A Shorter and Simple Approach to Study Fixed Point Results via b-Simulation Functions

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Abstract. The purpose of this short note is to consider much shorter and nicer proofs about fixed point results on b-metric spaces via b-simulation function introduced very recently by Demma et al. [M. Demma, R. Saadati, P. Vetro, Fixed point results on b-metric space via Picard sequences and b-simulation functions, Iranian J. Math. Sci. Infor. 11 (1) (2016) 123-136].

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1. Introduction and Preliminaries

In 2015, Khojasteh et al. [4] gave a new approach to study fixed point results in the framework of metric spaces via simulation function as follows:

A mapping $\zeta : [0, +\infty)^2 \to \mathbb{R}$ is called a simulation function if it satisfies the following:

\begin{align*}
(\zeta_1) & \quad \zeta(0,0) = 0; \\
(\zeta_2) & \quad \zeta(t,s) < s - t \text{ for all } t, s > 0; \\
(\zeta_3) & \quad \text{if } \{t_n\}, \{s_n\} \text{ are sequences in } (0, +\infty) \text{ such that } \lim_{n \to \infty} t_n = \lim_{n \to \infty} s_n > 0, \text{ then } \lim_{n \to \infty} \zeta(t_n, s_n) < 0.
\end{align*}

Also, they denoted the set of all simulation functions by $\mathcal{Z}$.

It is worth noticing that Argoubi et al. [1] revised the above definition by withdrawing the condition $(\zeta_1)$ (also, see [7]). Also, Roldan et al. [8] revised $(\zeta_3)$ by taking $t_n < s_n$. Hence, we can say that a mapping $\zeta : [0, +\infty)^2 \to \mathbb{R}$ is called a simulation function if it satisfies:

\begin{align*}
(\zeta_2) & \quad \zeta(t,s) < s - t \text{ for all } t, s > 0; \\
(\zeta_3) & \quad \text{if } \{t_n\}, \{s_n\} \text{ are sequences in } (0, +\infty) \text{ such that } \lim_{n \to \infty} t_n = \lim_{n \to \infty} s_n > 0 \text{ and } t_n < s_n \text{ for all } n \in \mathbb{N}, \text{ then } \lim_{n \to \infty} \zeta(t_n, s_n) < 0.
\end{align*}

For several examples of simulation functions, see [1, 2, 4, 6, 7, 8].

**Definition 1.1.** [4] Let $(X,d)$ be a metric space and $\zeta \in \mathcal{Z}$. Then a mapping $T : X \to X$ is called a $\mathcal{Z}$-contraction with respect to $\zeta$ if the following condition is satisfied:

$$\zeta(d(Tx, Ty), d(x,y)) \geq 0 \quad \forall x, y \in X. \quad (1.1)$$

Now, it is clear that $\zeta(t, t) < 0$ when $t > 0$; further (1.1) implies that $d(Tx, Ty) < d(x, y)$ when $x \neq y$ for each $x, y \in X$. This means that each $\mathcal{Z}$-contraction with respect to $\zeta$ is continuous.

**Theorem 1.2.** [4] Let $(X,d)$ be a complete metric space and $T : X \to X$ be a $\mathcal{Z}$-contraction with respect to $\zeta$. Then $T$ has a unique fixed point in $X$ and for every $x_0 \in X$, the Picard sequence $\{x_n\}$, where $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$, converges to the fixed point of $T$.

One very important and significant kind of generalized (standard) metric spaces are so-called b-metric spaces (or metric type spaces). Namely, $(X,d)$ is b-metric space if $X \neq \emptyset$ and $d : X \times X \to [0, +\infty)$ be a mapping such that for all $x, y, z \in X$ hold: $d(x, y) = 0 \iff x = y; d(x, y) = d(y, x)$ and $d(x, y) \leq b(d(x, y) + d(y, z))$ for $b \geq 1$. Then $d$ is called b–metric. For more details on b-metric spaces, see [2, 3, 5] and the references contained therein.

Recently, Demma et al. [2] introduced the b-simulation function in the framework of b-metric spaces as follows.

**Definition 1.3.** Let $(X,d)$ be a b-metric space. A b-simulation function is a function $\zeta : [0, +\infty)^2 \to \mathbb{R}$ satisfying the following:
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(ξ₁) $ξ(t, s) < s - t$ for all $t, s > 0$;

(ξ₂) if $\{t_n\}, \{s_n\}$ are sequences in $(0, +\infty)$ such that

$$0 < \lim_{n \to +\infty} t_n \leq \lim_{n \to +\infty} s_n \leq \lim_{n \to +\infty} b s_n \leq b \lim_{n \to +\infty} t_n < +\infty,$$  \hspace{1cm} (1.2)

then $\lim_{n \to +\infty} \xi(bt_n, s_n) < 0$.

It is clear if $b = 1$, then $b$-simulation function is in fact the simulation function in the framework of (standard) metric spaces.

Example 1.4. [2] Let $ξ : [0, +\infty)^2 \to \mathbb{R}$ be defined by

(i) $ξ(t, s) = \lambda s - t$ for all $t, s \in [0, +\infty)$, where $\lambda \in (0, 1)$.

(ii) $ξ(t, s) = \psi(s) - \varphi(t)$ for all $t, s \in [0, +\infty)$, where $\varphi, \psi : [0, +\infty) \to [0, +\infty)$ are two continuous functions such that $\psi(t) = \varphi(t) = 0$ if and only if $t = 0$ and $\psi(t) < t \leq \varphi(t)$ for all $t > 0$.

(iii) $ξ(t, s) = s - \frac{f(t, s)}{g(t, s)} t$ for all $t, s \in [0, +\infty)$, where $f, g : [0, +\infty)^2 \to (0, +\infty)$ are two continuous functions with respect to each variable such that $f(t, s) > g(t, s)$ for all $t, s > 0$.

(iv) $ξ(t, s) = s - \varphi(s) - t$ for all $t, s \in [0, +\infty)$, where $\varphi : [0, +\infty) \to [0, +\infty)$ is a lower semi-continuous function such that $\varphi(t) = 0$ if and only if $t = 0$.

(v) $ξ(t, s) = s\varphi(s) - t$ for all $t, s \in [0, +\infty)$, where $\varphi : [0, +\infty) \to [0, 1]$ is such that $\lim_{t \to +\infty} \varphi(t) < 1$ for all $r > 0$.

Each of the function considered in (i)-(v) is a $b$-simulation function.

The following important and very interesting results are proved in [2].

Lemma 1.5. Let $(X, d)$ be a $b$-metric space and $f : X \to X$ be a mapping.
Suppose that there exists a $b$-simulation function $ξ$ such that following condition holds.

$$ξ(b d(f x, f y), d(x, y)) \geq 0 \hspace{1cm} \forall x, y \in X.$$  \hspace{1cm} (1.3)

Let $\{x_n\}$ be a sequence of Picard of initial at point $x_0 \in X$ and $x_{n-1} \neq x_n$ for all $n \in \mathbb{N}$. Then

$$\lim_{n \to +\infty} d(x_{n-1}, x_n) = 0.$$  

Lemma 1.6. Let $(X, d)$ be a $b$-metric space and $f : X \to X$ be a mapping.
Suppose that there exists a $b$-simulation function $ξ$ such that (1.3) holds. Let $\{x_n\}$ be a sequence of Picard of initial at point $x_0 \in X$ and $x_{n-1} \neq x_n$ for all $n \in \mathbb{N}$. Then $\{x_n\}$ is a bounded sequence.

Lemma 1.7. Let $(X, d)$ be a $b$-metric space and $f : X \to X$ be a mapping.
Suppose that there exists a $b$-simulation function $ξ$ such that (1.3) holds. Let $\{x_n\}$ be a sequence of Picard of initial at point $x_0 \in X$ and $x_{n-1} \neq x_n$ for all $n \in \mathbb{N}$. Then $\{x_n\}$ is a Cauchy sequence.
Theorem 1.8. Let \((X,d)\) be a complete b-metric space and let \(f : X \to X\) be a mapping. Suppose that there exists a b-simulation function \(\xi\) such that (1.3) holds; that is,
\[
\xi(bd(fx, fy), d(x, y)) \geq 0 \quad \forall x, y \in X.
\]
Then \(f\) has a unique fixed point.

For the proof of Theorem 1.8, Demma et al. [2] used Lemmas 1.5-1.7.

2. Main results

In this section we improve the main result from [2]; that is, we prove Theorem 1.8 without using all three lemmas 1.5-1.7. At the first, we quote some well known results from b-metric spaces. The following lemma was used (and proved) in the course of proofs of several fixed point results in the framework of b-metric spaces in [3].

Lemma 2.1. Let \(\{y_n\}\) be a sequence in a b-metric space \((X,d)\) such that
\[
d(y_n, y_{n+1}) \leq \lambda d(y_{n-1}, y_n)
\]
for some \(\lambda, 0 \leq \lambda < \frac{1}{b}\) and each \(n = 1, 2, \ldots\). Then \(\{y_n\}\) is a Cauchy sequence in \((X,d)\).

By utilizing Lemma 2.1, Jovanović et al. [3] proved following result.

Theorem 2.2. Let \((X,d)\) be a complete b-metric space and \(f : X \to X\) be a map such that
\[
d(fx, fy) \leq \lambda d(x, y)
\]
holds for all \(x, y \in X\), where \(0 \leq \lambda < \frac{1}{b}\). Then \(f\) has a unique fixed point \(z\) and for every \(x_0 \in X\), the sequence \(\{f^n x_0\}\) converges to \(z\).

Now we formulate and prove Theorem 1.8 via a shorter and simple approach.

Theorem 2.3. Let \((X,d)\) be a complete b-metric space and \(f : X \to X\) be a mapping. Suppose that there exists a b-simulation function \(\xi\) such that (1.3) holds; that is,
\[
\xi(bd(fx, fy), d(x, y)) \geq 0 \quad \forall x, y \in X.
\]
Then \(f\) has a unique fixed point.

Proof. It is enough clear that (2.3) implies
\[
bd(fx, fy) \leq d(x, y) \quad \forall x, y \in X.
\]
Indeed, (2.4) holds if \(x = y\). In the case that \(x \neq y\) there are two possibilities, either \(fx = fy\) or \(fx \neq fy\). In the first case we have that \(b \cdot d(fx, fy) = 0 < d(x, y)\), while in second case the result follows from (\(\xi_1\)). This means that (2.3) implies (2.4) for all \(x, y \in X\). Further, obviously, (2.4) implies that
\[
d(f^2 x, f^2 y) \leq \frac{1}{b^2} d(x, y) = \lambda d(x, y).
\]
Since \( \lambda = \frac{1}{b^2} \in [0, \frac{1}{b}) \), then according to Theorem 2.2, \( f^2 \) has a unique fixed point (say \( z \)) in \( X \). This further means that \( f \) has a unique fixed point \( z \) in \( X \). Now, the proof of this theorem is complete. \( \square \)

Obviously, our proof is much shorter than the corresponding ones from Demma et al.’s work [2]. It is very interesting that all four Corollaries 4.1-4.4 from [2] follows immediately according to our easy approach. Thus we have following corollary.

**Corollary 2.4.** Let \( (X, d) \) be a complete \( b \)-metric space and let \( f : X \to X \) be a mapping. Suppose that

(i) \( \lambda \in [0, 1) \) such that \( bd(fx, fy) \leq \lambda d(x, y) \);

(ii) a lower semi-continuous function \( \varphi : [0, +\infty) \to [0, +\infty) \) with \( \varphi^{-1}(0) = \{0\} \) such that \( bd(fx, fy) \leq d(x, y) - \varphi(d(x, y)) \);

(iii) \( \varphi : [0, +\infty) \to [0, 1) \) with \( \lim_{t \to r^+} \varphi(t) < 1 \) for all \( r > 0 \) such that \( bd(fx, fy) \leq d(x, y) \) for all \( x, y \in X \).

(iv) \( \eta : [0, +\infty) \to [0, +\infty) \) with \( \eta(t) < t \) for all \( t > 0 \) and \( \eta(0) = 0 \) such that \( bd(fx, fy) \leq \eta(d(x, y)) \)

for all \( x, y \in X \). Then \( f \) has a unique fixed point in each one of above condition.

**Proof.** Obviously, each one of mentioned conditions implies the condition (2.4) by selecting the appropriate \( b \)-simulation function in Example 1.4. Hence, we obtain that \( bd(fx, fy) \leq d(x, y) \) for all \( x, y \in X \). The result then follows according to Theorem 2.3. \( \square \)

**Example 2.5.** Now, we consider Example 4.5 from [2]. Let \( X = [0, 1] \) and \( d : X \times X \to \mathbb{R} \) be defined by \( d(x, y) = |x - y|^2 \). Then \( (X, d) \) is a complete \( b \)-metric space with \( b = 2 \). Consider a mapping \( f : X \to X \) by

\[
fx = \frac{ax}{1 + x}
\]

for all \( x \in X \), where \( a \in [0, \frac{1}{\sqrt{2}}] \). Now, we have

\[
2d(fx, fy) = 2 \left| \frac{ax}{1 + x} - \frac{ay}{1 + y} \right|^2 = 2a^2 \frac{|x - y|^2}{(1 + x)^2 (1 + y)^2} \leq |x - y|^2 = d(x, y)
\]

for all \( x, y \in X \). Further, (2.6) implies that

\[
d(f^2x, f^2y) \leq \frac{1}{4} d(x, y);
\]

that is, \( f^2 \) has a unique fixed point according to Theorem 2.2. This means that \( f \) has a unique fixed point. Here it is \( z = 0 \).

The next result is probably known, but our proof is very condensed.
Theorem 2.6. Let $(X, d)$ be a complete $b$-metric space and let $f : X \to X$ be a mapping such that
\[
d(fx, fy) \leq \lambda d(x, y)
\]
for all $x, y \in X$, where $\lambda \in [0, 1)$. Then $f$ has a unique fixed point (say $z$) in $X$ and for $x_0 \in X$ the sequence $\{f^n x_0\}_{n \in \mathbb{N}}$ converges to $z$.

Proof. The condition (2.7) implies that
\[
d(f^n x, f^n y) \leq \lambda d(f^{n-1} x, f^{n-1} y) \leq \cdots \leq \lambda^n d(x, y)
\]
for all $x, y \in X$ and $n \in \mathbb{N}$. Since $\lambda^n \to 0$ as $n \to \infty$, there is $k \in \mathbb{N}$ such that $\lambda^k < \frac{1}{b}$. Therefore, we have
\[
d(f^{k+1} x, f^{k+1} y) \leq \frac{1}{b^2} d(x, y).
\]
The result now follows by Theorem 2.2. \qed

Question 1. Does Theorem 2.3 holds if $\xi(d(fx, fy), d(x, y)) \geq 0$ for all $x, y \in X$, where $(X, d)$ is a given complete $b$-metric space and $f : X \to X$ be a mapping and $\xi$ a given $b$-simulation function?

Question 2. Can you obtain this results by considering ordered $b$-metric spaces or cone $b$-metric spaces instead of $b$-metric spaces?

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