

## Groups whose Bipartite Divisor Graph for Character Degrees Has Five Vertices

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**ABSTRACT.** Let  $G$  be a finite group and  $\text{cd}^*(G)$  be the set of nonlinear irreducible character degrees of  $G$ . Suppose that  $\rho(G)$  denotes the set of primes dividing some element of  $\text{cd}^*(G)$ . The bipartite divisor graph for the set of character degrees which is denoted by  $B(G)$ , is a bipartite graph whose vertices are the disjoint union of  $\rho(G)$  and  $\text{cd}^*(G)$ , and a vertex  $p \in \rho(G)$  is connected to a vertex  $a \in \text{cd}^*(G)$  if and only if  $p|a$ . In this paper, we investigate the structure of a group  $G$  whose graph  $B(G)$  has five vertices. Especially we show that all these groups are solvable.

**Keywords:** Bipartite divisor graph, Character degree, Solvable group.

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### 1. INTRODUCTION

Throughout this paper,  $G$  is a finite group. We write  $\text{cd}(G)$  to denote the set of irreducible character degrees of the group  $G$ , and we use  $\text{cd}^*(G)$  for the set  $\text{cd}(G) \setminus \{1\}$ . Suppose that  $\rho(G)$  is the set of primes dividing some element of  $\text{cd}^*(G)$ . Exploring the interplay between the structure of a finite group  $G$  and the set  $\text{cd}(G)$  is a favorite research field in group theory. One of the questions that was studied extensively is the graphs attached to the set  $\text{cd}(G)$ . A comprehensive survey on this topic can be found in [6]. The prime graph  $\Delta(G)$  and the common divisor graph  $\Gamma(G)$  are two important graphs associated to  $\text{cd}(G)$ . The prime graph  $\Delta(G)$  is the graph with vertex set  $\rho(G)$

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and there is an edge between two vertices  $p$  and  $q$  if there exist some  $n \in \text{cd}(G)$  which is divisible by  $pq$ . The common-divisor graph  $\Gamma(G)$  is the graph with vertex set  $\text{cd}^*(G)$  and two vertices  $m$  and  $n$  are connected if  $\gcd(m, n) > 1$ . In this paper we focus on bipartite divisor graph for  $\text{cd}^*(G)$ . The bipartite divisor graph  $B(G)$  is a bipartite graph whose vertices are the disjoint union of  $\rho(G)$  and  $\text{cd}^*(G)$  and a vertex  $p \in \rho(G)$  is connected to a vertex  $a \in \text{cd}^*(G)$  if and only if  $p|a$ . Groups whose  $\Delta(G)$  or  $\Gamma(G)$  has few vertices has been studied by many authors. For example, the prime graphs with four or fewer vertices are considered in papers [4, 8, 9]. In this note, we do an analogous work for bipartite divisor graph. The notion of  $B(G)$  is introduced in [11] and groups whose  $B(G)$  is a path or cycle are discussed in [2]. In this paper, we consider groups whose bipartite divisor graph has five vertices and obtain some group theoretical properties of these groups. We also provide examples of each possible graph.

## 2. PRELIMINARIES

The following theorems will be used throughout the paper.

**Theorem 2.1** (corollary 12.34 of [5]). *Let  $G$  be solvable. Then  $G$  has a normal abelian Sylow  $p$ -subgroup iff every element of  $\text{cd}(G)$  is relatively prime to  $p$ .*

**Theorem 2.2** (corollary 12.2 of [5]). *Suppose  $p|\chi(1)$  for every nonlinear  $\chi \in \text{Irr}(G)$ , where  $p$  is a prime. Then  $G$  has a normal  $p$ -complement.*

**Theorem 2.3** (Theorem 4.5 of [11]). *Let  $G$  be a group whose  $B(G)$  is a complete bipartite graph. Then one of the following cases occurs:*

- (a)  $G = AH$ , where  $A$  is an abelian normal Hall subgroup of  $G$  and  $H$  is abelian, i.e.  $G$  is metabelian.
- (b)  $G = AH$ , where  $A$  is an abelian normal Hall subgroup of  $G$  and  $H$  is a non-abelian  $p$ -group for some prime  $p$ . In particular,  $\rho(G) = \{p\}$ .

*Remark 2.4.* Theorem 2.3 implies that the subgroup  $H$  is a  $\rho(G)$ -subgroup and  $A$  is a  $\rho(G)'$ -subgroup of  $G$ .

The following theorem from [12], helps us to obtain some examples of groups with a given set of character degrees.

**Theorem 2.5** (Theorem 4.1 of [12]). *Let  $1 < m_1 < \dots < m_r$  be integers such that  $m_i$  divides  $m_{i+1}$  for all  $i = 1, 2, \dots, r-1$ . Then there exists a group  $G$  with  $\text{cd}(G) = \{1, m_1, \dots, m_r\}$ .*

We also use the library of the small groups in GAP [1] for many examples and the  $k$ th group of order  $n$  in this library is recognizable by command `SmallGroup( $n, k$ )` which is the symbol we use for this group.

3. GROUPS WHOSE  $B(G)$  HAS FIVE VERTICES

By definition of  $\rho(G)$  and  $\text{cd}^*(G)$ , each  $p \in \rho(G)$  divides some element in  $\text{cd}^*(G)$  and every element  $n \in \text{cd}^*(G)$  is divisible by a prime number which lies in  $\rho(G)$ . Therefore, the graph  $B(G)$  has no isolated vertex. Let  $G$  be a group whose graph  $B(G)$  has five vertices. Using the fact that vertices of  $B(G)$  are disjoint union of  $\rho(G)$  and  $\text{cd}^*(G)$ , one of the following cases occur:

- (i)  $|\rho(G)| = 1$  and  $\text{cd}^*(G) = 4$ .
- (ii)  $|\rho(G)| = 4$  and  $\text{cd}^*(G) = 1$ .
- (iii)  $|\rho(G)| = 2$  and  $\text{cd}^*(G) = 3$ .
- (iv)  $|\rho(G)| = 3$  and  $\text{cd}^*(G) = 2$ .

We investigate each case separately and determine the structure of groups of each possible case.

**Theorem 3.1.** *Let  $G$  be a group whose  $B(G)$  has five vertices. If  $|\rho(G)| = 1$  then  $G = AP$ , where  $P$  is a Sylow  $p$ -subgroup for some prime  $p$  and  $A$  is a normal abelian  $p$ -complement.*

*Proof.* By hypothesis,  $|\rho(G)| = 1$ , hence  $\text{cd}(G) = \{1, p^a, p^b, p^c, p^d\}$  where  $p$  is a prime number. Since  $p$  divides the degree of every character, by Theorem 2.2, we conclude that  $G$  has a normal  $p$ -complement. Therefore  $G = AP$  where  $P$  is a Sylow  $p$ -subgroup of  $G$  and  $A$  is a normal  $p$ -complement. In addition, every prime divisor of  $|A|$  is coprime to  $p$  and by corollary 12.34 of [5],  $A$  is abelian and we are done.  $\square$

**Theorem 3.2.** *Let  $G$  be a group whose  $B(G)$  has five vertices. If  $|\rho(G)| = 4$  then  $G'$  is abelian,  $G' \cap Z(G) = 1$  and  $G/Z(G)$  is a Frobenius group with cyclic complement.*

*Proof.* Since  $B(G)$  has five vertices and  $|\rho(G)| = 4$ , therefore  $\text{cd}(G) = \{1, m\}$  where  $m$  is divisible by exactly four prime numbers. By corollary 12.6 of [5],  $G'$  is abelian. Assume that  $G$  is nilpotent. Since  $|\text{cd}(G)| = 2$ ,  $G$  is nonabelian. Let  $P$  be a nonabelian Sylow  $p$ -subgroup of  $G$ . If  $G$  has another nonabelian Sylow  $q$ -subgroup for some prime  $q \neq p$ , then  $|\text{cd}(G)| > 2$  which is a contradiction. Hence  $P$  is the only nonabelian Sylow subgroup of  $G$  which implies that every element of  $\text{cd}(G)$  must be a prime power which is a contradiction. Therefore  $G$  is not nilpotent. Now by using Theorem (C) of [3], we obtain the results.  $\square$

**EXAMPLE 3.3.** Groups that satisfy hypothesis of Theorems 3.1 and 3.2 exist. For example, let  $P$  be a  $p$ -group of order  $p^3$  with  $\text{cd}(P) = \{1, p\}$  and let  $G_1 = P \times P \times P \times P$ , then

$$\text{cd}(G_1) = \{1, p, p^2, p^3, p^4\}.$$

Therefore  $G$  is a group that satisfy hypothesis of Theorem 3.1. Furthermore, if we replace  $P$  by the direct product of an abelian group  $A$  and the group  $P$ ,

then we have an example of Theorem 3.1 with non-trivial subgroup  $A$ . For a group which satisfy Theorem 3.2, suppose that  $F = GF(211)$  is a finite field with 211 elements and let  $H$  be the multiplicative group of  $F$  with 210 elements. Then  $H$  act Frobeniously on  $F$  and so the corresponding semidirect product  $G_2 = FH$  is a Frobenius group with abelian kernel and complement. It is easy to check that  $\text{cd}(G_2) = \{1, 210\}$  and the results of Theorem 3.2 hold.

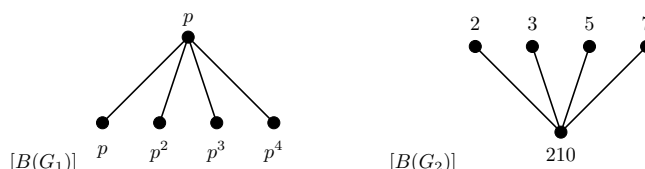


FIGURE 1. Graphs of example 3.3.

**Theorem 3.4.** *Let  $G$  be a group whose  $B(G)$  has five vertices. If  $|\rho(G)| = 2$  then  $G$  is solvable and one of the following cases occurs:*

- (i)  $G = HN$  where  $H$  is a Sylow  $p$ -subgroup of  $G$  or a Hall  $\{p, q\}$ -subgroup and  $N$  is a normal complement.
- (ii)  $G$  is one of the families stated in [7].

*Proof.* Since  $B(G)$  has no isolated vertex, it is easy to check that  $B(G)$  is one of the graphs in Figure 2.

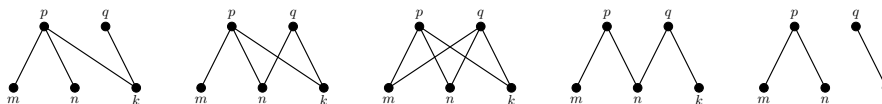


FIGURE 2. Possible graphs with  $|\rho(G)| = 2$ .

First, suppose that  $B(G)$  is as graph (a) or (b) in Figure 2 where  $p$  and  $q$  are prime numbers and  $\text{cd}(G) = \{1, m, n, k\}$ . Since  $p$  divides every nonlinear character degree of  $G$ , by Theorem 2.2,  $G$  has a normal  $p$ -complement, therefore  $G = PN$  where  $P$  is a Sylow  $p$ -subgroup of  $G$  and  $N$  is a normal  $p$ -complement and case (i) occurs. We claim that in both cases  $G$  is solvable. Suppose that  $G$  is not solvable. Note that in both graphs (a) and (b) of Figure 2 there is a prime which divides every nonlinear character degree. Since  $|\text{cd}(G)| = 4$ , using Theorem A and B of [10] we have  $\text{cd}(G) = \{1, r-1, r, r+1\}$  for some prime power  $r$  or  $\text{cd}(G) = \{1, 9, 10, 16\}$ . In both cases there is no prime which divides every nonlinear character degree, thus graphs (a) and (b) can not occur

for these groups. Therefore  $G$  is a solvable group. If  $B(G)$  is as graph (c) in Figure 2, then  $B(G)$  is a complete bipartite graph and by Theorem 2.3 case (i) holds. Furthermore, by corollary 4.2 of [11],  $G$  is solvable. If  $B(G)$  is as graph (d) in Figure 2, then  $B(G)$  is a path of length four and by proposition 2 of [2],  $G$  is a solvable group. Since for every prime  $r \neq p, q$ ,  $r$  divides no character degree, using Theorem 2.1, we see that case (i) occurs. Now suppose that  $B(G)$  is as graph (e) in Figure 2. Therefore  $B(G)$  is a disconnected graph and has two connected components. We prove that  $G$  is solvable. Assume that  $G$  is not solvable. Again by Theorem A and B of [10], we must have  $\text{cd}(G) = \{1, r-1, r, r+1\}$  for some prime power  $r$ . Since  $\text{cd}(G) = \{1, p^a, p^b, q^c\}$  it follows that  $p$  divides two consecutive numbers which is impossible. Hence  $G$  is solvable and by Theorem 2.1 of [11],  $G$  belongs to a family of groups stated in [7] and case (ii) holds.  $\square$

The following example shows that all graphs in Figure 2 occur as  $B(G)$  for some group  $G$ .

EXAMPLE 3.5. Let  $G_1 = \text{SmallGroup}(108, 17)$ , then  $\text{cd}(G_1) = \{1, 2, 4, 6\}$  and  $B(G_1)$  is the same as graph (a) in Figure 2.

By Theorem 2.5, there is a group  $G_2$  with  $\text{cd}(G_2) = \{1, 2, 6, 12\}$  and  $B(G_2)$  is as graph (b) in Figure 2.

Suppose that  $G_3 = \text{SmallGroup}(108, 17)$ , then  $\text{cd}(G_3) = \{1, 6, 12, 18\}$  and  $B(G_3)$  is the graph (c) in Figure 2.

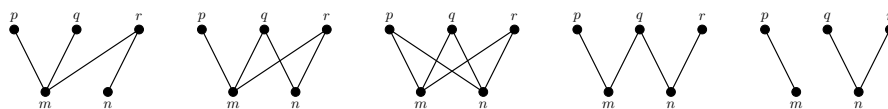
Assume that  $G_4 = \text{SmallGroup}(72, 15)$ , then we have  $\text{cd}(G_4) = \{1, 2, 3, 6\}$  and  $B(G_4)$  is the graph (d) in Figure 2.

Put  $G_5 = \text{SmallGroup}(48, 28)$ , then  $\text{cd}(G_5) = \{1, 2, 3, 4\}$  and  $B(G_5)$  is the graph (e) in Figure 2.

**Theorem 3.6.** *Let  $G$  be a group whose  $B(G)$  has five vertices. If  $|\rho(G)| = 3$  then  $G$  is solvable and one of the following cases holds:*

- (i)  $G = HN$  where  $H$  is a Sylow  $p$ -subgroup or a Hall  $\{p, q\}$ -subgroup or a Hall abelian  $\{p, q, r\}$ -subgroup of  $G$  and  $N$  is its normal complement.
- (ii)  $G = QN$  where  $Q$  is an abelian Sylow  $q$ -subgroup of  $G$  and  $N$  is its normal complement.
- (iii)  $G$  is one of the families stated in [7].

*Proof.* It's easy to verify that  $B(G)$  is one of the graphs in Figure 3. Since  $|\text{cd}(G)| = 3$  Theorem 12.15 of [5] shows that in all cases  $G$  is a solvable group. Now we investigate each graph separately.

FIGURE 3. Possible graphs with  $|\rho(G)| = 3$ .

First suppose that  $B(G)$  is as graph (a) in Figure 3. Then  $\text{cd}(G) = \{1, p^a q^b r^c, r^d\}$  and  $r$  divides every nonlinear character degree. Thus Theorem 2.2 implies that  $G$  has a normal  $r$ -complement. Therefore  $G = HN$  where  $H$  is a Sylow  $r$ -subgroup of  $G$  and  $N$  is a normal  $r$ -complement and case (i) of theorem holds. Now assume that  $B(G)$  is the graph (b) in Figure 3. In this case, we have  $\text{cd}(G) = \{1, p^a q^b r^c, q^d r^h\}$  and every nonlinear character degree is divisible by both  $q$  and  $r$ . Again Theorem 2.1 applies and  $G = HN$  which  $H$  is a Hall  $\{r, q\}$ -subgroup and  $N$  is its normal complement. Hence case (i) occurs. If  $B(G)$  is the graph (c) in Figure 3, then  $B(G)$  is a complete graph. Applying Theorem 2.3, we have  $G = HN$  where  $H$  is a Hall abelian  $\rho(G)$ -subgroup and  $N$  is an abelian normal complement of  $H$ . Therefore again case (i) of theorem holds. Now suppose that  $B(G)$  is the graph (d) in Figure 3. Then  $\text{cd}(G) = \{1, p^a q^b, q^c r^d\}$ . Since  $q$  divides every nonlinear character degree,  $G$  has a normal  $q$ -complement. Therefore,  $G = QN$  where  $Q$  is a Sylow  $q$ -subgroup and  $N$  is its normal complement. Since  $Q \cong G/N$  and  $\text{cd}(G)$  contains no powers of  $q$ , therefore  $Q$  is abelian. Hence case (ii) of theorem holds. Finally, suppose that  $B(G)$  is the graph (e) in Figure 3. Then  $B(G)$  is disconnected and has two connected components. Since  $G$  is solvable, Theorem 2.1 of [11] implies that  $G$  belongs to a family of groups stated in [7] and case (iii) holds.  $\square$

EXAMPLE 3.7. We show that all graphs in Figure 3 really occur as  $B(G)$  for some group  $G$ .

By Theorem 2.5 there exists a group  $G_1$  with  $\text{cd}(G_1) = \{1, 5, 30\}$  and the graph  $B(G_1)$  is the graph (a) in Figure 3.

Since  $15|30|60$ , Theorem 2.5 implies that there is a group  $G_2$  with  $\text{cd}(G_2) = \{1, 15, 30\}$  and there is a group  $G_3$  with  $\text{cd}(G) = \{1, 30, 60\}$ . Thus the graphs  $B(G_2)$  and  $B(G_3)$  are the graphs (b) and (c) in Figure 3, respectively.

Suppose that  $G_4 = \text{SmallGroup}(960, 5748)$ , then  $\text{cd}(G_4) = \{1, 12, 15\}$  and  $B(G_4)$  is the graph (d) in Figure 3.

Let  $G_5 = \text{SmallGroup}(480, 1188)$ , then  $\text{cd}(G_5) = \{1, 2, 15\}$  and the graph  $B(G_5)$  is the graph (e) in Figure 3.

Suppose that  $G$  is a group which satisfy hypothesis of Theorem 3.6, then  $\text{cd}(G) = \{1, m, n\}$ . If we apply Theorems of [12], then we can obtain more information about the structure of  $G$ , depending on the relation between  $\pi(m)$

and  $\pi(n)$ , where  $\pi(l)$  denotes the prime divisors of  $l$ . For details see [12].

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