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Homomorphisms on Topological Groups from the Perspective of Bourbaki-boundedness

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Abstract. In this note we study some topological properties of bounded sets and Bourbaki-bounded sets. Also we introduce two types of Bourbaki-bounded homomorphisms on topological groups including, n−Bourbaki-bounded homomorphisms and B−Bourbaki-bounded homomorphisms. We compare them to each other and with the class of continuous homomorphisms. So, two topologies are presented on them and we determine some properties on domain and range spaces led to Bourbaki-completeness some of these classes of homomorphisms with the given topologies. At the end of this note we focus on n-compact homomorphisms and B-compact homomorphisms briefly.

Keywords: Topological group, Bounded set, Bourbaki-bounded set, Bourbaki-bounded homomorphism, Continuous homomorphism, Compact homomorphism, Bourbaki-completeness.

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1. Introduction

Boundedness is one of the most important tools in the study of metric spaces. According to this, it has been tried to develop this concept on topological spaces which are similar to the metric spaces such as uniform spaces and topological groups. Boundedness in uniform spaces and topological groups can not be defined as boundedness in metric spaces. So, some new boundedness such as totally boundedness and Bourbaki-boundedness are defined in the metric spaces which can be generalized to uniform spaces and topological groups. Totally bounded subsets and Bourbaki-bounded subsets of metric spaces are bounded. While the reverse inclusions are not true in general. For instance if we consider in $\mathbb{R}$ the $0-1$ discrete metric d, then every subset is bounded but only the fnite ones are totally bounded and Bourbaki-bounded. Morever the class of all totally bounded subsets and the class of all Bourbaki-bounded subsets of uniform spaces and topological groups are bornology. In recent years many research has been done on totally boundedness and Bourbaki-boundedness in uniform spaces and topological groups $[2, 5, 6, 7, 9, 11, 12, 13, 14, 15]$. Atkin [2] has proved that Bourbaki-boundedness in topological vector spaces is equivalent to the Von Neumann-boundedness. Troitsky [18] and Hejazian et al. [11] have studied some classes of operators on topological vector spaces from the perspective of Von Neumann-boundedness. Kocinac and Zabeti [17] have tried to discuss similar concepts on topological groups.

In this note we investigate some topological properties of group bounded sets and Bourbaki-bounded sets. Also we introduce two types of homomorphisms on topological groups, G and H, from the perspective of Bourbaki-boundedness, including, n$-$Bourbaki-bounded homomorphisms and B$-$Bourbaki-bounded homomorphisms. We compare them to each other and with the class of continuous homomorphisms. So, two topologies are presented on homomorphisms between the topological groups G and H. We determine some properties on domain and range spaces led to Bourbaki-completeness some of these classes of homomorphisms with the given topologies. At the end of this note we focus on n-compact homomorphisms and B-compact homomorphisms briefly.

Recall that, a subset B of a topological group G is called *Bourbaki-bounded* if for each neighborhood U of the identity element e_G there exist a positive integer n and a finite collection of points $x_1, x_2, ..., x_k$ in the group G such that $B \subset \bigcup_{i=1}^{k} x_i U^n$. In this definition, if $n = 1$, then B is called *totally bounded* and if $k = 1$, then B is called *gruop bounded*.

All topological groups in this note are assumed to be Hausdorff and Abelian.

2. Group bounded sets and Bourbaki$-$bounded sets

In this section we investigate some topological properties of group bounded sets and Bourbaki-bounded sets.

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Totally boundedness and Bourbaki-boundedness alone do not imply group boundedness. For example, finite sets with more than one member in discrete topological groups are totally bounded and so are Bourbaki-bounded but they are not group bounded. In this respect, the following proposition can be expressed:

Proposition 2.1. *Any Bourbaki-bounded, connected topological group is group bounded. In particular, any totally bounded, connected topological group is group bounded*

Proof. Let G be a Bourbaki-bounded, connected topological group, and let U be a neighborhood of the identity element e_G. Choose symmetric neighborhood V of e_G with $V^2 \subset U$. Then by the assumed Bourbaki-boundedness of G, there exists a positive integer n and a finite set $A \subset G$ such that $G = AU^n$, where $AU^n = \bigcup_{x\in A} xU^n$. If $\{A_1, A_2\}$ is any partition of A ($A_1 \neq \emptyset, A_2 \neq \emptyset$), then by the assumed connectedness, $A_1V^n \bigcap A_2V^n \neq \emptyset$. Now, we prove the group boundedness of G. For this purpose, let x, y be two arbitrary points of G and x be a fix point. Without loss of generality, let $x \in x_1V^n$, where $x_1 \in A$. If $y \in x_1V^n$ then

$$yx^{-1} \in x_1V^n(x_1V^n)^{-1} = x_1x_1^{-1}V^{2n} = V^{2n} \subset U^n \Rightarrow y \in xU^n.$$

If $y \notin x_1V^n$, choose the point x_2 in A such that $x_2 \neq x_1$ and $x_1V^n \bigcap x_2V^n \neq \emptyset$. Choose $z_1 \in x_1V^n \bigcap x_2V^n$. If $y \in x_2V^n$ then

$$yx^{-1} = yz_1^{-1}z_1x_1^{-1}x_1x^{-1} \in U^{3n} \Rightarrow y \in xU^{3n}.$$

If $y \notin x_2V^n$, choose the point x_3 in A such that $x_3 \neq x_1$, $x_3 \neq x_2$ and $(x_1V^n \bigcup x_2V^n) \bigcap x_3V^n \neq \emptyset$. Choose $z_2 \in (x_1V^n \bigcup x_2V^n) \bigcap x_3V^n$. By continuing this process, we obtain a sequence $y, z_1, x_1, z_2, ...x_m, x$ of point such that

$$yx^{-1} = yz_1^{-1}z_1x_1^{-1}x_1z_2^{-1}...\, x_kx^{-1} \in U^{(k+2)n} \Rightarrow y \in xU^{(k+2)n}.$$

This implies the group boundedness of G. □

In general, every compact subset C of a topological group G is not group bounded. According to the following proposition, it's necessary that C must be a component of some $x \in G$.

Proposition 2.2. *Let x be an arbitrary point of topological group G and C_x be it's component in G. If C_x be compact, then it is group bounded.*

Proof. According to [8, p.163], the component C_x of x in topological group G is as $C_x = \bigcap C_{x,U}$, where U is a neighborhood of e_G and $C_{x,U} = \bigcup_{n=1}^{\infty} xU^n$. Since for every natural number n, $U^n \subseteq U^{n+1}$ and C_x is campact, there exists a natural number k such that $C_x \subset xU^k$ and so it is group bounded. □

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Proposition 2.3. *Let G be a topological group and H is a subgroup of G. If H and G/H are Bourbaki-bounded, then G is also.*

Proof. Let U be an arbitrary neighborhood of the identity element e_G. Put $V := U \cap H$. Bourbaki-boundedness of H implies that there exist a positive integer m and a finite collection of points $x_1, x_2, ..., x_l \in H$ such that $H \subset \bigcup_{i=1}^{l} x_i V^m$. Also Bourbaki-boundedness of G/H implies that there exist a positive integer n and a finite collection of points $[y_1], [y_2], ..., [y_t] \in G/H$ such that $G/H \subset \bigcup_{j=1}^{t} [y_i](G/H)^n$. Now, let x be an arbitrary element in G. If $x \in H$, then there exist $1 \leq i \leq l$ such that

$$x \in x_i V^m \subseteq x_i U^m \subseteq x_i U^{m+n}.$$

If $x \notin H$, then

$$[x] = xH \in G/H \subset \bigcup_{j=1}^{t} [y_i](U/H)^n$$

and so $xH \in [y_j](U/H)^n$ for some $1 \leq j \leq t$ and therefore, there exist $u_1, u_2, ..., u_n \in U$ such that

$$xH = y_j H.u_1.u_2...u_n.H = y_j.u_1.u_2...u_n H$$

and so

$$\begin{aligned} x \in y_j U^n H &\subseteq y_j U^n \bigcup_{i=1}^{l} x_i V^m \\ &= \bigcup_{i=1}^{l} x_i y_j U^n V^m \\ &\subseteq \bigcup_{i=1}^{l} x_i y_j U^n U^m \\ &= \bigcup_{i=1}^{l} x_i y_j U^{m+n} \end{aligned}$$

□

3. Bourbaki−bounded homomorphisms

In this section we introduce two types of homomorphisms on topological groups from the perspective of Bourbaki-boundedness. We compare them to each other and with the class of continuous homomorphisms.

Definition 3.1. A homomorphism T from a topological group G to a topological group H is said to be

(1) $n-$*Bourbaki-bounded* if there exists a neighborhood U of e_G such that $T(U)$ is Bourbaki-bounded in H.
(2) $B-$*Bourbaki-bounded* if for every Bourbaki-bounded set $B \subset G$, the set $T(B)$ is Bourbaki-bounded in H.

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The class of all n−Bourbaki-bounded homomorphisms (B−Bourbaki-bounded homomorphisms) from a topological group G to a topological group H is denoted by $Hom_{nB}(G,H)$ ($Hom_{BB}(G,H)$). Also we denote by $Hom_c(G,H)$ the class of all continuous homomorphisms from a topological group G to a topological group H.

Proposition 3.2. *For topological groups G and H the following holds:*

$$Hom_{nB}(G,H) \subset Hom_{BB}(G,H).$$

Proof. Let $T : G \to H$ be an n−Bourbaki-bounded homomorphism. Then it is B−Bourbaki-bounded. Suppose B is a Bourbaki-bounded set in G. We prove that $T(B)$ is Bourbaki-bounded in H. Since T is n−Bourbaki-bounded, there exists a neighborhood U of e_G such that $T(U)$ is Bourbaki-bounded in H. Bourbaki-boundedness of B implies that $B \subset \bigcup_{i=1}^{k} x_i U^n$ for some positive integer n and a finite collection of points $x_1, x_2, ..., x_k \in G$. Now let V be an arbitrary neighborhood of e_H. Bourbaki-boundedness of $T(U)$ implies that there exist a positive integer m and a finite collection of points $y_1, y_2, ..., y_j \in H$ such that $T(U) \subset \bigcup_{i=1}^{j} y_i V^m$, and therefore

$$\begin{aligned} T(B) \subset T(\bigcup_{i=1}^{k} x_i U^n) &= \bigcup_{i=1}^{k} T(x_i)T(U)^n \\ &\subset \bigcup_{i=1}^{k} T(x_i)(\bigcup_{i=1}^{j} y_i V^m)^n \\ &= \bigcup_{i=1}^{k} (T(x_i) \bigcup_{i_1,...,i_n=1}^{j} y_{i_1}...y_{i_n} V^{mn}) \\ &= \bigcup_{i=1}^{k} \bigcup_{i_1,...,i_n=1}^{j} T(x_i) y_{i_1}...y_{i_n} V^{mn}. \end{aligned}$$

This implies that $T(B)$ is Bourbaki-bounded set in H. □

Proposition 3.3. *For topological groups G and H the following holds:*

$$Hom_c(G,H) \subset Hom_{BB}(G,H).$$

Proof. Let $T : G \to H$ be a continuous homomorphism. Then it is B−Bourbaki-bounded. Suppose B is a Bourbaki-bounded set in G. We prove that $T(B)$ is Bourbaki-bounded in H. Let V be an arbitrary neighborhood of e_H. There exists a neighborhood U of e_G such that $T(U) \subset V$. Bourbaki-boundedness of B implies that $B \subset \bigcup_{i=1}^{k} x_i U^n$ for some positive integer n and a finite collection of points $x_1, x_2, ..., x_k \in G$. Therefore

$$T(B) \subset T(\bigcup_{i=1}^{k} x_i U^n) = \bigcup_{i=1}^{k} T(x_i)T(U)^n \subset \bigcup_{i=1}^{k} T(x_i) V^n.$$

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This implies that $T(B)$ is Bourbaki-bounded set in H. □

Corollary 3.4. *Let B be a Bourbaki-bounded subset of topological group G and H is a subgroup of G. Then B/H is Bourbaki-bounded subset of G/H.*

Recall that, a topological group G is said to be *locally Bourbaki-bounded* if there exists a Bourbaki-bounded neighborhood of e_G.

Note that, if G is a locally Bourbaki-bounded topological group, then $Hom_{nB}(G,H) = Hom_{BB}(G,H)$.
The inclution $Hom_{nB}(G,H) \subset Hom_{BB}(G,H)$ was expressed in proposition 3.2. To prove the inclution $Hom_{BB}(G,H) \subset Hom_{nB}(G,H)$, let $T : G \to H$ be a B−Bourbaki-bounded homomorphism. Consider U as a Bourbaki-bounded neighborhood of e_G. Then $T(U)$ is Bourbaki-bounded in H and so T is n−Bourbaki-bounded.

Also note, the topological group G is locally Bourbaki-bounded if and only if the identity homomorphism $i : G \to G$ is n−Bourbaki-bounded. For, let G be a locally Bourbaki-bounded topological group. Consider U as a Bourbaki-bounded neighborhood of e_G. Then $i(U) = U$ is Bourbaki-bounded in G and so i is n−Bourbaki-bounded. Conversely, if the identity homomorphism $i : G \to G$ is n−Bourbaki-bounded, then there exists a Bourbaki-bounded neighborhood U of e_G such that $i(U) = U$ is Bourbaki-bounded in G and so the topological group G is locally Bourbaki-bounded.

Note that the converse of proposition 3.2 is not true as the following example shows.

Example 3.5. Let G be an infinite countable topological group and H be a locally Bourbaki-bounded but not Bourbaki-bounded topological group. Consider the topological group $Hom(G,H)$ with the topology of pointwise convergence and with the operation of pointwise multiplication such that $Hom(G,H)$ is an infinite group. Since the topology of pointwise convergence on $Hom(G,H)$ coincides with the topology of a subspaces of the cartesian product $\prod_{x\in G} H_x$ where $H_x = H$ for every $x \in G$ and since all open sets in $\prod_{x\in G} H_x$ are in the form $\prod_{x\in G} W_x$, where W_x is an open subset of H_x and $W_x \neq H_x$ only for finitely many $x \in G$, the topological group $Hom(G,H)$ is not locally Bourbaki-bounded. Now, the identity homomorphism $1_{Hom(G,H)}$ on $Hom(G,H)$ is B−Bourbaki-bounded but it is not n−Bourbaki-bounded since $Hom(G,H)$ is not locally Bourbaki-bounded. Note, if the group $Hom(G,H)$ is finite then it is locally Bourbaki-bounded and so the assumption $|Hom(G,H)| = \infty$ is necessary.

Nevertheless, unlike in the case of bounded operators on topological vector spaces, there is no more relation between continuous homomorphisms on topological groups and Bourbaki-bounded ones. The following example shows this

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fact. Before presenting the example, note that there are many spaces which are Bourbaki-bounded. For example we can assume S^1 with the topology inherited from complex number, as a Bourbaki-bounded topological group.

Example 3.6. Let $(G, \mathcal{T}_{id})$ be the indiscrete topological group and let $(G, \mathcal{T})$ be a Bourbaki-bounded topological group. Now, consider the identity homomorphism $i : (G, \mathcal{T}_{id}) \to (G, \mathcal{T})$. This homomorphism is not continuous since the inverse image of each open set in $(G, \mathcal{T})$ is not open set in $(G, \mathcal{T}_{id})$ but it is n$-$Bourbaki-bounded since G is open set in $(G, \mathcal{T})$ and $i(G) = G$ is Bourbaki-bounded set in $(G, \mathcal{T})$ and also it is B$-$Bourbaki-bounded since every subset of a Bourbaki-bounded topological group is Bourbaki-bounded.

4. Bourbaki$-$completeness of homomorphism spaces

Let G and H be two topological groups. For every two homomorphisms T, S in $Hom(G, H)$ define TS, T^{-1} by

$$TS(x) := T(x)S(x) \ \ and \ \ T^{-1}(x) := (T(x))^{-1}, \ \ x \in G.$$

Clearly, $Hom(G, H)$ with these operations, forms a group with constant mapping $E : G \to H$ defined by $E(x) = e_H$ as the identity element. Also it is easy to see that $Hom_{nB}(G, H)$, $Hom_{BB}(G, H)$ and $Hom_c(G, H)$ are subgroups of the group $Hom(G, H)$. Now we equip the group $Hom(G, H)$ with two topologies: One of them is Bourbaki-bounded$-$open topology which introduced by Hejecman in [12]. Bourbaki-bounded$-$open topology on $Hom(G, H)$ is the topology generated by the subbase consisting of all sets $M(B, V) = \{T \in Hom(G, H) : T(B) \subset V\}$, where B is an arbitrary Bourbaki-bounded subset of G and V is an arbitrary neighborhood of e_H in H. Another one is the famous topology of uniform convergence. According to the following propositions, $Hom(G, H)$ with each of these topologies forms a topological group. We can assume these topologies as subspace topology on the families $Hom_{nB}(G, H)$, $Hom_{BB}(G, H)$ and $Hom_c(G, H)$.

Proposition 4.1. *The group $Hom(G, H)$ is a topological group with respect to the topology of uniform convergence.*

Proof. We must prove that the operations of multiplication and inversion are continuous. First, let $\{T_\alpha\}_{\alpha\in I}$, $\{S_\alpha\}_{\alpha\in I}$ are two nets in $Hom(G, H)$ which are convergent to homomorphisms T, S respectively. Let W be an arbitrary symmetric neighborhood of e_H. Choose a neighborhood V of e_H with $V^2 \subset W$. There exist $\alpha_1, \alpha_2 \in I$ such that for each $\alpha \geq \alpha_1$, $T_\alpha T^{-1}(x) \in V$ for every $x \in G$ and for each $\alpha \geq \alpha_2$, $S_\alpha S^{-1}(x) \in V$ for every $x \in G$. Choose α_0 with $\alpha_0 \geq \alpha_1$ and $\alpha_0 \geq \alpha_2$. Now, for each $\alpha \geq \alpha_0$,

$$(T_\alpha S_\alpha)(TS)^{-1}(x) = (T_\alpha T^{-1}(x))(S_\alpha S^{-1}(x)) \in V^2 \subset W.$$

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This implies the continuity of multiplication operation. Also, since for sufficently large α, $T_\alpha T^{-1}(x) \in W$ for every $x \in G$ and W is symmetric neighborhood, then,

$$(T_\alpha^{-1}T)(x) = (T_\alpha T^{-1}(x))^{-1} \in W$$

for every $x \in G$. Therefore the operation of inversion is continuous. □

Proposition 4.2. *The group $Hom(G, H)$ is a topological group with respect to the Bourbaki-bounded−open topology.*

Proof. We must prove that the operations of multiplication and inversion are continuous. First, let $\{T_\alpha\}_{\alpha\in I}$, $\{S_\alpha\}_{\alpha\in I}$ are two nets in $Hom(G, H)$ which are convergent to homomorphisms T, S respectively. Fix a Bourbaki-bounded set $B \subset G$. Let W be an arbitrary symmetric neighborhood of e_H. Choose a neighborhood V of e_H with $V^2 \subset W$. There exist $\alpha_1, \alpha_2 \in I$ such that for each $\alpha \geq \alpha_1$, $T_\alpha T^{-1}(B) \subset V$ and for each $\alpha \geq \alpha_2$, $S_\alpha S^{-1}(B) \subset V$. Choose α_0 with $\alpha_0 \geq \alpha_1$ and $\alpha_0 \geq \alpha_2$. Now, for each $\alpha \geq \alpha_0$,

$$(T_\alpha S_\alpha)(TS)^{-1}(B) \subset (T_\alpha T^{-1}(B))(S_\alpha S^{-1}(B)) \subset V^2 \subset W.$$

This implies the continuity of multiplication operation. Also, since for sufficently large α, $T_\alpha T^{-1}(B) \subset W$ and W is symmetric neighborhood, then,

$$(T_\alpha^{-1}T)(B) \subset (T_\alpha T^{-1}(B))^{-1} \subset W.$$

Therefore the operation of inversion is continuous. □

In this section we investigate whether or not each class of homomorphisms with the appropriate topologies is Bourbaki-complete. Also at the end of this section we focus on *n−compact* homomorphisms and *B−compact* homomorphisms briefly.

Definition 4.3. A net $\{T_\alpha\}_{\alpha\in I}$ in the topological group $Hom(G, H)$ is called *Bourbaki-Cauchy* if for every neighborhood V of the identity element e_H there exist a positive integer m and $\alpha_0 \in I$ such that for each $\alpha, \beta \geq \alpha_0$, $T_\alpha T_\beta^{-1}(x) \in V^m$ for every $x \in G$.

Since $Hom_c(G, H)$, the set of continuous group homomorphisms, is closed with the topology of uniform convergence, the following lemma is trivial.

Lemma 4.4. *Let $\{T_\alpha\}_{\alpha\in I}$ be a net of continuous homomorphisms and let S be a cluster point of this net in the uniform convergence topology. Then S is also a continuous homomorphism.*

Lemma 4.5. *Let $\{T_\alpha\}_{\alpha\in I}$ be a net of $B-$Bourbaki-bounded homomorphisms and let S be a cluster point of this net in the uniform convergence topology. Then S is also a $B-$Bourbaki-bounded homomorphism.*

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Proof. It is easy to prove that S is an algebraic homomorphism. We prove that S is B$-$Bourbaki-bounded. Let V be an arbitrary neighborhood of e_H and B be a Bourbaki-bounded set in G. Since S is a cluster point of the net $\{T_\alpha\}_{\alpha\in I}$, there exist a subnet $\{S_\beta\}_{\beta\in J}$ of the net $\{T_\alpha\}_{\alpha\in I}$ and $\beta_0 \in J$ such that for each $\beta \geq \beta_0$, $S_\beta^{-1}S(x) \in V$ for every $x \in G$. Fix $\beta \geq \beta_0$. Since S_β is a B$-$Bourbaki-bounded homomorphism, there exist a positive integer n and a finite collection of points $y_1, y_2, ..., y_k \in H$ such that $S_\beta(B) \subset \bigcup_{i=1}^k y_i V^n$, and therefore

$$S(B) \subset S_\beta(B)V \subset (\bigcup_{i=1}^k y_i V^n)V = \bigcup_{i=1}^k y_i V^{n+1}.$$

This implies that S is B$-$Bourbaki-bounded. □

Lemma 4.6. *Let G be a locally Bourbaki-bounded topological group. Let $\{T_\alpha\}_{\alpha\in I}$ be a net of continuous homomorphisms and let S be a cluster point of this net in the Bourbaki-bounded$-$open topology. Then S is also a continuous homomorphism.*

Proof. It is easy to prove that S is an algebraic homomorphism. Let us prove it's continuity at zero. Let U be a Bourbaki-bounded neighborhood of e_G and let W be a neighborhood of e_H. Choose a neighborhood V of e_H with $V^2 \subset W$. Since S is a cluster point of the net $\{T_\alpha\}_{\alpha\in I}$, there exist a subnet $\{S_\beta\}_{\beta\in J}$ of the net $\{T_\alpha\}_{\alpha\in I}$ and $\beta_0 \in J$ such that for each $\beta \geq \beta_0$, $S_\beta S^{-1} \in M(U,V)$. Fix $\beta \geq \beta_0$. Since S_β is a continuous homomorphism, there exists $U_1 \subset U$ such that $S_\beta(U_1) \subset V$, and therefore

$$S(U_1) \subset S_\beta(U_1)V \subset V^2 \subset W.$$

This implies that S is continuous. □

Lemma 4.7. *Let $\{T_\alpha\}_{\alpha\in I}$ be a net of B$-$Bourbaki-bounded homomorphisms and let S be a cluster point of this net in the Bourbaki-bounded $-$open topology. Then S is also a B$-$Bourbaki-bounded homomorphism.*

Proof. It is easy to prove that S is an algebraic homomorphism. We prove that S is B$-$Bourbaki-bounded. Let V be an arbitrary neighborhood of e_H and let B be a Bourbaki-bounded set in G. Since S is a cluster point of the net $\{T_\alpha\}_{\alpha\in I}$, there exist a subnet $\{S_\beta\}_{\beta\in J}$ of the net $\{T_\alpha\}_{\alpha\in I}$ and $\beta_0 \in J$ such that for each $\beta \geq \beta_0$, $S_\beta S^{-1} \in M(B,V)$. Fix $\beta \geq \beta_0$. Since S_β is a B$-$Bourbaki-bounded homomorphism, there exist a positive integer n and a finite collection of points $y_1, y_2, ..., y_k \in H$ such that $S_\beta(B) \subset \bigcup_{i=1}^k y_i V^n$, and therefore

$$S(B) \subset S_\beta(B)V \subset (\bigcup_{i=1}^k y_i V^n)V = \bigcup_{i=1}^k y_i V^{n+1}.$$

This implies that S is B$-$Bourbaki-bounded. □

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The class $Hom_{nB}(G,H)$ can contain a Bourbaki-Cauchy net that it is not clusters, $i.e$ it has not any subnet whose limit is an n$-$Bourbaki-bounded homomorphism. On the other word, $Hom_{nB}(G,H)$ is not Bourbaki-complete in the assumed topology. The following example shows this fact:

Example 4.8. Let $Hom(G,H)$ be as in example 3.5. Consider the sequence of homomorphisms $T_n : Hom(G,H) \to Hom(G,H)$ with $T_n(f) = g$, where g is a homomorphism in $Hom(G,H)$ as follows:

$$\begin{aligned} g(x_1) &= f(x_1) \\ g(x_2) &= f(x_2) \\ &\vdots \\ g(x_n) &= f(x_n) \\ g(x_{n+1}) &= e_H \\ g(x_{n+2}) &= e_H \\ &\vdots \end{aligned}$$

Each T_n is n$-$Bourbaki-bounded homomorphism. For, let V be a Bourbaki-bounded neighborhood of e_H and U_n be a neighborhood of $e_{Hom(G,H)}$ defined by,

$$U_n = \{f \in Hom(G,H) : f(x_n)^{-1}f(x_i) \in V, \;\; i = 1, 2, .., n\}.$$

Then, it's easy to see that $T_n(U_n)$ is Bourbaki-bounded set in $Hom(G,H)$. On the other hand it's easy to see that the sequence $\{T_n\}_{n\in N}$ is convergent to the identity homomorphism $1_{Hom(G,H)}$ on $Hom(G,H)$. But we have seen in Example 3.5 that $1_{Hom(G,H)}$ is not n$-$Bourbaki-bounded.

Now, we are going to find some conditions under which each class of considered homomorphisms is topologically Bourbaki-complete.

Definition 4.9. A group G is said to be Bourbaki-complete if every Bourbaki-Cauchy net in G clusters ($i.e$, it has some convergent subnet).

Theorem 4.10. *Let H be a Bourbaki-complete group. Then $Hom_c(G,H)$ is Bourbaki-complete with respect to the topology of uniform convergence.*

Proof. Let $\{T_\alpha\}_{\alpha\in I}$ be a Bourbaki-Cauchy net in $Hom_c(G,H)$. Then for every $x \in G$, $\{T_\alpha(x)\}_{\alpha\in I}$ is a Bourbaki-Cauchy net in H. Since H is a Bourbaki-complete group, by definition 4.9, the net $\{T_\alpha(x)\}_{\alpha\in I}$ clusters and so it has some convergent subnet in H. Let $\{S_\beta(x)\}_{\beta\in J}$ be a convergent subnet of the net $\{T_\alpha(x)\}_{\alpha\in I}$ in H. Put $S(x) := lim\, S_\alpha(x)$. By lemma 4.4, S is also a continuous homomorphism. Let W be an arbitrary neighborhood of e_H. Choose a neighborhood V of e_H with $V^2 \subset W$. Since $\{S_\beta(x)\}_{\beta\in J}$ is a cauchy net in H, there exists $\beta_0 \in J$ such that for each $\beta, \gamma \geq \beta_0$, $S_\beta S_\gamma^{-1}(x) \in V$ for

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every $x \in G$. On the other hand for sufficiently large γ, $S_\gamma^{-1}S(x) \in V$, and therefore for each $\beta \geq \beta_0$,

$$S_\beta S^{-1}(x) = S_\beta S_\gamma^{-1}(x) S_\gamma(x) S^{-1}(x) \in V^2 \subset W,$$

i.e the subnet $\{S_\beta\}_{\beta \in J}$ of the net $\{T_\alpha\}_{\alpha \in I}$ is convergent. This completes the proof. □

Theorem 4.11. *Let H be a Bourbaki-complete group. Then $Hom_{BB}(G, H)$ is Bourbaki-complete with respect to the topology of uniform convergence.*

Proof. Let $\{T_\alpha\}_{\alpha \in I}$ be a Bourbaki-Cauchy net in $Hom_{BB}(G, H)$. Then for every $x \in G$, $\{T_\alpha(x)\}_{\alpha \in I}$ is a Bourbaki-Cauchy net in H. Since H is a Bourbaki-complete group, by definition 4.9, the net $\{T_\alpha(x)\}_{\alpha \in I}$ clusters and so it has some convergent subnet in H. Let $\{S_\beta(x)\}_{\beta \in J}$ be a convergent subnet of the net $\{T_\alpha(x)\}_{\alpha \in I}$ in H. Put $S(x) := lim\ S_\alpha(x)$. By lemma 4.5, S is also a B$-$Bourbaki-bounded homomorphism. Let W be an arbitrary neighborhood of e_H. Choose a neighborhood V of e_H with $V^2 \subset W$. Since $\{S_\beta(x)\}_{\beta \in J}$ is a cauchy net in H, there exists $\beta_0 \in J$ such that for each $\beta, \gamma \geq \beta_0$, $S_\beta S_\gamma^{-1}(x) \in V$ for every $x \in G$. On the other hand for sufficiently large γ, $S_\gamma^{-1}S(x) \in V$, and therefore for each $\beta \geq \beta_0$,

$$S_\beta S^{-1}(x) = S_\beta S_\gamma^{-1}(x) S_\gamma(x) S^{-1}(x) \in V^2 \subset W,$$

i.e the subnet $\{S_\beta\}_{\beta \in J}$ of the net $\{T_\alpha\}_{\alpha \in I}$ is convergent. This completes the proof. □

Theorem 4.12. *Let G be a locally Burbaki-bounded group and H be a Bourbaki-complete group. Then $Hom_c(G, H)$ is Bourbaki-complete, with respect to the Bourbaki-bounded$-$open topology.*

Proof. Let $\{T_\alpha\}_{\alpha \in I}$ be a Bourbaki-Cauchy net in $Hom_c(G, H)$. Then for every $x \in G$, $\{T_\alpha(x)\}_{\alpha \in I}$ is a Bourbaki-Cauchy net in H. Since H is a Bourbaki-complete group, by definition 4.9, the net $\{T_\alpha(x)\}_{\alpha \in I}$ clusters and so it has some convergent subnet in H. Let $\{S_\beta(x)\}_{\beta \in J}$ be a convergent subnet of the net $\{T_\alpha(x)\}_{\alpha \in I}$ in H. Put $S(x) := lim\ S_\alpha(x)$. By lemma 4.6, S is also a continuous homomorphism. Let W be an arbitrary neighborhood of e_H. Choose a neighborhood V of e_H with $V^2 \subset W$. Also let B be a Bourbaki-bounded set in G. Since $\{S_\beta(x)\}_{\beta \in J}$ is a cauchy net in H, there exists $\beta_0 \in J$ such that for each $\beta, \gamma \geq \beta_0$, $S_\beta S_\gamma^{-1}(x) \in V$ for every $x \in B$. On the other hand for sufficiently large γ, $S_\gamma^{-1}S(x) \in V$ for every $x \in B$, and therefore for each $\beta \geq \beta_0$,

$$S_\beta S^{-1}(x) = S_\beta S_\gamma^{-1}(x) S_\gamma(x) S^{-1}(x) \in V^2 \subset W; \ \ \forall\, x \in B.$$

This implies that $S_\beta S^{-1} \in M(B, W)$, *i.e* the subnet $\{S_\beta\}_{\beta \in J}$ of the net $\{T_\alpha\}_{\alpha \in I}$ is convergent. This completes the proof. □

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Theorem 4.13. *Let H be a Bourbaki-complete group. Then $Hom_{BB}(G,H)$ is Bourbaki-complete with respect to the Bourbaki-bounded$-$open topology.*

Proof. Let $\{T_\alpha\}_{\alpha\in I}$ be a Bourbaki-Cauchy net in $Hom_{BB}(G,H)$. Then for every $x \in G$, $\{T_\alpha(x)\}_{\alpha\in I}$ is a Bourbaki-Cauchy net in H. Since H is a Bourbaki-complete group, by definition 4.9, the net $\{T_\alpha(x)\}_{\alpha\in I}$ clusters and so it has some convergent subnet in H. Let $\{S_\beta(x)\}_{\beta\in J}$ be a convergent subnet of the net $\{T_\alpha(x)\}_{\alpha\in I}$ in H. Put $S(x) := lim\ S_\alpha(x)$. By lemma 4.7, S is also a B$-$Bourbaki-bounded homomorphism. Let W be an arbitrary neighborhood of e_H. Choose a neighborhood V of e_H with $V^2 \subset W$. Also let B be a Bourbaki-bounded set in G. Since $\{S_\beta(x)\}_{\beta\in J}$ is a cauchy net in H, there exists $\beta_0 \in J$ such that for each $\beta, \gamma \geq \beta_0$, $S_\beta S_\gamma^{-1}(x) \in V$ for every $x \in B$. On the other hand for sufficiently large γ, $S_\gamma^{-1}S(x) \in V$ for every $x \in B$, and therefore for each $\beta \geq \beta_0$,

$$S_\beta S^{-1}(x) = S_\beta S_\gamma^{-1}(x) S_\gamma(x) S^{-1}(x) \in V^2 \subset W;\ \ \forall\, x \in B.$$

This implies that $S_\beta S^{-1} \in M(B,W)$, *i.e* the subnet $\{S_\beta\}_{\beta\in J}$ of the net $\{T_\alpha\}_{\alpha\in I}$ is convergent. This completes the proof. □

Note that when the group G and H has the same conditions in the above theorems, then $Hom_{nB}(G,H)$ might fail to be a Bourbaki-complete topological group. For, look Example 3.5 and Example 4.8.

Definition 4.14. A homomorphism T from a topological group G to a topological group H is said to be

(1) *$n-$compact* if there exists a neighborhood U of e_G such that $\overline{T(U)}$ is compact in H.
(2) *$B-$compact* if for every Bourbaki-bounded set $B \subset G$, the set $\overline{T(B)}$ is compact in H.

Lemma 4.15. *Every closed subset of Bourbaki-complete topological group, is Bourbaki-complete.*

Proof. Let G be a Bourbaki-complete topological group and $B \subset G$ be a closed subset of G. If $\{x_\alpha\}_{\alpha\in I}$ be a Bourbaki-Cauchy net in B, then it is Bourbaki-Cauchy net in G. Since G is Bourbaki-complete, by definition 4.9, it has some convergent subnet as $\{y_\beta\}_{\beta\in J}$. Let $\{y_\beta\}_{\beta\in J}$ is convergent to y. The subnet $\{y_\beta\}_{\beta\in J}$ is in B and since B is a closed set, it contains y. Therefore the subnet $\{y_\beta\}_{\beta\in J}$ is convergent in B. This implies Bourbaki-completeness of B. □

Theorem 4.16. *Let T be an $n-$Bourbaki-bounded homomorphism from topological group G to topological group H. If H is Bourbaki-complete, then T is $n-$compact.*

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Proof. Since T is an $n-Bourbaki$*-bounded* homomorphism, there exists a neighborhood U of e_G such that $T(U)$ is Bourbaki-bounded set in H. Bourbaki-boundedness of $T(U)$ implies the Bourbaki-boundedness of $\overline{T(U)}$. Since H is a Bourbaki-complete topological group, by lemma 4.15, $\overline{T(U)}$ is also Bourbaki-complete. Now, by using [15 ,$Theorem$ 17], proof is hold. □

Theorem 4.17. *Let T be a continuous homomorphism from topological group G to topological group H. If H is Bourbaki-complete, then T is $B-compact$.*

Proof. Let B be a Bourbaki-bounded subset of G. By proposition 3.3, $T(B)$ is Bourbaki-bounded set in H. Bourbaki-boundedness of $T(B)$ implies the Bourbaki-boundedness of $\overline{T(B)}$. Since H is a Bourbaki-complete topological group, by lemma 4.15, $\overline{T(B)}$ is also Bourbaki-complete. Now, by using [15 ,$Theorem$ 17], proof is hold. □

Corollary 4.18. *Let T be a $B-Bourbaki$-bounded homomorphism from topological group G to topological group H. If H is Bourbaki-complete, then T is $B-compact$.*

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