Iranian Journal of Mathematical Sciences and Informatics
Vol. 14, No. 2 (2019), pp 43-60
DOI: 10.7508/ijmsi.2019.02.005

Approximation by (p, q)-Lupaş Stancu Operators

Asif Khan, Vinita Sharma*

Department of Mathematics, Aligarh Muslim University, Aligarh–202002, India.
E-mail: asifjnu07@gmail.com
E-mail: vinita.sha23@gmail.com

Abstract. In this paper, (p, q)-Lupas Bernstein Stancu operators are constructed. Statistical as well as other approximation properties of (p, q)-Lupaş Stancu operators are studied. Rate of statistical convergence by means of modulus of continuity and Lipschitz type maximal functions has been investigated.

Keywords: (p, q)-Integers, Lupaş (p, q)-Bernstein Stancu operators, Statistical approximation, Korovkin's type approximation.

2000 Mathematics subject classification: 65D17, 41A10, 41A25, 41A36.

1. Introduction and Preliminaries

In 1912, S.N. Bernstein [6] introduced his famous operators $B_n : C[0, 1] \to C[0, 1]$ defined for any $n \in \mathbb{N}$ and for any function $f \in C[0, 1]$

$$B_n(f; x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1 - x)^{n-k} f\left(\frac{k}{n}\right), \; x \in [0, 1]. \tag{1.1}$$

and named it Bernstein polynomials to provide a constructive proof of the Weierstrass theorem [20].

*Corresponding Authors

Received 29 January 2017; Accepted 11 August 2017

Further, based on q-integers, Lupaş [21] introduced the first q-Bernstein operators [6] and investigated its approximating and shape-preserving properties. Another q-analogue of the Bernstein polynomials is due to Phillips [38]. Since then several generalizations of well-known positive linear operators based on q-integers have been introduced and their approximation properties studied.

Recently, the applications of (p,q)-calculus (post quantum calculus) emerged as a new area in the field of approximation theory [20]. The development of post quantum calculus has led to the discovery of various generalizations of Bernstein polynomials involving (p,q)-integers. The aim of these generalizations is to provide appropriate and powerful tools to application areas such as numerical analysis, computer-aided geometric design [7] and solutions of differential equations.

Mursaleen *et al* [27] introduced the concept of post quantum calculus in approximation theory and constructed the (p,q)-analogue of Bernstein operators defined as follows for $0 < q < p \leq 1$:

$$B_{n,p,q}(f;x) = \frac{1}{p^{\frac{n(n-1)}{2}}} \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array} \right]_{p,q} p^{\frac{k(k-1)}{2}} x^k \prod_{s=0}^{n-k-1} (p^s - q^s x)\, f\left(\frac{[k]_{p,q}}{p^{k-n}[n]_{p,q}} \right),\ x \in [0,1]. \tag{1.2}$$

Note when $p = 1$, (p,q)-Bernstein Operators given by (1.2) turns out to be Phillips q-Bernstein Operators [38].

Also, we have

$$\begin{aligned}(1-x)^n_{p,q} &= \prod_{s=0}^{n-1} (p^s - q^s x) = (1-x)(p-qx)(p^2-q^2x)...(p^{n-1}-q^{n-1}x) \\ &= \sum_{k=0}^{n} (-1)^k p^{\frac{(n-k)(n-k-1)}{2}} q^{\frac{k(k-1)}{2}} \left[\begin{array}{c} n \\ k \end{array} \right]_{p,q} x^k.\end{aligned}$$

Further, they applied the concept of (p,q)-calculus in approximation theory and studied approximation properties based on (p,q)-integers for Bernstein-Stancu operators, (p,q)-analogue of Bernstein-Kantorovich, (p,q)-analogue of Bernstein-Shurer operators, (p,q)-analogue of Bleimann-Butzer-Hahn operators and (p,q)-analogue of Lorentz polynomials on a compact disk in [28, 31, 32, 33, 35].

On the other hand, Khalid and Lobiyal defined (p,q)-analogue of Lupaş Bernstein operators [17] as follows :

For any $p > 0$ and $q > 0$, the linear operators $L^n_{p,q} : C[0,1] \to C[0,1]$ as

$$L^n_{p,q}(f;x) = \sum_{k=0}^{n} \frac{f\left(\frac{p^{n-k}\ [k]_{p,q}}{[n]_{p,q}}\right) \left[\begin{array}{c} n \\ k \end{array}\right]_{p,q} p^{\frac{(n-k)(n-k-1)}{2}} q^{\frac{k(k-1)}{2}}\ x^k\ (1-x)^{n-k}}{\prod\limits_{j=1}^{n} \{p^{j-1}(1-x) + q^{j-1}x\}}, \tag{1.3}$$

are (p,q)-analogue of Lupaş Bernstein operators.

Again when $p = 1$, Lupaş (p,q)-Bernstein operators turns out to be Lupaş q-Bernstein operators as given in [22, 37].

When $p = q = 1$, Lupaş (p,q)-Bernstein operators turns out to be classical Bernstein operators [6].

They studied two different techniques as de-Casteljau's algorithm and Korovkin's type approximation properties [17]: de-Casteljau's algorithm and related results of degree elevation reduction for Bèzier curves and surfaces holds for all $p > 0$ and $q > 0$. However to study Korovkin's type approximation properties for Lupaş (p,q)-analogue of the Bernstein operators, $0 < q < p \leq 1$ is needed.

Based on Korovkin's type approximation, they proved that the sequence of (p,q)-analogue of Lupaş Bernstein operators $L^n_{p_n,q_n}(f,x)$ converges uniformly to $f(x) \in C[0,1]$ if and only if $0 < q_n < p_n \leq 1$ such that $\lim\limits_{n\to\infty} q_n = 1$, $\lim\limits_{n\to\infty} p_n = 1$ and $\lim\limits_{n\to\infty} q_n^n = 1$. On the other hand, for any $p > 0$ fixed and $p \neq 1$, the sequence $L^n_{p,q}(f,x)$ converges uniformly to $f(x) \in C[0,1]$ if and only if $f(x) = ax + b$ for some $a, b \in \mathbb{R}$.

Furthermore, in comparison to q-Bèzier curves and surfaces based on Lupaş q-Bernstein rational functions, their generalization gives more flexibility in controlling the shapes of curves and surfaces.

Some advantages of using the extra parameter p have been discussed in the field of approximations on compact disk [35] and in computer aided geometric design [17].

For more details related to approximation theory [20], one can refer [1, 2, 3, 5, 8, 9, 12, 13, 14, 15, 18, 19, 22, 23, 24, 34, 36, 39, 40, 42, 43, 44, 45, 46, 47, 48].

Let us recall certain notations of (p,q)-calculus.

For any $p > 0$ and $q > 0$, the (p,q) integers $[n]_{p,q}$ are defined by

$$[n]_{p,q}=p^{n-1}+p^{n-2}q+p^{n-3}q^2+...+pq^{n-2}+q^{n-1}=\begin{cases}\frac{p^n-q^n}{p-q}, \text{ when } \ p\neq q\neq 1\\ \\ n\ p^{n-1}, \ \text{ when } p=q\neq 1\\ \\ [n]_q, \text{ when } p=1\\ n, \ \text{ when } p=q=1\end{cases}$$

where $[n]_q$ denotes the q-integers and $n=0,1,2,\cdots$.

The formula for (p,q)-binomial expansion is as follows:

$$(ax+by)^n_{p,q}:=\sum_{k=0}^{n}p^{\frac{(n-k)(n-k-1)}{2}}q^{\frac{k(k-1)}{2}}\left[\begin{array}{c}n\\k\end{array}\right]_{p,q}a^{n-k}b^kx^{n-k}y^k,$$

$$(x+y)^n_{p,q}=(x+y)(px+qy)(p^2x+q^2y)\cdots(p^{n-1}x+q^{n-1}y),$$

$$(1-x)^n_{p,q}=(1-x)(p-qx)(p^2-q^2x)\cdots(p^{n-1}-q^{n-1}x),$$

where (p,q)-binomial coefficients are defined by

$$\left[\begin{array}{c}n\\k\end{array}\right]_{p,q}=\frac{[n]_{p,q}!}{[k]_{p,q}![n-k]_{p,q}!}.$$

Details on (p,q)-calculus can be found in [10, 11, 27].

Also, we have (p,q)-analogue of Euler's identity as:

$$(1-x)^n_{p,q}=\prod_{s=0}^{n-1}(p^s-q^sx)=(1-x)(p-qx)(p^2-q^2x)...(p^{n-1}-q^{n-1}x)$$

$$=\sum_{k=0}^{n}(-1)^kp^{\frac{(n-k)(n-k-1)}{2}}q^{\frac{k(k-1)}{2}}\left[\begin{array}{c}n\\k\end{array}\right]_{p,q}x^k.$$

Again by some simple calculations and using the property of (p,q)-integers, we get (p,q)-analogue of Pascal's relation as follows:

$$\left[\begin{array}{c}n\\k\end{array}\right]_{p,q}=q^{n-k}\left[\begin{array}{c}n-1\\k-1\end{array}\right]_{p,q}+p^k\left[\begin{array}{c}n-1\\k\end{array}\right]_{p,q} \tag{1.4}$$

$$\left[\begin{array}{c}n\\k\end{array}\right]_{p,q}=p^{n-k}\left[\begin{array}{c}n-1\\k-1\end{array}\right]_{p,q}+q^k\left[\begin{array}{c}n-1\\k\end{array}\right]_{p,q}. \tag{1.5}$$

We recall some results from [17] for Lupas (p,q)-Bernstein operators, which reproduces linear and constant functions.

Some auxillary results:

(1) $L^n_{p,q}(1, \frac{u}{u+1}) = 1$

(2) $L^n_{p,q}(t, \frac{u}{u+1}) = \frac{u}{u+1}$

(3) $L^n_{p,q}(t^2, \frac{u}{u+1}) = \frac{u}{u+1}\frac{p^{n-1}}{[n]_{p,q}} + \frac{qu}{u+1}(\frac{qu}{p+qu})\frac{[n-1]_{p,q}}{[n]_{p,q}}$

or equivalently for $x = \frac{u}{u+1}$

$$L^n_{p,q}(1, x) = 1, \tag{1.6}$$

$$L^n_{p,q}(t, x) = x, \tag{1.7}$$

$$L^n_{p,q}(t^2, x) = \frac{p^{n-1}x}{[n]_{p,q}} + \frac{q^2\ x^2}{p(1-x)+qx}\frac{[n-1]_{p,q}}{[n]_{p,q}}. \tag{1.8}$$

2. Construction of (p,q)-Lupaş Stancu Operators

In this section, we introduce (p,q)-Lupaş Stancu operators as follows:

For any $p > 0$ and $q > 0$, the linear operators $L^n_{p,q} : C[0,1] \to C[0,1]$

$$L^{\alpha,\beta}_{n,p,q}(f;x) = \sum_{k=0}^{n} f\left(\frac{p^{n-k}\ [k]_{p,q} + \alpha}{[n]_{p,q} + \beta}\right) b^{k,n}_{p,q}(t) \tag{2.1}$$

and $b^{k,n}_{p,q}(t)$ is given by

$$b^{k,n}_{p,q}(t) = \frac{\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\frac{(n-k)(n-k-1)}{2}} q^{\frac{k(k-1)}{2}}\ t^k\ (1-t)^{n-k}}{\prod_{j=1}^{n}\{p^{j-1}(1-t) + q^{j-1}t\}}, \tag{2.2}$$

where $0 < \alpha < \beta$.

We give some equalities for operators (2.1) in the following lemma.

Lemma 4.1. The following equalities are true:

(i) $L^{\alpha,\beta}_{n,p,q}(1;x) = 1$,

(ii) $L^{\alpha,\beta}_{n,p,q}(t;x) = \frac{[n]_{p,q}x+\alpha}{[n]_{p,q}+\beta}$,

(iii) $L^{\alpha,\beta}_{n,p,q}(t^2;x) = \frac{1}{([n]_{p,q}+\beta)^2}\frac{q^2[n]_{p,q}[n-1]_{p,q}}{p(1-x)+qx}x^2 \ + \ \frac{[n]_{p,q}(2\alpha+p^{n-1})}{([n]_{p,q}+\beta)^2}x + \frac{\alpha^2}{([n]_{p,q}+\beta)^2}$.

Proof. Proof of part (i) is obvious.

$$L_{n,p,q}^{\alpha,\beta}(t;x) = \sum_{k=0}^{n} \left(\frac{p^{n-k}\ [k]_{p,q} + \alpha}{[n]_{p,q} + \beta}\right) b_{p,q}^{k,n}(t)$$

$$= \frac{[n]_{p,q}}{[n]_{p,q} + [\beta]} L_{p,q}^{n}(t;x) \ + \ \frac{[\alpha]}{[n]_{p,q} + [\beta]} L_{p,q}^{n}(1;x).$$

So from inequalities (1.6) and (1.7), we get the result.

Proof (iii)

$$L_{n,p,q}^{\alpha,\beta}(t^2;x) = \sum_{k=0}^{n} \left(\frac{p^{n-k}\ [k]_{p,q} + \alpha}{[n]_{p,q} + \beta}\right) b_{p,q}^{k,n}(t)$$

$$= \frac{1}{([n]_{p,q} \ + \ \beta)^2} \Bigg[p^{2n-2k} [k]_{p,q}^2 b_{p,q}^{k,n}(t)$$

$$+ \ 2\alpha p^{n-k} [k]_{p,q} b_{p,q}^{k,n}(t) \ + \ \alpha^2 b_{p,q}^{k,n}(t) \Bigg]$$

$$= \frac{1}{([n]_{p,q} \ + \ \beta)^2} \Bigg[\ A \ + \ B \ + \ C \ \Bigg].$$

$$A \ = \ p^{2n} \sum_{k=0}^{n} \frac{[k]^2}{p^{2k}} \frac{\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\frac{(n-k)(n-k-1)}{2}} q^{\frac{k(k-1)}{2}} \ t^k \ (1-t)^{n-k}}{\prod_{j=1}^{n} \{p^{j-1}(1-t) + q^{j-1}t\}}$$

$$A \ = \ [n] p^{2n} \sum_{k=1}^{n} \frac{[k]}{p^{2k}} \frac{\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_{p,q} p^{\frac{(n-k)(n-k-1)}{2}} q^{\frac{k(k-1)}{2}} \ u^k}{\prod_{j=1}^{n} \{p^{j-1} + q^{j-1}u\}}.$$

On shifting the limits and on replacing k by $k+1$, we get

$$A \ = \ [n] p^{2n} \sum_{k=1}^{n} \frac{[k+1]}{p^{2k+2}} \frac{\begin{bmatrix} n-1 \\ k \end{bmatrix}_{p,q} p^{\frac{(n-k-1)(n-k-2)}{2}} q^{\frac{k(k+1)}{2}} \ u^k}{\prod_{j=1}^{n-1} \{p^j + q^j u\}},$$

$$= \ [n] p^n \frac{u}{u+1} \sum_{k=0}^{n-1} \frac{[k+1]}{p^{k+2}} \frac{\begin{bmatrix} n-1 \\ k \end{bmatrix}_{p,q} p^{\frac{(n-k-1)(n-k-2)}{2}} q^{\frac{k(k+1)}{2}} \ (\frac{qu}{p})^k}{\prod_{j=0}^{n-2} \{p^j + q^j (\frac{qu}{p})\}}.$$

Using $[k+1]_{p,q} \ = p^k \ + \ q[k]_{p,q}$, we get our desired result:

$$\begin{aligned} A &= [n]p^n \frac{u}{u+1} \sum_{k=0}^{n-1} \frac{[p^k+q[k]]}{p^{k+2}} \frac{\begin{bmatrix} n-1 \\ k \end{bmatrix}_{p,q} p^{\frac{(n-k-1)(n-k-2)}{2}} q^{\frac{k(k+1)}{2}} \ (\frac{qu}{p})^k}{\prod\limits_{j=0}^{n-2} \{p^j + q^j(\frac{qu}{p})\}}, \\ &= [n]_{p,q} p^{n-1} \frac{u}{u+1} \ + \ \frac{q^2 u^2 [n]_{p,q}[n-1]_{p,q}}{(u+1)(p+qu)}, \end{aligned}$$

equivalently

$$A \ = \ [n]_{p,q} p^{n-1} x \ + \ \frac{q^2 [n]_{p,q}[n-1]_{p,q}}{(p(1-x)+qx)} x^2.$$

Similarly

$$\begin{aligned} B &= 2\alpha p^n \sum_{k=0}^{n} \frac{[k]}{p^k} \frac{\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\frac{(n-k)(n-k-1)}{2}} q^{\frac{k(k-1)}{2}} \ t^k \ (1-t)^{n-k}}{\prod\limits_{j=1}^{n} \{p^{j-1}(1-t) + q^{j-1}t\}}, \\ &= 2\alpha [n] p^n \sum_{k=1}^{n} \frac{1}{p^k} \frac{\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_{p,q} p^{\frac{(n-k)(n-k-1)}{2}} q^{\frac{k(k-1)}{2}} \ u^k}{\prod\limits_{j=1}^{n} \{p^{j-1} + q^{j-1}u\}}. \end{aligned}$$

After shifting the limits and on replacing k by $k+1$, we get

$$B \ = \ 2\alpha [n] p^n \frac{u}{u+1} \sum_{k=0}^{n-1} \frac{1}{p} \frac{\begin{bmatrix} n-1 \\ k \end{bmatrix}_{p,q} p^{\frac{(n-k-1)(n-k-2)}{2}} q^{\frac{k(k-1)}{2}} \ (\frac{qu}{p})^k}{\prod\limits_{j=0}^{n-2} \{p^j + q^j(\frac{qu}{p})\}},$$

which implies

$$B \ = \ 2\alpha [n]_{p,q} x.$$

Similarly

$$\begin{aligned} C &= \alpha^2 \sum_{k=0}^{n} \frac{\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\frac{(n-k)(n-k-1)}{2}} q^{\frac{k(k-1)}{2}} \ t^k \ (1-t)^{n-k}}{\prod\limits_{j=1}^{n} \{p^{j-1}(1-t) + q^{j-1}t\}}, \\ &= \alpha^2. \end{aligned}$$

□

Theorem 2.1. *Let* $0 < q_n < p_n \leq 1$ *such that* $\lim_{n\to\infty} p_n = 1$, $\lim_{n\to\infty} q_n = 1$, $\lim_{n\to\infty} q_n^n = 1$ *and for* $f \in C[0,1]$, *we have* $\lim_n |L_{n,p,q}^{\alpha,\beta}(f;x) - f(x)| = 0.$

Proof. Let us recall the following Korovkin's theorem see [20]. Let (T_n) be a sequence of positive linear operators from $C[a,b]$ into $C[a,b]$. Then $\lim_n \|T_n(f,x) - f(x)\|_{C[a,b]} = 0$, for all $f \in C[a,b]$ if and only if $\lim_n \|T_n(f_i,x) - f_i(x)\|_{\mathcal{C}[a,b]} = 0$, for $i = 0,1,2$, where $f_0(t) = 1$, $f_1(t) = t$ and $f_2(t) = t^2$.

□

3. The Rate of Convergence

In this section, we compute the rates of convergence of the operators $L_{n,p,q}^{\alpha,\beta}(f;x)$ to the functions f by means of modulus of continuity, elements of Lipschitz class and peetre's K-functional.

Let $f \in C[0,1]$. The modulus of continuity of f denoted by $\omega(f,\delta)$ is defined as:

$$\omega(f,\delta) = \sup_{y,x\in[0,1],\ |y-x|<\delta} |f(y) - f(x)|.$$

where $w(f;\delta)$ satisfies the following conditions: for all $f \in C[0,1]$,

$$\lim_{\delta\to 0}\ w(f;\delta) = 0. \tag{3.1}$$

and

$$|f(y) - f(x)| \leq w(f;\delta)\bigg(\frac{|y-x|}{\delta} + 1\bigg). \tag{3.2}$$

Theorem 3.1. *Let* $0 < q < p \leq 1$, *and* $f \in C[0,1]$, *and* $\delta > 0$, *we have*

$$\|L_{n,p,q}^{\alpha,\beta}(f;x) - f(x)\|_{C[0,1]} \leq 2\omega(f;\delta_n)$$

where

$$\delta_n = \Bigg[\bigg(\frac{q^2[n]_{p,q}[n-1]_{p,q}}{([n]_{p,q}\ +\ \beta)^2(p(1-x)+qx)} - \frac{[n]_{p,q} - \beta}{[n]_{p,q}\ +\ \beta}\bigg)\ +\ \bigg(\frac{p^{n-1}[n]_{p,q} - 2\alpha\beta}{([n]_{p,q}+\beta)^2}\bigg) + \frac{\alpha^2}{([n]_{p,q}+\beta)^2}\Bigg]^{\frac{1}{2}}.$$

Proof. From lemma (4.1) we have

$$|L_{n,p,q}^{\alpha,\beta}(t-x)^2;x) = \left(\frac{q^2[n]_{p,q}[n-1]_{p,q}}{([n]_{p,q} \ + \ \beta)^2(p(1-x)+qx)} - \frac{[n]_{p,q}-\beta}{[n]_{p,q} \ + \ \beta}\right)x^2 + \left(\frac{p^{n-1}[n]_{p,q}-2\alpha\beta}{([n]_{p,q}+\beta)^2}\right)x + \frac{\alpha^2}{([n]_{p,q}+\beta)^2}. \tag{3.3}$$

For $x \in [0,1]$, we take

$$|L_{n,p,q}^{\alpha,\beta}(f;x) - f(x)| \leq w(f;\delta)\left\{1 + \frac{1}{\delta}(L_{n,p,q}^{\alpha,\beta}(t-x)^2 : x)^{\frac{1}{2}}\right\},$$

then we get

$$\begin{aligned}\|L_{n,p,q}^{\alpha,\beta}(f;x) - f(x)\|_{C[0,1]} &\leq w(f;\delta)\left\{1 + \frac{1}{\delta}(L_{n,p,q}^{\alpha,\beta}(t-x)^2 : x)^{\frac{1}{2}}\right\}\\ &\leq w(f;\delta)\left\{1 + \frac{1}{\delta}\left((\frac{1}{([n]_{p,q}+\beta)^2}\frac{q^2[n]_{p,q}[n-1]_{p,q}}{(p(1-x)+qx)}\right.\right.\\ &\quad - \frac{[n]_{p,q}-\beta}{[n]_{p,q} \ + \ \beta}) \ + \ (\frac{p^{n-1}[n]_{p,q}-2\alpha\beta}{([n]_{p,q}+\beta)^2})\\ &\quad \left.\left. + \frac{\alpha^2}{([n]_{p,q}+\beta)^2}\right)^{\frac{1}{2}}\right\}.\end{aligned}$$

If we choose

$$\delta_n = \left[\left(\frac{q^2[n]_{p,q}[n-1]_{p,q}}{([n]_{p,q} \ + \ \beta)^2(p(1-x)+qx)} - \frac{[n]_{p,q}-\beta}{[n]_{p,q} \ + \ \beta}\right) + \ \left(\frac{p^{n-1}[n]_{p,q}-2\alpha\beta}{([n]_{p,q}+\beta)^2}\right) + \frac{\alpha^2}{([n]_{p,q}+\beta)^2}\right]^{\frac{1}{2}}.$$

Then we have

$$\|L_{n,p,q}^{\alpha,\beta}(f;x) - f(x)\|_{C[0,1]} \ \leq \ 2\omega(f;\delta_n).$$

So we have the desired result. □

Now we compute the approximation order of operator $L_{n,p,q}^{\alpha,\beta}$ in term of the elements of the usual Lipschitz class.

Let $f \in C[0,1]$ and $0 < \rho \leq 1$. We recall that f belongs to $Lip_M(\rho)$ if the inequality

$$|f(x) - f(y)| \leq M|x-y|^{\rho};\ for\ all\ x,y \in [0,1] \tag{3.4}$$

holds.

Theorem 3.2. *For all $f \in Lip_M(\rho)$*

$\|L^{\alpha,\beta}_{n,p,q}(f;x) - f(x)\|_{C[0,1]} \ \leq\ M\delta_n^{\rho}$
where

$$\delta_n = \Bigg[\Bigg(\frac{q^2[n]_{p,q}[n-1]_{p,q}}{([n]_{p,q}\ +\ \beta)^2(p(1-x)+qx)} - \frac{[n]_{p,q}-\beta}{[n]_{p,q}\ +\ \beta}\Bigg) + \left(\frac{p^{n-1}[n]_{p,q}-2\alpha\beta}{([n]_{p,q}+\beta)^2}\right) + \frac{\alpha^2}{([n]_{p,q}+\beta)^2}\Bigg]^{\frac{1}{2}}$$

and M is a positive constant.

Proof. Let$f \in Lip_M(\rho)$ and $0 < \rho \leq 1$. by (3.4) and linearity and monotonicity of $L^{\alpha,\beta}_{n,p,q}$ then we have

$$\begin{aligned} |L^{\alpha,\beta}_{n,p,q}(f;x) - f(x)| &\leq L^{\alpha,\beta}_{n,p,q}(|f(t)-f(x)|;x) \\ &\leq L^{\alpha,\beta}_{n,p,q}(|t-x|^{\rho};x). \end{aligned}$$

Applying the Holder inequality with $m = \frac{2}{\rho}$ and $n = \frac{2}{2-\rho}$, we get

$$|L^{\alpha,\beta}_{n,p,q}(f;x) - f(x)| \leq (L^{\alpha,\beta}_{n,p,q}((t-x)^2;x))^{\frac{\rho}{2}}. \tag{3.5}$$

if we choose $\delta = \delta_n$ as above, then proof is completed.

Finally, we will study the rate of convergence of the positive linear operators $L^{\alpha,\beta}_{n,p,q}$ by means of the Peetre's K-functionals.

$C^2[0,1]$: The space of those functions f for which $f, f', f'' \in C[0,1]$. we recall the following norm in the space $C^2[0,1]$:

$$\|f\|_{C^2[0,1]}\ =\ \|f\|_{C[0,1]}\ +\ \|f'\|_{C[0,1]}\ +\ \|f''\|_{C[0,1]}.$$

We consider the following Peetre's K-functional

$$K(f,\delta) := \inf_{g \in C^2[0,1]}\left\{\|f-g\|_{C[0,1]}\ +\ \delta\|g\|_{C^2[0,1]}\right\}.$$

$\square$

Theorem 3.3. *Let $f \in C[0,1]$. Then we have*

$$\|L_{n,p,q}^{\alpha,\beta}(f;x) - f(x)\|_{C[0,1]} \;\leq\; 2K(f;\delta_n)$$

Where $K(f;\delta_n)$ is Peetre's functional and

$$\delta_n \;=\frac{1}{4}\left(\frac{q^2[n]_{p,q}[n-1]_{p,q}}{([n]_{p,q}\;+\;\beta)^2(p(1-x)+qx)}-\frac{[n]_{p,q}-\beta}{[n]_{p,q}\;+\;\beta}\right) + \frac{1}{4}\left(\frac{p^{n-1}[n]_{p,q}-2\alpha\beta}{([n]_{p,q}+\beta)^2}\right)+\frac{1}{4}\frac{\alpha^2}{([n]_{p,q}+\beta)^2}\;+\;\frac{1}{2}\frac{[\alpha]+[\beta]}{[n]_{p,q}+[\beta]}.$$

Proof. Let $g \in C^2[0,1]$. If we use the Taylor's expansion of the function g at $s = x$, we have

$$g(s) \;=\; g(x) \;+\; (s-x)g'(x) \;+\; \frac{(s-x)^2}{2}g''(x).$$

Hence we get

$$\|L_{n,p,q}^{\alpha,\beta}(f;x) - f(x)\|_{C[0,1]} \leq \|L_{n,p,q}^{\alpha,\beta}((s-x);x)\|_{C[0,1]}\|g(x)\|_{C^2[0,1]} + \frac{1}{2}\|L_{n,p,q}^{\alpha,\beta}((s-x)^2;x)\|_{C[0,1]}\|g(x)\|_{C^2[0,1]}. \tag{3.6}$$

From the lemma (2.1) we have

$$\|L_{n,p,q}^{\alpha,\beta}((s-x);x)\|_{C[0,1]} \leq \frac{[\alpha]+[\beta]}{[n]_{p,q}+[\beta]}. \tag{3.7}$$

So if we use (3.3) and (3.7) in (3.6), then we get

$$\|L_{n,p,q}^{\alpha,\beta}(g;x) - g(x)\|_{C[0,1]} \leq \Bigg[\frac{1}{2}\left(\frac{q^2[n]_{p,q}[n-1]_{p,q}}{([n]_{p,q}\;+\;\beta)^2(p(1-x)+qx)}-\frac{[n]_{p,q}-\beta}{[n]_{p,q}\;+\;\beta}\right) + \left(\frac{1}{2}\frac{p^{n-1}[n]_{p,q}-2\alpha\beta}{([n]_{p,q}+\beta)^2}\right)+\frac{1}{2}\frac{\alpha^2}{([n]_{p,q}+\beta)^2} \quad (3.8)$$
$$+\;\frac{[\alpha]+[\beta]}{[n]+[\beta]}\Bigg]\|g(x)\|_{C[0,1]}. \tag{3.9}$$

On the other hand, we can write

$$|L_{n,p,q}^{\alpha,\beta}(f;x)-f(x)\| \ \leq |L_{n,p,q}^{\alpha,\beta}(f-g;x)\| \ + \ |L_{n,p,q}^{\alpha,\beta}(g;x)-g(x)| \\ + \ |f(x)-g(x)|.$$

If we take the maximum on $[0,1]$, we have

$$\|L_{n,p,q}^{\alpha,\beta}(f;x)-f(x)\|_{C[0,1]} \leq 2\|f-g\|_{C[0,1]}+\|L_{n,p,q}^{\alpha,\beta}(g;x)-g(x)\|_{C[0,1]}. \quad (3.10)$$

If we consider (3.8) in (3.10), we obtain

$$\begin{aligned}\|L_{n,p,q}^{\alpha,\beta}(f;x)-f(x)\|_{C[0,1]} \ \leq & 2\|f-g\|_{C[0,1]}+\Bigg[\frac{1}{4}\bigg(\frac{q[n]_{p,q}[n-1]_{p,q}}{([n]_{p,q} \ + \ \beta)^2(p(1-x)+qx)} \\ & -\frac{[n]_{p,q}-\beta}{[n]_{p,q} \ + \ \beta}\bigg)+\bigg(\frac{1}{4}\frac{p^{n-1}[n]_{p,q}-2\alpha\beta}{([n]_{p,q}+\beta)^2}\bigg) \\ & +\frac{1}{4}\frac{\alpha^2}{([n]_{p,q}+\beta)^2}+ \ \frac{1}{2}\frac{[\alpha]+[\beta]}{[n]_{p,q}+[\beta]}\Bigg]\|g(x)\|_{C^2[0,1]}.\end{aligned}$$

If we choose

$$\begin{aligned}\delta_n \ = & \frac{1}{4}\bigg(\frac{q^2[n]_{p,q}[n-1]_{p,q}}{([n]_{p,q} \ + \ \beta)^2(p(1-x)+qx)}-\frac{[n]_{p,q}-\beta}{[n]_{p,q} \ + \ \beta}\bigg) \\ & +\bigg(\frac{1}{4}\frac{p^{n-1}[n]_{p,q}-2\alpha\beta}{([n]_{p,q}+\beta)^2}\bigg)+\frac{1}{4}\frac{\alpha^2}{([n]_{p,q}+\beta)^2} \ + \ \frac{1}{2}\frac{[\alpha]+[\beta]}{[n]_{p,q}+[\beta]},\end{aligned}$$

then we get

$$\|L_{n,p,q}^{\alpha,\beta}(f;x)-f(x)\|_{C[0,1]} \leq 2\bigg\{\|f-g\|_{C[0,1]} \ + \ \delta_n\|g(x)\|_{C^2[0,1]}\bigg\}. \quad (3.11)$$

Finally, one can observe that if we take the infimum of both side of above inequality for the function $g \in C^2[0,1]$, we can find

$$\|L_{n,p,q}^{\alpha,\beta}(f;x)-f(x)\|_{C[0,1]} \ \leq \ 2K(f,\delta_n).$$

□

4. The Rates of Statistical Convergence

At this point, let us recall the concept of statistical convergence. The statistical convergence which was introduced by Fast [41] in 1951, is an important research area in approximation theory. In [41], Gadjiev and Orhan used the concept of statistical convergence in approximation theory. They proved a Bohman-Korovkin type theorem for statistical convergence.

Recently, statistical approximation properties of many operators are investigated in [4, 25, 26, 29, 30].

A sequence $x = (x_k)$ is said to be statistically convergent to a number L if for every $\epsilon > 0$,

$$\delta\{K \in \mathbf{N} : |x_k - L| \geq \varepsilon\} = 0,$$

where $\delta(K)$ is the natural density of the set $K \subseteq \mathbf{N}$.

The density of subset $K \subseteq N$ is defined by

$$\delta(K) := \lim_n \frac{1}{n}\{\text{the number } k \leq n : k \in K\}$$

whenever the limit exists.

For instance, $\delta(\mathbf{N}) = 1,\ \delta\{2K : k \in \mathbf{N}\} = \frac{1}{2}$ and $\delta\{k^2 : K \in \mathbf{N}\} = 0.$

To emphasize the importance of the statistical convergence, we have an example: The sequence

$$X_k = \begin{cases} L_1;\ if\ k = m^2, \\ \\ L_2;\ if\ k \neq m^2. \end{cases} \quad where\ m \in \mathbf{N} \tag{4.1}$$

is statistically convergent to L_2 but not convergent in ordinary sense when $L_1 \neq L_2$. We note that any convergent sequence is statiscally convergent but not conversley.

Now we consider sequences $q = q_n$ and $p = p_n$ such that:

$$st - \lim_n q_n = 1,\ st - \lim_n p_n = 1,\ \text{and } st - \lim_n q_n^n = 1. \tag{4.2}$$

Gadjiev and Orhan [41] gave the following theorem for linear positive operators which is about statistically Korovkin type theorem. Now, we recall this theorem.

Theorem 4.1. *If A_n be the sequence of linear positive operators from $C[a,b]$ to $C[a,b]$ satisfies the conditions*
$st - \lim_{n} \|A_n((t^\nu; x)) - (x)^\nu\|_C[0,1] = 0$ *for* $\nu = 0,\ 1,\ 2.$

then for any function $f \in C[a,b]$,

$st - \lim_{n} \|A_n(f; .) - f\|_C[a,b] = 0.$

Now we will discuss the rates of statistical convergence of $L^{\alpha,\beta}_{n,p,q}$ operators.

Remark 4.2. *For $q \in (0,1)$ and $p \in (q,1]$, it is obvious that*

$$\lim_{n\to\infty} [n]_{p,q} = \begin{cases} 0, \ when \ \ p, q \in (0,1) \\ \\ \frac{1}{1-q}, \quad when \ p = 1 \ and \ q \in (0,1). \end{cases}$$

In order to reach to convergence results of the operator $L^n_{p,q}(f;x)$, we take a sequence $q_n \in (0,1)$ and $p_n \in (q_n, 1]$ such that $\lim_{n\to\infty} p_n = 1$, $\lim_{n\to\infty} q_n = 1$. So we get $\lim_{n\to\infty} [n]_{p_n,q_n} = \infty$.

Theorem 4.3. *Let $L^{\alpha,\beta}_{n,p,q}$ be the sequence of operators and the sequences $p = p_n$ and $q = q_n$ satisfies Remark 4.2 then for any function $f \in C[0,1]$*

$$st - \lim_{n} \| \ L^{\alpha,\beta}_{n,p_n,q_n}(f, .) - f\| = 0. \tag{4.3}$$

Proof. Clearly for $\nu = 0$,

$$L^{\alpha,\beta}_{n,p_n,q_n}(1, x) = 1,$$

which implies

$$st - \lim_{n} \|L^{\alpha,\beta}_{n,p_n,q_n}(1; x) \ - 1 \ \| \ = \ 0.$$

For $\nu \ = \ 1$

$$\begin{aligned} \|L^{\alpha,\beta}_{n,p_n,q_n} \ (t; x) \ - \ x \ \| &\leq \left| \frac{[n]_{p_n,q_n}}{[n]_{p_n,q_n} + \beta} x \ + \ \frac{\alpha}{[n]_{p_n,q_n} + \beta} \ - \ x \right| \\ &= \left| \left(\frac{[n]_{p_n,q_n}}{[n]_{p_n,q_n} + \beta} \ - \ 1 \right) x \ + \ \frac{\alpha}{[n]_{p_n,q_n} + \beta} \right| \\ &\leq \left| \frac{[n]_{p_n,q_n}}{[n]_{p_n,q_n} + \beta} \ - \ 1 \right| \ + \ \left| \frac{\alpha}{[n]_{p_n,q_n} + \beta} \right|. \end{aligned}$$

For a given $\epsilon > 0$, let us define the following sets.

$$U = \{n : \|L^{\alpha,\beta}_{n,p_n,q_n}(t;x) - x\| \geq \epsilon\}$$
$$U' = \{n : 1 - \frac{[n]_{p_n,q_n}}{[n]_{p_n,q_n} + \beta}\} \geq \epsilon$$
$$U'' = \{n : \frac{\alpha}{[n]_{p_n,q_n} + \beta} \geq \epsilon\}.$$

It is obvious that $U \subseteq U'' \cup U'$,

So using

$$\delta\{k \leq n : 1 - \frac{[n]_{p_n,q_n}}{[n]_{p_n,q_n} + \beta} \geq \epsilon\},$$

then we get

$$st - \lim_n \|L^{\alpha,\beta}_{n,p_n,q_n}(t;x) - x\| = 0. \tag{4.4}$$

Lastly for $\nu = 2$, we have

$$\|L^{\alpha,\beta}_{n,p_n,q_n}(t^2 : x) - x^2\| \leq \left|\frac{q^2[n]_{p_n,q_n}[n-1]_{p_n,q_n}}{p(1-x)+qx}\frac{1}{\left([n]_{p_n,q_n}+\beta\right)^2} - 1\right| + \left|\frac{[n]_{p_n,q_n}\left(2\alpha + p^{n-1}\right)^2}{[n]_{p_n,q_n}+\beta}x\right| + \left|\frac{\alpha^2}{\left([n]_{p_n,q_n}+\beta\right)^2}\right|.$$

If we choose

$$\alpha_n = \frac{q^2[n]_{p_n,q_n}[n-1]_{p_n,q_n}}{p(1-x)+qx}\frac{1}{\left([n]_{p_n,q_n}+\beta\right)^2} - 1$$
$$\beta_n = \frac{[n]_{p_n,q_n}\left(2\alpha + p^{n-1}\right)^2}{[n]_{p_n,q_n}+\beta}$$
$$\gamma_n = \frac{\alpha^2}{\left([n]_{p_n,q_n}+\beta\right)^2}.$$

Then

$$st - \lim_n \alpha_n = st - \lim_n \beta_n = st - \lim_n \gamma_n = 0.$$

Now given $\epsilon > 0$, we define the following four sets:

$$U = \|L^{\alpha,\beta}_{n,p_n,q_n}(t^2 : x) - x^2\| \geq \epsilon,$$
$$U_1 = \{n : \alpha_n \geq \frac{\epsilon}{3}\},$$
$$U_2 = \{n : \beta_n \geq \frac{\epsilon}{3}\},$$

$$U_3 = \{n : \gamma_n \geq \frac{\epsilon}{3}\}.$$

It is obvious that $U \subseteq U_1 \bigcup U_2 \bigcup U_3$. Thus we obtain

$$\delta\{K \leq n : \|L^{\alpha,\beta}_{n,p,q}(t^2 : x) - x^2\| \geq \epsilon\}$$
$$\leq \delta\{K \leq n : \alpha_n \geq \frac{\epsilon}{3}\} \ + \ \delta\{K \leq n : \beta_n \geq \frac{\epsilon}{3}\} + \delta\{K \leq n : \gamma_n \geq \frac{\epsilon}{3}\}.$$

So the right hand side of the inequalities is zero.

Then

$$st - \lim_n \|L^{\alpha,\beta}_{n,p_n,q_n}(t; x) - x\| = 0$$

holds and thus the proof is completed.

□

Acknowledgement

The authors are thankful to the learned referees for their valuable comments and suggestions leading to improvement of the paper.

References

1. T. Acar, (p, q)-Generalization of Szsz-Mirakyan operators, *Mathematical Methods in the Applied Sciences*, DOI: 10.1002/mma.3721.
2. T. Acar, A. Aral, S. A. Mohiuddine, On Kantorovich modification of (p, q)-Baskakov operators, *Journal of Inequalities and Applications*, **98**(2016).
3. A. Aral, V. Gupta, R. P. Agarwal, *Applications of q-Calculus in Operator Theory*, Springer, New York, 2013.
4. O. H. H. Edely, S. A. Mohiuddine, Abdullah K. Noman, Korovkin type approximation theorems obtained through generalized statistical convergence, *Appl. Math. Lett*, **23**(11), (2010), 1382-1387.
5. M. R. Eslahchi , S. Amani, The best uniform polynomial approximation of two classes of rational functions, *Iranian Journal of Mathematical Sciences and Informatics* Vol. 7, No. 2 (2012), pp 93-102.
6. S. N. Bernstein, constructive proof of Weierstrass approximation theorem, *Comm. Kharkov Math. Soc.* (1912)
7. P.E. Bèzier, *Numerical Control-Mathematics and applications*, John Wiley and Sons, London, 1972.
8. L.-W. Han, Y. Chu, Zh.-Y. Qiu, Generalized Bèzier curves and surfaces based on Lupaş q-analogue of Bernstein operator, *Jour. Comput. Appl. Math.* **261**, (2014), 352-363.
9. Q.-B. Cai, G. Zhoub, On (p, q)-analogue of kantorovich type Bernstein-stancu-schurer operators, *Appl. Math. Comput.*, **276**, (2016), 1220.
10. M. Norbert Hounkonnou, Joseph Désiré Bukweli Kyemba, $\mathcal{R}(p, q)$-calculus: differentiation and integration, *SUT Journal of Mathematics,* **49**(2), (2013), 145-167.

11. R. Jagannathan, K. Srinivasa Rao, Two-parameter quantum algebras, twin-basic numbers and associated generalized hypergeometric series, *Proceedings of the International Conference on Number Theory and Mathematical Physics*, (2005), 20-21.
12. U. Kadak, On weighted statistical convergence based on (p, q)-integers and related approximation theorems for functions of two variables, *Journal of Mathematical Analysis and Applications,* (2016), DOI: 10.1016/j.jmaa.2016.05.062.
13. U. Kadak, Naim L. Braha, H. M. Srivastava, Statistical weighted B-summability and its applications to approximation theorems, *Appl. Math. Comput.*, **302**, (2017), 80-96.
14. U. Kadak, Generalized weighted invariant mean based on fractional difference operator with applications to approximation theorems for functions of two variables, *Results in Mathematics*, (2016), doi:10.1007/s00025-016-0634-8.
15. U. Kadak, Weighted statistical convergence based on generalized difference operator involving (p, q)-Gamma function and its applications to approximation theorems, *Journal of Mathematical Analysis and Applications*, **448**(2), (2017), 1633-1650.
16. Kh. Khan, D. K. Lobiyal, A. Kilicman, A de casteljau algorithm for bernstein type polynomials based on (p, q)-integers, *Appl. Appl. Math.*, **13**(2) (2018).
17. Kh. Khan, D.K. Lobiyal, Bézier curves based on lupas (p, q)-analogue of Bernstein functions in CAGD, *Jour. Comput. Appl. Math.*, **317**, (2017), 458-477.
18. Kh. Khan, D.K. Lobiyal, Bézier curves and surfaces based on modified Bernstein polynomials, *Azerb. j. Math.*, **9**(1), 2019.
19. V. Kac, P. Cheung, *Quantum Calculus*, in: Universitext Series, vol.IX, Springer-Verlag, (2002).
20. P. P. Korovkin, Linear operators and approximation theory, *Hindustan Publishing Corporation*, Delhi, 1960.
21. A. Lupaş, A q-analogue of the Bernstein operator, Seminar on Numerical and Statistical Calculus, *University of Cluj-Napoca*, **9**(1987), 85-92.
22. N. I. Mahmudov, P. Sabancigil, Some approximation properties of Lupaş q-analogue of Bernstein operators, *arXiv*:1012.4245v1 [math.FA] 20 Dec 2010.
23. V. N. Mishra, S. Pandey, On (p, q) Baskakov-Durrmeyer-Stancu Operators, *Advances in Applied Clifford Algebras*, (2016), DOI: 10.1007/s00006-016-0738-y.
24. V. N. Mishra, S. Pandey; On Chlodowsky variant of (p, q) Kantorovich-Stancu-Schurer operators, *International Journal of Analysis and Applications*, **11**(1), (2016), 28-39.
25. N. L. Braha, H. M. Srivastava, S. A. Mohiuddine, A Korovkin's type approximation theorem for periodic functions via the statistical summability of the generalized de la Valle Poussin mean.*Appl. Math. Comput.*, **228**, (2014), 162-169.
26. C. Belen, S. A. Mohiuddine: Generalized weighted statistical convergence and application. *Appl. Math. Comput.,* **219**(18), (2013), 9821-9826 .
27. M. Mursaleen, K. J. Ansari, A. Khan, On (p, q)-analogue of Bernstein Operators, *Appl. Math. Comput.,* **266**(2015) 874-882, [Erratum: 278 (2016) 70-71].
28. M. Mursaleen, Md. Nasiruzzaman, A. Nurgali, Some approximation results on Bernstein-Schurer operators defined by (p, q)-integers, *Journal of Inequalities and Applications*, **249**, (2015), DOI 10.1186/s13660-015-0767-4.
29. M. Mursaleen, A. Khan, Statistical Approximation Properties of Modified q-Stancu-Beta Operators, *Bull. Malays. Math. Sci. Soc.*, (2) **36**(3) (2013), 683690.
30. M. Mursaleen, F. Khan, A. Khan, Statistical approximation for new positive linear operators of Lagrange type, *Appl. Math. Comput.,* **232**(2014) 548558.

31. M. Mursaleen, K. J. Ansari, A. Khan, Some Approximation Results by (p, q)-analogue of Bernstein-Stancu Operators, *Appl. Math. Comput.*, **264**, (2015), 392-402.
32. M. Mursaleen, K. J. Ansari, A. Khan, Some approximation results for Bernstein-Kantorovich operators based on (p, q)-calculus, *U.P.B. Sci. Bull., Series A,* **78**(4), (2016).
33. M. Mursaleen, Md. Nasiruzzaman, A. Khan, K. J. Ansari, Some approximation results on Bleimann-Butzer-Hahn operators defined by (p, q)-integers, *Filomat*, **30**(3), (2016), 639-648, DOI 10.2298/FIL1603639M.
34. M. Mursaleen, A. Khan, Generalized q-Bernstein-Schurer Operators and Some Approximation Theorems, *Journal of Function Spaces and Applications* **2013**, Article ID 719834, 7 pages http://dx.doi.org/10.1155/2013/719834
35. M. Mursaleen, F. Khan, A. Khan, Approximation by (p, q)-Lorentz polynomials on a compact disk, *Complex Anal. Oper. Theory*, DOI: 10.1007/s11785-016-0553-4.
36. M. Mursaleen, Md. Nasiruzzaman, F. Khan, A. Khan, On (p, q)-analogue of divided differences and Bernstein operators, *Journal of Nonlinear Functional Analysis*, **2017**, (2017), Article ID 25, pp. 1-13.
37. S. Ostrovska, On the Lupaş q-analogue of the bernstein operator, *Rocky mountain journal of mathematics*, **36**(5), 2006.
38. G. M. Phillips, Bernstein polynomials based on the q-integers, The heritage of P.L.Chebyshev, *Ann. Numer. Math.*, **4**, (1997), 511-518.
39. G. M. Phillips, A generalization of the Bernstein polynomials based on the q-integers *ANZIAMJ* **42**, (2000), 79-86.
40. A. Rababah, S. Manna, Iterative process for G2-multi degree reduction of Bèzier curves, *Appl. Math. Comput.,* 217 (2011) 8126-8133.
41. A. D. Gadjiev, C. Orhan,Some approximation theorems bystatistical convergence, *Rocky Mountain J.math.*, **32**, (2002) 129-138.
42. A. Wafi, N. Rao, Bivariate-Schurer-Stancu operators based on (p, q)-integers, *Filomat*, **32**(4), (2018), 1251-1258.
43. A. Wafi, N. Rao, (p, q)-Bivariate-Bernstein-Chlowdosky Operators, *Filomat*, **32**(2), (2018), 369378.
44. A. Wafi, N. Rao, Approximation properties of (p, q)-variant of Stancu-Schurer, *Boletim da Sociedade Paranaense de Matematica* **37**(4), (2017).
45. A. Wafi, N. Rao, A generalization of Szasz-type operators which preserves constant and quadratic test functions, *cogent mathematics,* **3**(1), 2016, 1227023.
46. A. Wafi, N. Rao Modified Szasz operators involving Charlier polynomials based on two parameters, *Thai J. of Maths.*, 2016.
47. A. Wafi, N. Rao Approximation properties by generalized Baskakov Kantorovich Stancu type operators, *Appl. Maths Inf. Sci. Lett.*, **4**(3) (2016), PP:111-118.
48. A. Wafi, N. Rao, Stancu-variant of generalized Baskakov operators, *Filomat*, **31**(9), (2017).