Then, by applying Lemma 1.1 for the coefficients p_{2m} and q_{2m} in last equation, we obtain the desired bound for $|a_{m+1}|$ as asserted in (2.3).

Next, in order to find the bound on $|a_{2m+1}|$ by subtracting (2.12) from (2.10), we obtain

$$2[1+\lambda(2m-1)](2m+1)a_{2m+1} - [1+\lambda(2m-1)](2m+1)(m+1)a_{m+1}^2$$
$$= \alpha(p_{2m} - q_{2m}) + \frac{1}{2}\alpha(\alpha-1)(p_m^2 - q_m^2). \tag{2.15}$$

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It follows from (2.13), (2.14) and (2,15) that

$$|a_{2m+1}| = \frac{(m+1)\alpha^2(p_m^2+q_m^2)}{4([1+\lambda(m-1)](m+1))^2} + \frac{2\alpha}{2[1+\lambda(2m-1)](2m+1)}.$$

Then, applying Lemma 1.1 once again for the coefficients p_m ,p_{2m} ,q_m and q_{2m} in last equation, we obtain the desired bound for $|a_{2m+1}|$ as asserted in (2.4). This completes the proof of Theorem 2.2 . □

2.2. **Coefficient estimates of the function class** $H_{\Sigma_m}(\lambda, \beta)$**.**

Definition 2.3. A function f given by (1.4) is said to be in the class $H_{\Sigma_m}(\lambda, \beta)$ if the following conditions are satisfied:

$$f \in \Sigma_m \text{ and } Re\{(1-\lambda)f^{'}(z) + \lambda(1+(\frac{zf^{''}(z)}{f^{'}(z)}))\} > \beta, \tag{2.16}$$

$$(0 \le \lambda < 2, 0 < \beta \le 1, z \in U)$$

.

$$f \in \Sigma_m \text{ and } Re\{(1-\lambda)g^{'}(w) + \lambda(1+(\frac{wg^{''}(w)}{g^{'}(w)}))\} > \beta, \tag{2.17}$$

$$(0 \le \lambda < 2, 0 < \beta \le 1, w \in U),$$

where the function g is given by (1.5).

We note that for $m = 1$, the class $H_{\Sigma_m}(\lambda, \beta)$ reduces to the class $S_\Sigma(\lambda, \beta)$ introduced and study by Azizi et al. [4] and for $m = 1$, $\lambda = 0$ and $m = 1$, $\lambda = 1$ the class $H_{\Sigma_m}(\lambda, \beta)$ reduces to the class $H_\Sigma(\beta)$ and $K_\Sigma(\beta)$ studied by Srivastsva et al. [11] and Brannan and Taha [5], respectively.

Theorem 2.4. *Let the function f given by (1.4) be in the class $H_{\Sigma_m}(\lambda, \beta), 0 \le \lambda < 2$, and $0 < \beta \le 1$. Then*

$$|a_{m+1}| \le \sqrt{\frac{4(1-\beta)}{(m+1)[[1+\lambda(2m-1)](2m+1) - 2\lambda m(m+1)]}}, \tag{2.18}$$

and

$$|a_{2m+1}| \le \frac{2(m+1)(1-\beta)^2}{([1+\lambda(m-1)](m+1))^2} + \frac{2(1-\beta)}{[1+(2m-1)](2m+1)}. \tag{2.19}$$

Proof. It follows from (2.16) and (2.17) that there exists $p(z) \in P$ and $q(z) \in P$ such that

$$((1-\lambda)f^{'}(z) + \lambda(1+(\frac{zf^{''}(z)}{f^{'}(z)})) = \beta + (1-\beta)p(z), \tag{2.20}$$

and

$$((1-\lambda)g^{'}(w) + \lambda(1+(\frac{wg^{''}(w)}{g^{'}(w)})) = \beta + (1-\beta)q(z), \tag{2.21}$$

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where $p(z)$ and $q(w)$ have the forms (2.7) and (2.8), respectively. Equating the coefficient in (2.20) and (2.21) yields

$$[1+\lambda(m-1)](m+1)a_{m+1} = (1-\beta)p_m, \tag{2.22}$$

$$[1+\lambda(2m-1)](2m+1)a_{2m+1} - \lambda m(m+1)^2 a_{m+1}^2 = (1-\beta)p_{2m}, \tag{2.23}$$

$$-[1+\lambda(m-1)](m+1)a_{m+1} = (1-\beta)q_m, \tag{2.24}$$

and

$$[1+\lambda(2m-1)](2m+1)[(m+1)a_(^2m+1) - a_{2m+1}] - \lambda m(m+1)^2 a_{m+1}^2$$
$$= (1-\beta)q_{2m}. \tag{2.25}$$

From (2.22) and (2.24), we obtain

$$p_m = -q_m, \tag{2.26}$$

and

$$2([1+\lambda(m-1)](m+1))^2 a_{m+1}^2 = (1-\beta)^2(p_m^2 + q_m^2). \tag{2.27}$$

Also, from (2.23) and (2.25), we find that

$$[(m+1)[[1+\lambda(2m-1)](2m+1) - 2\lambda m(m+1)]]a_{m+1}^2 = (1-\beta)(p_{2m}+q_{2m}),$$

Then, by applying Lemma 1.1 for the coefficients p_{2m} and q_{2m} in last equation, we obtain the desired bound for $|a_{m+1}|$ as asserted in (2.18).
Next, in order to find the bound on $|a_{2m+1}|$ by subtracting (2.25) from (2.23), we obtain

$$2[1+\lambda(2m-1)](2m+1)a_{2m+1} - [1+\lambda(2m-1)](2m+1)(m+1)a_{m+1}^2$$
$$= (1-\beta)(p_{2m}-q_{2m}),$$

or, equivalent

$$a_{2m+1} = \frac{(m+1)a_{m+1}^2}{2} + \frac{(1-\beta)(p_{2m}-q_{2m})}{2[1+\lambda(2m-1)](2m+1)}.$$

Upon substituting the value of a_{m+1}^2 from (2.27), we have

$$a_{2m+1} = \frac{(m+1)(1-\beta)^2(p_m^2+q_m^2)}{4([1+\lambda(m-1)](m+1))^2} + \frac{(1-\beta)(p_{2m}-q_{2m})}{2[1+\lambda(2m-1)](2m+1)}.$$

Then, applying Lemma 1.1 once again for the coefficients p_m ,p_{2m} ,q_m and q_{2m} in last equation, we obtain the desired bound for $|a_{2m+1}|$ as asserted in (2.19). This completes the proof of Theorem 2.4 . □

Remark 2.5. If we put $m = 1$ in Theorem 2.4, we obtain the corresponding results due to Azizi et al. [4].

Remark 2.6. If we put $m = 1, \lambda = 0$ in Theorem 2.2 and Theorem 2.4, we obtain the corresponding results due to Srivastsva et al. [11].

Remark 2.7. If we put $m = 1, \lambda = 1$ in Theorem 2.2 and Theorem 2.4, we obtain the corresponding results due to Brannan and Taha [5].

Remark 2.8. For all coefficients of functions belonging to the classes investigated in this paper sharp estimates for it are yet open problems.

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