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Arithmetic Deformation Theory of Lie Algebras

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Abstract. This paper is devoted to deformation theory of graded Lie algebras over $\mathbb{Z}$ or $\mathbb{Z}_l$ with finite dimensional graded pieces. Such deformation problems naturally appear in number theory. In the first part of the paper, we use Schlessinger criteria for functors on Artinian local rings in order to obtain universal deformation rings for deformations of graded Lie algebras and their graded representations. In the second part, we use a version of Schlessinger criteria for functors on the Artinian category of nilpotent Lie algebras which is formulated by Pridham, and explore arithmetic deformations using this technique.

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1. Introduction

To a hyperbolic smooth curve defined over a number-field one naturally associates an "anabelian" representation of the absolute Galois group of the base field landing in outer automorphism group of the algebraic fundamental group [4]. It would be very fruitful to deform this object, rather than the abelian version coming from the action of Galois group on the Tate module of the Jacobian variety. On the other hand, there is no formalism available for

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deforming such a representation. This is why we translate it to the language of graded Lie algebras which are still non-linear enough to carry more information than the abelian version. For more details see [9] and [10].

In this paper, we introduce several deformation problems for Lie-algebra versions of the above representation. In particular, we deform the representation of certain Galois-Lie algebras landing in the graded Lie algebra associated to the weight filtration on outer automorphism group of the pro-l fundamental group, and we construct a deformation ring parameterizing all deformations fixing the mod-l Lie-algebra representation, which is the typical thing to do in the world of Galois representations.

Organization of the paper is as follows: First, we review methods of associating Lie algebras to profinite groups. Then, we introduce some deformation problems for representations landing in graded Lie algebras. Afterwards, we use the classical Schlessinger criteria for representability of functors on Artin local rings for deformation of the above representation. In some cases, universal deformations exist and in some others, we are only able to construct a hull which parameterizes all possible deformations. finally, we use a graded version of Pridham's adaptation of Schlessinger criteria for functors on finite dimensional nilpotent Lie algebras.

2. Lie algebras associated to profinite groups

Exponential map on the tangent space of an algebraic group defined over a field k of characteristic zero, gives an equivalence of categories between nilpotent Lie algebras of finite dimension over k and unipotent algebraic groups over k. This way, one can associate a Lie algebra to the algebraic unipotent completion $\Gamma^{\mathrm{alg}}(\mathbb{Q})$ of any profinite group Γ.

On the other hand, Malčev defines an equivalence of categories between nilpotent Lie algebras over $\mathbb{Q}$ and uniquely divisible nilpotent groups. Inclusion of such groups in nilpotent groups has a right adjoint $\Gamma \to \Gamma_{\mathbb{Q}}$. For a nipotent group Γ, torsion elements form a subgroup T and $\Gamma_{\mathbb{Q}} = \cup(\Gamma/T)^{1/n}$. In fact we have $\Gamma_{\mathbb{Q}} = \Gamma^{\mathrm{alg}}(\mathbb{Q})$.

Any nilpotent finite group is a product of its sylow subgroups. Therefore, the profinite completion $\Gamma^{\wedge}$ factors to pro-l completions $\Gamma_l^{\wedge}$, each one a compact open subgroup of the corresponding $\Gamma^{alg}(\mathbb{Q}_l)$ and we have the following isomorphisms of l-adic Lie groups

$$\mathrm{Lie}(\Gamma_l^{\wedge}) = \mathrm{Lie}(\Gamma^{\mathrm{alg}}(\mathbb{Q}_l)) = \mathrm{Lie}(\Gamma^{\mathrm{alg}}(\mathbb{Q})) \otimes \mathbb{Q}_l.$$

In fact, the adelic Lie group associated to Γ can be defined as $\mathrm{Lie}(\Gamma^{\mathrm{alg}}(\mathbb{Q})) \otimes \mathbb{A}^f$ which is the same as $\prod \mathrm{Lie}(\Gamma_l^{\wedge})$.

Suppose we are given a nilpotent representation of Γ on a finite dimensional vector space V over k, which means that for a filtration F on V respecting the

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action, the induced action of $Gr_F(V)$ is trivial. The subgroup

$$\{\sigma \in \mathrm{GL}(V) | \sigma F = F, \mathrm{Gr}_F(\sigma) = 1\}$$

is a uniquely divisible group and one obtains a morphism

$$\mathrm{Lie}(\Gamma_{\mathbb{Q}}) \to \{\sigma \in \mathrm{GL}(V) | \sigma F = F, \mathrm{Gr}_F(\sigma) = 0\}$$

which is an equivalence of categories between nilpotent representations of Γ and representations of $\Gamma^{\mathrm{alg}}(\mathbb{Q})$ and nilpotent representations of the Lie algebra $\mathrm{Lie}(\Gamma_{\mathbb{Q}})$ over the field k. The above equivalence of categories extends to an equivalence between linear representations of Γ and representations of its algebraic envelope

The notion of weighted completion of a group developed by Hain and Matsumoto generalizes the concept of algebraic unipotent completion. Suppose that R is an algebraic k-group and $w : \mathbb{G}_m \to R$ is a central cocharacter. Let G be an extension of R by a unipotent group U in the category of algebraic k-groups

$$0 \longrightarrow U \longrightarrow G \longrightarrow R \longrightarrow 0.$$

The first homology of U is an R-module, and therefore an $\mathbb{G}_m$-module via w, which naturally decomposes to to direct sum of irreducible representations each isomorphic to a power of the standard character. We say that our extension is negatively weighted if only negative powers of the standard character appear in $H_1(U)$. The weighted completion of Γ with respect to the representation ρ with Zariski dense image $\rho : \Gamma \to R(\mathbb{Q}_l)$ is the universal $\mathbb{Q}_l$-proalgebraic group $\mathcal{G}$ which is a negative weighted extension of R by a prounipotent group $\mathcal{U}$ and a continuous lift of ρ to $\mathcal{G}(\mathbb{Q}_l)$ [6]. The Lie algebra of $\mathcal{G}(\mathbb{Q}_l)$ is a more sophisticated version of $\mathrm{Lie}(\Gamma_{\mathbb{Q}}) \otimes \mathbb{Q}_l$.

3. Cohomology theories for graded Lie algebras

We shall first review cohomology of Lie algebras with the adjoint representation as coefficients. Let L be a graded $\mathbb{Z}_l$-algebra and let $C^q(L, L)$ denote the space of all skew-symmetric q-linear forms on a Lie algebra L with values in L. Define the differential

$$\delta : C^q(L, L) \longrightarrow C^{q+1}(L, L)$$

where the action of δ on a skew-symmetric q-linear form γ is a skew-symmetric $(q+1)$-linear form which takes $(l_1, ..., l_{q+1}) \in L^{q+1}$ to

$$\sum (-1)^{s+t-1} \gamma([l_s, l_t], l_1, ..., \hat{l}_s, ..., \hat{l}_t, ..., l_{q+1}) + \sum [l_u, \gamma(l_1, ..., \hat{l}_u, ..., l_{q+1})]$$

where the first sum is over s and t with $1 \leq s < t \leq q+1$ and the second sum is over u with $1 \leq u \leq q+1$. Then $\delta^2 = 0$ and we can define $H^q(L, L)$ to be the cohomology of the complex $\{C^q(L, L), \delta\}$. If we put $C^m = C^{m+1}(L, L)$ and $H^m = H^{m+1}(L, L)$, then there exists a natural bracket operation which makes $C = \oplus C^m$ a differential graded algebra and $H = \oplus H^m$ a graded Lie

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algebra. Look in [3]. If L is $\mathbb{Z}$-graded, $L = \oplus L(m)$, we say $\phi \in C^q(L,L)(m)$ if for $l_i \in L(g_i)$ we have $\phi(l_1, ..., l_q) \in L(g_1 + ... + g_q - m)$. Then, there exists a grading induced on the Lie algebra cohomology $H^q(L,L) = \oplus H^q(L,L)(m)$.

The computational tool used by geometers to study deformations of Lie algebras is a cohomology theory of K-algebras where K is a field, which is developed by Harrison [5]. This cohomology theory is generalized by Barr to algebras over general rings [1]. Here we use a more modern version of the latter introduced independently by Andre and Quillen which works for general algebras [8].

Let A be a commutative algebra with identity over $\mathbb{Z}_l$ or any ring R and let M be an A-module. By an n-long singular extension of A by M we mean an exact sequence of A-modules

$$0 \to M \to M_{n-1} \to ... \to M_1 \to T \to R \to 0$$

where T is a commutative $\mathbb{Z}_l$-algebra and the final map a morphism of $\mathbb{Z}_l$-algebras whose kernel has square zero. It is trivial how to define morphisms and isomorphisms between n-long singular extensions. Barr defines a group structure on these isomorphism classes [1], which defines $H^n_{\text{Barr}}(A,M)$ for $n > 1$ and we put $H^1_{\text{Barr}}(A,M) = \text{Der}(A,M)$. Barr proves that for a multiplicative subset S of R not containing zero

$$H^n_{\text{Barr}}(A,M) \cong H^n_{\text{Barr}}(A_S, M)$$

for all n and any A_S-module M. According to this isomorphism, the cohomology of the algebra A over $\mathbb{Z}_l$ is the same after tensoring A with $\mathbb{Q}_l$ if M is $A \otimes \mathbb{Q}_l$-module. Thus one could assume that we are working with an algebra over a field, and then direct definitions given by Harrison would serve our computations better. Consider the complex

$$0 \to \text{Hom}(A,M) \to \text{Hom}(S^2A, M) \to Hom(A \otimes A \otimes A, M)$$

where $\psi \in \text{Hom}(A,M)$ goes to

$$d_1\psi : (a,b) \mapsto a\psi(b) - \psi(ab) + b\psi(a)$$

and $\phi \in \text{Hom}(S^2A, M)$ goes to

$$d_2\phi(a,b,c) \mapsto a\phi(b,c) - \phi(ab,c) + \phi(a,bc) - c\phi(a,b)$$

The cohomology of this complex defines $H^i_{\text{Harr}}(R,M)$ for $i = 1, 2$. If A is a local algebra with maximal ideal m and residue field k, the Harrison cohomology $H^1_{\text{Harr}}(A,k) = (m/m^2)'$, which is the space of homomorphisms $A \to k[t]/t^2$ such that m is the kernel of the composition $A \to k[t]/t^2 \to k$.

Andre-Quillen cohomology is the same as Barr cohomology in low dimensions and can be described directly in terms of derivations and extensions. For any morphism of commutative rings $A \to B$ and B-module M we denote the B-module of A-algebra derivations of B with values in M by $\text{Der}_A(B,M)$. Let

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$\mathrm{Ext}^{\inf}_A(B,M)$ denote the B-module of infinitesimal A-algebra extensions of B by M. The functors Der and $\mathrm{Ext}^{\inf}$ have transitivity property. Namely, given morphisms of commutative rings $A \to B \to C$ and a C-module M, there is an exact sequence

$$0 \longrightarrow \mathrm{Der}_B(C,M) \longrightarrow \mathrm{Der}_A(C,M) \longrightarrow \mathrm{Der}_A(B,M)$$
$$\longrightarrow \mathrm{Ext}^{\inf}_B(C,M) \longrightarrow \mathrm{Ext}^{\inf}_A(C,M) \longrightarrow \mathrm{Ext}^{\inf}_A(B,M).$$

The two functors Der and $\mathrm{Ext}^{\inf}$ also satisfy flat base-change property. Namely, given morphisms $A \to B$ and $A \to A'$ if $Tor_1^A(A',B) = 0$, then there are isomorphisms $\mathrm{Der}_{A'}(A' \otimes_A B, M) \cong \mathrm{Der}_A(B,M)$ and $\mathrm{Ext}^{\inf}_{A'}(A' \otimes_A B, M) \cong \mathrm{Ext}^{\inf}_A(B,M)$. Andre-Quillen cohomology associates $\mathrm{Der}_A(B,M)$ and $\mathrm{Ext}^{\inf}_A(B,M)$ to any morphism of commutative rings $A \to B$ and B-module M as the first two cohomologies and extends it to higher dimensional cohomologies such that transitivity and flat base-change extend in the obvious way.

4. Several deformation problems

Let X denote a hyperbolic smooth algebraic curve defined over a number field K. Let S denote the set of bad reduction places of X together with places above l. We shall construct Lie algebra versions of the pro-l outer representation of the Galois group

$$\rho^l_X : \mathrm{Gal}(K_S/K) \longrightarrow \mathrm{Out}(\pi_1(\bar{X})^{(l)}).$$

Let I_l denote the decreasing filtration on $Out(\pi_1(X)^{(l)})$ induced by the central series filtration of $\pi_1(\bar{X})^{(l)}$. By abuse of notation, we also denote the filtration on $\mathrm{Gal}(K_S/K)$ by I_l. We get an injection of the associated graded $\mathbb{Z}_l$-Lie algebras on both sides

$$\mathcal{G}al(K_S/K) \longrightarrow \mathcal{O}ut(\pi_1(\bar{X})^{(l)}).$$

One can also start with the l-adic unipotent completion of the fundamental group and the outer representation of Galois group on this group.

$$\rho^{un,l}_X : \mathrm{Gal}(K_S/K) \longrightarrow \mathrm{Out}(\pi_1(\bar{X})^{un}_{/\mathbb{Q}_l}).$$

By [6] 8.2 the associated Galois Lie algebra would be the same as those associated to I_l. Let U_S denote the prounipotent radical of the Zariski closure of the image of $\rho^{un,l}_X$. The image of $\mathrm{Gal}(K_S/K)$ in $\mathrm{Out}(\pi_1(\bar{X})^{un}_{/\mathbb{Q}_l})$ is a negatively weighted extension of $\mathbb{G}_m$ by U_S with respect to the central cocharacter $w : x \mapsto x^{-2}$. By [6] 8.4 the weight filtration induces a graded Lie algebra $\mathcal{U}_S$ which is isomorphic to $\mathcal{G}al(K_S/K) \otimes \mathbb{Q}_l$.

There are several deformation problem in this setting which are interesting. For example, the action of Galois group on unipotent completion of the fundamental group induces an action of the Galois group on the corresponding

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nilpotent $\mathbb{Q}_l$-Lie algebra

$$\rho_X^{ni,l} : \mathrm{Gal}(K_S/K) \longrightarrow \mathrm{Aut}(\mathcal{U}_S).$$

which could be deformed. Another possibility is deforming the following representation

$$\mathrm{Gal}(K_S/K) \longrightarrow \mathrm{Aut}(\pi_1(\bar{X})^{un}_{/\mathbb{Q}_l}) \longrightarrow \mathrm{Aut}(H_1(\mathcal{P}))$$

where $\mathcal{P}$ denote the nilpotent Lie algebra associated to $\pi_1(\bar{X})^{un}_{/\mathbb{Q}_l}$. This time, the Schlessinger criteria may not help us in finding a universal representation. There exists also a derivation version, which is a Schlessinger friendly $\mathbb{Z}_l$-Lie algebras representation

$$\mathcal{G}al(K_S/K) \longrightarrow \mathrm{Der}(\mathcal{P})/\mathrm{Inn}(\mathcal{P}).$$

One could also deform the following morphism, fixing its mod-l reduction

$$\mathcal{G}al(K_S/K) \longrightarrow \mathcal{O}ut(\pi_1(\bar{X})^{(l)}).$$

5. Deformations of local graded Lie algebras

In this section, we are only concerned with deformations of Lie algebras and leave deformation of their representations for the next section. We are interested in deforming the coefficient ring of graded Lie algebras over $\mathbb{Z}_l$ of the form $L = Gr^\bullet_I\widetilde{\mathrm{Out}}(\pi_1^l(X))$ and then deforming representations of Galois graded Lie algebra

$$\mathrm{Gr}^\bullet_{X,l}\mathrm{Gal}(\bar{K}/K) \to \mathrm{Gr}^\bullet_I\widetilde{\mathrm{Out}}(\pi_1^l(X)).$$

One can reduce the coefficient ring $\mathbb{Z}_l$ modulo l and get a graded Lie algebra $\bar{L}$ over $\mathbb{F}_l$ and a representation

$$\mathrm{Gr}^\bullet_{X,l}\mathrm{Gal}(\bar{K}/K) \to \bar{L}.$$

We look for liftings of this representation which is landing in $\bar{L}$ among representations landing in graded Lie algebras over Artinian local rings A of the form $L = \oplus_i L^i$ where L^i is a finitely generated A-module for positive i.

Let k be a finite field of characteristic p and let Λ be any complete Noetherian local ring. For example Λ can be $W(k)$, the ring of Witt vectors of k, or O, the ring of integers of any local field with quotient field $K_\wp$ and residue field k. Let C denote the category of Artinian local Λ-algebras with residue field k. A covariant functor from C to $Sets$ is called pro-representable if it has the form

$$F(A) \cong \mathrm{Hom}_\Lambda(R_{\mathrm{univ}}, A) \qquad A \in C$$

where R_{univ} is a complete local Λ-algebra with maximal ideal m_{univ} such that $R_{\mathrm{univ}}/m^n_{\mathrm{univ}}$ is in C for all n.

There are a number of deformation functors related to our problem. For a local ring $A \in C$ with maximal ideal m, the set of deformations of $\bar{L}$ to A is denoted by $D_c(\bar{L}, A)$ and is defined to be the set of isomorphism classes of

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graded Lie algebras L/A of the above form which reduce to $\bar{L}$ modulo m. In this notation c stands for coefficients, since we are only deforming coefficients not the Lie algebra structure. The functor $D(\bar{L})$ as defined above is not a pro-representable functor. As we will see, there exists a "hull" for this functor (Schlessinger's terminology [11]) parameterizing all possible deformations. The idea of deforming the Lie structure of Lie algebras has been extensively used by geometers. For example, Fialowski studied this problem in double characteristic zero case [2]. In this paper, we are interested in double characteristic $(0, l)$-version.

Now we make an assumption for further constructions. Assume $H^2(L, L)(m)$ is finite dimensional for all m and consider the algebra $\mathbb{D}_1 = \mathbb{Z}_l \oplus \bigoplus_m H^2(L, L)(m)'$ where $'$ means the dual over $\mathbb{Z}_l$. Fix a graded homomorphism of degree zero

$$\mu : H^2(L, L) \longrightarrow C^2(L, L)$$

which takes any cohomology class to a cocycle representing this class. Now define a Lie algebra structure on

$$\mathbb{D}_1 \otimes L = L \oplus \mathrm{Hom}^0(H^2(L, L), L)$$

where Hom^0 means degree zero graded homomorphisms, by the following bracket

$$[(l_1, \phi_1), (l_2, \phi_2)] := ([l_1, l_2], \psi)$$

where $\psi(\alpha) = \mu(\alpha)(l_1, l_2) + [\phi_1(\alpha), l_2] + [\phi_2(\alpha), l_1]$. The Jacobi identity is implied by $\delta\mu(\alpha) = 0$. It is clear that this in an infinitesimal deformation of L and it can be shown that, up to an isomorphism, this deformation does not depend on the choice of μ. We shall denote this deformation by η_L after Fialowski and Fuchs [3].

Proposition 5.1. *Any infinitesimal deformation of $\bar{L}$ to a finite dimensional local ring A is induced by pushing forward η_L by a unique morphism*

$$\phi : \mathbb{Z}_l \oplus \bigoplus_m H^2(L, L)(m)' \longrightarrow A.$$

Proof. This is the double characteristic version of proposition 1.8 in [3]. □

Note that, in our case $H^2(L, L)$ is not finite dimensional. This is why we restrict our deformations to the space of graded deformations. Since $H^2(L, L)(0)$ is the tangent space of the space of graded deformations, and the grade zero piece $H^2(L, L)(0)$ is finite dimensional, the following version is more appropriate:

Proposition 5.2. *Any infinitesimal graded deformation of $\bar{L}$ to a finite dimensional local ring A is induced by pushing forward η_L^0 by a unique morphism*

$$\phi : \mathbb{Z}_l \oplus H^2(L, L)(0)' \longrightarrow A.$$

where η_L^0 denotes the restriction of η_L to $\mathbb{Z}_l \oplus H^2(L, L)(0)'$.

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Let A be a small extension of $\mathbb{F}_l$ and L be a graded deformation of $\bar{L}$ over the base A. The deformation space $D(\bar{L}, A)$ can be identified with $H^2(L, L)(0)$ which is finite dimensional. Therefore, by Schlessinger criteria, in the subcategory C' of C consisting of local algebras with $m^2 = 0$ for the maximal ideal m, the functor $D(\bar{L}, A)$ is pro-representable. This means that there exists a unique map

$$\mathbb{Z}_l \oplus H^2(L, L)(0)' \to R_{univ}$$

inducing the universal infinitesimal graded deformation.

5.1. **Obstructions to deformations.** Let A be an object in the category C and $L \in \mathrm{Def}(\bar{L}, A)$. The pair (A, L) defines a morphism of functors $\theta : \mathrm{Mor}(A, B) \to \mathrm{Def}(\bar{L}, B)$. We say that (A, L) is universal if θ is an isomorphism for any choice of B. We say that (A, L) is miniversal if θ is always surjective, and gives an isomorphism for $B = k[\varepsilon]/\varepsilon^2$. We intend to construct a miniversal deformation of $\bar{L}$.

Consider a graded deformation with base in a local algebra A with residue field $k = \mathbb{F}_l$. One can define a map

$$\Phi_A : \mathrm{Ext}^{\inf}_{\mathbb{Z}_l}(A, k) \longrightarrow H^3(\bar{L}, \bar{L}).$$

Indeed, choose an extension $0 \to k \to B \to A \to 0$ corresponding to an element in $\mathrm{Ext}^{\inf}_{\mathbb{Z}_l}(A, k)$. Consider the B-linear skew-symmetric operation $\{.,.\}$ on $\bar{L} \otimes_k B$ commuting with $[.,.]$ on $\bar{L} \otimes A$ defined by $\{l, l_1\} = [l, \bar{l}_1]$ for l in the kernel of $\bar{L} \otimes_k B \to \bar{L} \otimes_k A$ which can be identified by $\bar{L}$. Here $\bar{l}_l$ is the image of l_1 under the projection map $B \to k$ tensored with $\bar{L}$ whose kernel is the inverse image of the maximal ideal of A. The Jacobi expression induces a multilinear skew-symmetric form on $\bar{L}$ which could be regarded as a closed element in $C^3(\bar{L}, \bar{L})$. The image in $H^3(\bar{L}, \bar{L})$ is independent of the choices made.

Theorem 5.3. *(Fialowski) One can deform the Lie algebra structure on $L \otimes_k A$ to $L \otimes_k B$ if and only if the image of the above extension vanishes under the morphism $\mathrm{Ext}^{\inf}_{\mathbb{Z}_l}(A, k) \to H^3(\bar{L}, \bar{L})$.*

Proof. The proof presented in [2] and [3] works for algebras over fields of finite characteristic. □

Using the above criteria for extending deformations, one can follow the methods of Fialowski and Fuchs to introduce a miniversal deformation for $\bar{L}$.

Proposition 5.4. *Given a local commutative algebra A over $\mathbb{Z}_l$ there exists a universal extension*

$$0 \longrightarrow \mathrm{Ext}^{\inf}_{\mathbb{Z}_l}(A, k)' \longrightarrow C \longrightarrow A \longrightarrow 0$$

among all extensions of A with modules M over A with $mM = 0$ where m is the maximal ideal of A.

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This is proposition 2.6 in [3]. Consider the canonical split extension

$$0 \longrightarrow H^2(L,L)(0)' \longrightarrow \mathbb{D}_1 \longrightarrow k \longrightarrow 0.$$

We will initiate an inductive construction of $\mathbb{D}_k$ such that

$$0 \longrightarrow \mathrm{Ext}^{\inf}_{\mathbb{Z}_l}(\mathbb{D}_k,k)' \longrightarrow \bar{\mathbb{D}}_{k+1} \longrightarrow \mathbb{D}_k \longrightarrow 0.$$

together with a deformation η_k of L to the base $\mathbb{D}_k$. For η_1 take η_L, and assume $\mathbb{D}_i$ and η_k is constructed for $i \leq k$. Given a local commutative algebra $\mathbb{D}_k$ with maximal ideal m, there exists a unique universal extension for all extensions of $\mathbb{D}_k$ by $\mathbb{D}_k$-modules M with $mM = 0$ of the following form

$$0 \longrightarrow \mathrm{Ext}^{\inf}_{\mathbb{Z}_l}(\mathbb{D}_k,k)' \longrightarrow C \longrightarrow \mathbb{D}_k \longrightarrow 0.$$

associated to the cocycle $f_k : S^2(\mathbb{D}_k) \to \mathrm{Ext}^{\inf}_{\mathbb{Z}_l}(\mathbb{D}_k,k)'$ which is dual to the homomorphism

$$\mu : \mathrm{Ext}^{\inf}_{\mathbb{Z}_l}(\mathbb{D}_k,k) \longrightarrow S^2(\mathbb{D}_k)'$$

which takes a cohomology class to a cocycle from the same class. The obstruction to extend η_k lives in $\mathrm{Ext}^{\inf}_{\mathbb{Z}_l}(\mathbb{D}_k,k)' \otimes H^3(\bar{L},\bar{L})$. Consider the composition of the associated dual map

$$\Phi'_k : H^3(\bar{L},\bar{L})' \longrightarrow \mathrm{Ext}^{\inf}_{\mathbb{Z}_l}(\mathbb{D}_k,k)'.$$

with $\mathrm{Ext}^{\inf}_{\mathbb{Z}_l}(\mathbb{D}_k,k)' \to C$ and define $\mathbb{D}_{k+1}$ to be the cokernel of this map. We get the following exact sequence

$$0 \longrightarrow (\ker\Phi_k)' \longrightarrow \mathbb{D}_{k+1} \longrightarrow \mathbb{D}_k \longrightarrow 0.$$

We can extend η_k to η_{k+1}. Now, taking a projective limit of $\mathbb{D}_k$ we get a base and a formal deformation of $\bar{L}$.

Theorem 5.5. *Let $\mathbb{D}$ denote the projective limit $\varprojlim \mathbb{D}_k$ which is a $\mathbb{Z}_l$-module. One can deform $\bar{L}$ uniquely to a graded Lie algebra with base $\mathbb{D}$ which is miniversal among all deformations of $\bar{L}$ to local algebras over $\mathbb{Z}_l$.*

Proof. This is the double characteristic version of theorem 4.5 in [3]. The same proof works here because theorems 11 and 18 in [5] which are used in the arguments of Fialowski and Fuchs work for algebras over any perfect field. □

Proposition 5.6. *(Fialowski-Fuchs) The base of the minversal deformation of $\bar{L}$ is the zero locus of a formal map $H^2(\bar{L},\bar{L})(0) \to H^3(\bar{L},\bar{L})(0)$.*

Proof. This is the graded version of proposition 7.2 in [3]. □

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5.2. **Deformations of graded Lie algebra representations.** In the previous section we discussed deformation theory of the mod l reduction of the Lie algebra $\mathrm{Gr}_I^\bullet\widetilde{\mathrm{Out}}(\pi_1^l(X))$. We shall mention the following

Theorem 5.7. *The cohomology groups $H^i(\mathrm{Gr}_I^\bullet\widetilde{\mathrm{Out}}(\pi_1^l(X)), \mathrm{Gr}_I^\bullet\widetilde{\mathrm{Out}}(\pi_1^l(X)))(0)$ are finite dimensional for all non-negative integer i.*

Proof. By a theorem of Labute $\mathrm{Gr}_I^\bullet\pi_1^l(X)$ is quotient of a finitely generated free Lie algebra with finitely generated module of relations [Lab]. Therefore, the cohomology groups $H^i(\mathrm{Gr}_I^\bullet\pi_1^l(X), \mathrm{Gr}_I^\bullet\pi_1^l(X))(0)$ are finite dimensional. Finite dimensionality of the cohomology of $\mathrm{Gr}_I^\bullet\widetilde{\mathrm{Out}}(\pi_1^l(X))$ follows from proposition 1.3. □

We are interested in deforming the following graded representation of the Galois graded Lie algebra

$$\rho : \mathrm{Gr}^\bullet_{X,l}\mathrm{Gal}(\bar{K}/K) \to \mathrm{Gr}_I^\bullet\widetilde{\mathrm{Out}}(\pi_1^l(X))$$

among all graded representations which modulo l reduce to the graded representation

$$\bar{\rho} : \mathrm{Gr}^\bullet_{X,l}\mathrm{Gal}(\bar{K}/K) \to \bar{L}$$

where the Lie algebra $\bar{L}$ over $\mathbb{F}_l$ is the mod-l reduction of $Gr_I^\bullet\widetilde{\mathrm{Out}}(\pi_1^l(X))$. There are suggestions from the classical deformation theory of Galois representations on how to get a representable deformation functor. Let $D(\bar{\rho}, A)$ denote the set of isomorphism classes of Galois graded Lie algebra representations to graded Lie algebras L/A of the above form which reduce to $\bar{\rho}$ modulo m. The first ingredient we need in order to prove representability of $D(\bar{\rho})$ is finite dimensionality of the tangent space of the functor. The tangent space of the deformation functor $D(\bar{\rho})$ for an object $A \in C$ is canonically isomorphic to

$$H^1(\mathrm{Gr}^\bullet_{X,l}\mathrm{Gal}(\bar{K}/K), Ad \circ \bar{\rho})$$

where the Lie algebra module is given by the composition of $\bar{\rho}$ with the adjoint representation of $\mathrm{Gr}_I^\bullet\widetilde{\mathrm{Out}}(\pi_1^l(X/K))$. To get finite dimensionality, we restrict ourselves to the graded deformations of the graded representation $\bar{\rho}$.

Theorem 5.8. *$H^1(\mathrm{Gr}^\bullet_{X,l}\mathrm{Gal}(\bar{K}/K), Ad \circ \bar{\rho})(0)$ is finite dimensional.*

Proof. The Galois-Lie representation $\mathrm{Gr}_I^\bullet\mathrm{Gal}(\bar{K}/K) \to \mathrm{Gr}_I^\bullet\widetilde{\mathrm{Out}}(\pi_1^l(X))$ is an injection. Derivation inducing cohomology commutes with inclusion of Lie algebras. Therefore $H^1(\mathrm{Gr}^\bullet_{X,l}\mathrm{Gal}(\bar{K}/K), Ad\circ\bar{\rho})(0)$ injects in $H^1(\mathrm{Gr}_I^\bullet\widetilde{\mathrm{Out}}(\pi_1^l(X)), Ad\circ\bar{\rho})(0)$ which is finite dimensional by previous theorem. □

For a surjective mapping $A_1 \to A_0$ of Artinian local rings in C such that the kernel $I \subset A_1$ satisfies $I.m_1 = 0$ and given any deformation ρ_0 of $\bar{\rho}$ to $L \otimes A_0$ one can associate a canonical obstruction class in

$$H^2(Gr^\bullet_{X,l}\mathrm{Gal}(\bar{K}/K), I \otimes Ad \circ \bar{\rho}) \cong H^2(\mathrm{Gr}^\bullet_{X,l}\mathrm{Gal}(\bar{K}/K), Ad \circ \bar{\rho}) \otimes I$$

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which vanishes if and only if ρ_0 can be extended to a deformation with coefficients A_1. Therefore, vanishing results on second cohomology are important.

Theorem 5.9. *Suppose* $\mathrm{Gr}^\bullet_{X,l}\mathrm{Gal}(\bar{K}/K)$ *is a free Lie algebra over* $\mathbb{Z}_l$*, then the Galois cohomology* $H^2(Gr^\bullet_{X,l}\mathrm{Gal}(\bar{K}/K), Ad\circ\bar{\rho})$ *vanishes.*

Proof. The free Lie algebra $G = Gr^\bullet_{X,l}\mathrm{Gal}(\bar{K}/K)$ is rigid, and therefore has trivial infinitesimal deformations. Thus, we get vanishing of its second cohomology:

$$H^2(\mathrm{Gr}^\bullet_{X,l}\mathrm{Gal}(\bar{K}/K), \mathrm{Gr}^\bullet_{X,l}\mathrm{Gal}(\bar{K}/K)) = 0.$$

The injection of G inside $L = \mathrm{Gr}^\bullet_I\widetilde{\mathrm{Out}}(\pi_1^l(X))$ as Lie-algebras over $\mathbb{Z}_l$ implies that, the cohomology group $H^2(\mathrm{Gr}^\bullet_{X,l}\mathrm{Gal}(\bar{K}/K), Ad\circ\rho)$ vanishes again by freeness of G. Let $\bar{G}$ denote the reduction modulo l of G which is a free Lie algebra over $\mathbb{F}_l$. The cohomology $H^2(G,\bar{G})$ is the mod-l reduction of $H^2(G,G)$, hence it also vanishes. So does the cohomology $H^2(\mathrm{Gr}^\bullet_{X,l}\mathrm{Gal}(\bar{K}/K), Ad\circ\bar{\rho})$ by similar reasoning. □

We have obtained conceptual conditions implying H_4 of [11]. What we have proved can be summarized as follows.

Main Theorem 5.10. *Suppose that* $\mathrm{Gr}^\bullet_{X,l}\mathrm{Gal}(\bar{K}/K)$ *is a free Lie algebra over* $\mathbb{Z}_l$*. There exists a universal deformation ring* $R_{\mathrm{univ}} = R(X,K,l)$ *and a universal deformation of the representation* $\bar{\rho}$

$$\rho^{\mathrm{univ}} : \mathrm{Gr}^\bullet_{X,l}\mathrm{Gal}(\bar{K}/K) \longrightarrow \mathrm{Gr}^\bullet_I\widetilde{\mathrm{Out}}(\pi_1^l(X)) \otimes R_{\mathrm{univ}}$$

which is unique in the usual sense. If $\mathrm{Gr}^\bullet_{X,l}\mathrm{Gal}(\bar{K}/K)$ *is not free, then a miniversal deformation exists which is universal among infinitesimal deformations of* $\bar{\rho}$*.*

Remark 5.1. Note that, freeness of $\mathrm{Gr}^\bullet_{X,l}\mathrm{Gal}(\bar{K}/K)$ in the special case of $K = \mathbb{Q}$ where filtration comes from punctured projective curve $X = \mathbb{P}^1 - \{0,1,\infty\}$ or a punctured elliptic curve $X = E - \{0\}$ is implied by Deligne's conjecture.

As in the classical case, the Lie algebra structure on $Ad\circ\bar{\rho}$ induces a graded Lie algebra structure on the cohomology $H^*(\mathrm{Gr}^\bullet_{X,l}\mathrm{Gal}(\bar{K}/K), Ad\circ\bar{\rho})$ via cup-product, and in particular, a symmetric bilinear pairing

$$H^1(\mathrm{Gr}^\bullet_{X,l}\mathrm{Gal}(\bar{K}/K), Ad\circ\bar{\rho}) \times H^1(\mathrm{Gr}^\bullet_{X,l}\mathrm{Gal}(\bar{K}/K), Ad\circ\bar{\rho})$$

$$\longrightarrow H^2(\mathrm{Gr}^\bullet_{X,l}\mathrm{Gal}(\bar{K}/K), Ad\circ\bar{\rho})$$

which gives the quadratic relations satisfied by the minimal set of formal parameters of $R_{\mathrm{univ}}/lR_{\mathrm{univ}}$ for characteristic l different from 2.

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6. Functors on nilpotent graded Lie algebras

In this section, we review Pridham's nilpotent Lie algebra version of Schlessinger criteria [7]. The only change we impose is to consider finitely generated graded nilpotent Lie algebras with finite dimensional graded pieces, instead of finite dimensional nilpotent Lie algebras.

Fix a field k and let $\mathcal{N}_k$ denote the category of finitely generated NGLAs (nilpotent graded Lie algebras) with finite dimensional graded pieces, and $\widehat{\mathcal{N}_k}$ denote the category of pro-NGLAs with finite dimensional graded pieces which are finite dimensional in the sense that $\dim L/[L,L] < \infty$. Given $\mathcal{L} \in \widehat{\mathcal{N}_k}$ define $\mathcal{N}_{\mathcal{L},k}$ to be the category of pairs $\{N \in \mathcal{N}_k, \phi : \mathcal{L} \to N\}$ and $\widehat{\mathcal{N}_{\mathcal{L},k}}$ to be the category of pairs $\{N \in \widehat{\mathcal{N}_k}, \phi : \mathcal{L} \to N\}$.

All functors on $\mathcal{N}_{\mathcal{L},k}$ should take the 0 object to a one point set. for a functor $F : \mathcal{N}_{\mathcal{L},k} \to \text{Set}$, define $\hat{F} : \widehat{\mathcal{N}_{\mathcal{L},k}} \to \text{Set}$ by

$$\hat{F}(L) = \lim_{\leftarrow} F(L/\Gamma_n(L)),$$

where $\Gamma_n(L)$ is the n-th term in the central series of L. Then for $h_L : \mathcal{N}_{\mathcal{L},k} \to \text{Set}$ defined by $N \to \text{Hom}(L, N)$ we have an isomorphism

$$\hat{F}(L) \longrightarrow \text{Hom}(h_L, F)$$

which can be used to define the notion of a pro-representable functor.

A morphism $p \in N \to M$ in $\mathcal{N}_{\mathcal{L},k}$ is called a small section if it is surjective with a principal ideal kernel (t) such that $[N, (t)] = (0)$.

Given $F : \mathcal{N}_{\mathcal{L},k} \to \text{Set}$, and morphisms $N' \to N$ and $N'' \to N$ in $\mathcal{N}_{\mathcal{L},k}$, consider the map

$$F(N' \times_N N'') \longrightarrow F(N') \times_{F(N)} F(N'').$$

Then, by the Lie algebra analogue of the Schlessinger theorem F has a hull if and only if it satisfies the following properties

(H1) The above map is surjective whenever $N'' \to N$ is a small section.

(H2) The above map is bijective when $N = 0$ and $N'' = L(\epsilon)$.

(H3) $\dim_k(t_F < \infty$.

F is pro-representable if and only if it satisfies the following additional property

(H4) The above map is an isomorphism for any small extension $N'' \to N$.

Note that, in case we are considering graded deformations of graded Lie algebras, only the zero grade piece of the cohomology representing the tangent space shall be checked to be finite dimensional.

7. Deformation of nilpotent graded Lie algebras

Let us concentrate on deforming the graded Lie algebra representation

$$\mathcal{G}al(K_S/K) \longrightarrow \mathcal{O}ut(\pi_1(\bar{X})^{(l)}).$$

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We could fix the mod-l representation, or fix restriction of this representation to decomposition Lie algebra $\mathcal{D}_p$ at prime p, which is induced by the same filtration as $\mathrm{Gal}(K_S/K)$ on the decomposition group. For each prime p of K we get a map

$$\mathcal{D}_p \longrightarrow \mathcal{G}al(K_S/K) \longrightarrow \mathcal{O}ut(\pi_1(\bar{X})^{(l)}).$$

Theorem 7.1. *For a graded $\mathbb{Z}_l$-Lie algebra L, let $D(L)$ be the set of representations of $\mathcal{G}al(K_S/K)$ to L which reduce to*

$$\bar{\rho} : \mathcal{G}al(K_S/K) \longrightarrow \mathcal{O}ut(\pi_1(\bar{X})^{(l)})/l\mathcal{O}ut(\pi_1(\bar{X})^{(l)})$$

after reduction modulo l. Assume that $\mathcal{G}al(K_S/K)$ is a free $\mathbb{Z}_l$-Lie algebra. Then, there exists a universal deformation graded $\mathbb{Z}_l$-Lie algebra L_{univ} and a universal representation

$$\mathcal{G}al(K_S/K) \longrightarrow L_{univ}$$

representing the functor D. In case $\mathcal{G}al(K_S/K)$ is not free, then one can find a hull for the functor D.

Proof. For free $\mathcal{G}al(K_S/K)$ by theorem 4.9, we have

$$H^2(\mathcal{G}al(K_S/K), Ad \circ \bar{\rho}) = 0$$

which implies that D is pro-representable. In case $\mathcal{G}al(K_S/K)$ is not free, we have constructed a miniversal deformation Lie algebra for another functor , which implies that the first three Schlessinger criteria hold. By a similar argument one could prove that there exists a hull for D. Note that $\mathcal{O}ut(\pi_1(\bar{X})^{(l)})$ is pronilpotent, and for deformation of such an object one should deform the truncated object and then take a limit to obtain a universal object in pro-NGLAs. □

Note that, by a conjecture of Deligne, there exists a graded Lie algebra over $\mathbb{Z}$ which gives rise to all $\mathcal{G}al(K_S/K)$ for different primes l, after tensoring with $\mathbb{Z}_l$. For deformation of the representations of this $\mathbb{Z}$-Lie algebra, one can not use representability for functors on Artinian local rings, and the nilpotent Lie algebra deformations become crucial.

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