

**On the 2-Adjointable Operators and Superstability of them
between 2-Pre Hilbert C^* -Module Spaces**

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ABSTRACT. In this paper, first, we introduce the new concept of 2-inner product on Banach modules over a C^* -algebra. Next, we present the concept of 2-linear operators over a C^* -algebra. Our result improve the main result of the paper [Z. Lewandowska, *On 2-normed sets*, *Glasnik Mat.*, **38(58)** (2003), 99-110]. In the end of this paper, we define the notions 2-adjointable mappings between 2-pre Hilbert C^* -modules and prove superstability of them in the spirit of Hyers-Ulam-Rassias.

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1. INTRODUCTION

The concept of 2-inner product has been intensively studied by many authors in the last three decades. The basic definitions and elementary properties of 2-inner product spaces can be found in [1] and [2].

Recently, M.Frank and e.t. defined the notion ϕ -perturbation of an adjointable mapping and proved the superstability of an adjointable mapping on Hilbert C^* -modules(see [3]).

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In this paper, first, we introduce the definition 2-pre Hilbert C^* -module spaces and give several important properties. Next, we present the concept of 2-linear operators over a C^* -algebra which coincides with Lewandowska's definition (see [4, 5]). Also, we define 2-adjoinable mappings between 2-pre Hilbert C^* -modules and prove an analogue of ϕ -perturbation of adjointable mappings in paper([3]).

We refer the interested reader to monographs [6, 7, 8, 9] and references therein for more information.

2. 2-PRE HILBERT MODULES

Let X be a left module over a C^* -algebra A . An action of $a \in A$ on X is denoted by $a.x \in X$, $x \in X$.

Definition 2.1. A 2-pre Hilbert A -module is a left A -module X equipped with A -valued function defined on $X \times X \times X$ satisfying the following conditions:

I_1) $(x, x|z)$ is a positive element in A for any $x, z \in X$ and $(x, x|z) = 0$ if and only if x and z are linearly dependent;

I_2) $(x, x|z) = (z, z|x)$ for any $x, z \in X$;

I_3) $(y, x|z) = (x, y|z)^*$ for any $x, y, z \in X$;

I_4) $(\alpha x + x', y|z) = \alpha(x, y|z) + (x', y|z)$ for any $\alpha \in \mathbb{C}$ and $x, x', y, z \in X$;

I_5) $(ax, y|z) = a(x, y|z)$ for any $x, y, z \in X$ and any $a \in A$.

The map $(\cdot, \cdot|\cdot)$ is called A -valued 2-inner product and $(X, (\cdot, \cdot|\cdot))$ is called 2-pre Hilbert C^* -module space.

EXAMPLE 2.2. Every 2-inner product space is a 2-pre Hilbert \mathbb{C} -module.

EXAMPLE 2.3. Let A be a C^* -algebra and $J \subseteq A$ be a left ideal. Then J can be equipped with the structure of 2-pre Hilbert A -module with A -valued inner product $(x, y|z) := xy^*zz^* - xz^*zy^*$ for any $x, y, z \in A$.

Definition 2.4. Let X be a 2-pre Hilbert A -module. we can define a function $\|\cdot\|_X$ on $X \times X$ by $\|x|z\|_X = \|(x, x|z)\|_A^{\frac{1}{2}}$ for all $x, z \in X$.

Lemma 2.5. $\|\cdot\|_X$ satisfies the following conditions:

N1) $\|ax|z\|_X \leq \|a\| \|x|z\|_X$ for any $x, z \in X$ and $a \in A$;

N2) $(x, y|z)(y, x|z) \leq \|y|z\|_X^2 (x, x|z)$ for any $x, y, z \in X$;

N3) $\|(x, y|z)\|^2 \leq \|(x, x|z)\| \|(y, y|z)\|$

Proof. N1 is obvious; N3 follows from N2, so let us prove N2.

Let ϕ be a positive linear functional on A . Then $\phi((\cdot, \cdot|\cdot))$ is usual 2-inner product on X . Applying the Schwartz inequality for 2-inner product (see [2],

page 3) we obtain for all $x, y, z \in X$,

$$\begin{aligned} \phi((x, y|z) (y, x|z)) &= \phi((x, y|z)y, x |z)) \\ &\leq \phi((x, x|z))^{\frac{1}{2}} \phi(((x, y|z)y, (x, y|z)y |z))^{\frac{1}{2}} \\ &\leq \phi((x, x|z))^{\frac{1}{2}} \phi((x, y|z) (y, y|z) (x, y|z)^*)^{\frac{1}{2}} \\ &\leq \phi((x, x|z))^{\frac{1}{2}} \|(y, y|z)\|^{\frac{1}{2}} \phi((x, y|z) (y, x|z))^{\frac{1}{2}}. \end{aligned}$$

Thus, for any positive linear functional ϕ , we have

$$\phi((x, y|z) (y, x|z)) \leq \|y|z\|_X^2 \phi((x, x|z))$$

hence

$$(x, y|z) (y, x|z) \leq \|y|z\|_X^2 (x, x|z).$$

□

Theorem 2.6. *The function $\|\cdot\|_X$ is a 2-norm on X .*

Proof. Now, we verify that $\|\cdot\|_X$ satisfies the following properties of 2-norms:

1) I_3 and I_4 show that $\|\alpha x|y\|_X = \|(\alpha x, \alpha x|y)\|^{\frac{1}{2}} = |\alpha| \|x|y\|_X$ for all $x, y \in X$ and $\alpha \in \mathbb{C}$.

2) I_1 follows that $\|x|y\|_X = 0$ if and only if x and y are linearly dependent for all $x, y \in X$.

3) it follows from I_2 that $\|x|y\|_X = \|(x, x|y)\|^{\frac{1}{2}} = \|y|x\|_X$ for all $x, y \in X$.

4) By proposition 2.5 ($N3$), we have

$$\begin{aligned} \|x + x'|y\|_X^2 &= \|(x + x', x + x'|y)\| = \|(x, x|y) + (x', x'|y) + (x, x'|y) + (x', x'|y)\| \\ &\leq \|(x, x|y)\| + 2\|(x, x'|y)\| + \|(x', x'|y)\| \\ &\leq (\|(x, x|y)\|^{\frac{1}{2}} + \|(x', x'|y)\|^{\frac{1}{2}})^2 = (\|x|y\|_X + \|x'|y\|_X)^2 \end{aligned}$$

for all $x, x', y \in X$. This show that $(X, \|\cdot\|_X)$ is a 2-normed space. □

3. 2-ADJOINTABLE MAPPINGS

In continue, we let A be a C^* -algebra. Now, we start with following definition.

Definition 3.1. Let X and Y be two 2-pre Hilbert A -modules. An operator $f : X \times X \rightarrow Y$ is said to be A -2 linear if it satisfies the following conditions:

- 1) $f(x + y, z + w) = f(x, z) + f(x, w) + f(y, z) + f(y, w)$ for all $x, y, z, w \in X$;
- 2) $f(\alpha x, \beta y) = \alpha \bar{\beta} f(x, y)$ for all $\alpha, \beta \in \mathbb{C}$ and $x, y \in X$;
- 3) $f(ax, by) = a \cdot b^* \cdot f(x, y)$ for all $x, y \in X$ and $a, b \in A$.

EXAMPLE 3.2. Let X be a 2-pre Hilbert A -module and $z \in X$. Define $f : X \times X \rightarrow A$ by $f(x, y) = (x, y|z)$. Then f is a A - 2 linear operator.

Definition 3.3. Let X and Y be two 2-pre Hilbert A -modules. A mapping $f : X \times X \rightarrow Y$ is called 2-adjointable if there exists a mapping $g : Y \times Y \rightarrow X$ such that

$$(f(x, y), s | t) = (x, y | g(s, t)) \quad (3.1)$$

for all $x, y \in X$ and $s, t \in Y$. The mapping g is denoted by f^* and is called the 2-adjointable of f .

Lemma 3.4. Let X be a 2-pre Hilbert A -module and $\dim(X) > 1$. If $(x, y|z) = 0$ for all $y, z \in X$, then $x = 0$.

Proof. Suppose $x \neq 0$. Let x and y be linearly independent. Then by hypothesis $(x, x|y) = 0$ and this is contradiction. \square

Lemma 3.5. Every 2-adjonable mapping is A -2 linear.

Proof. Let $f : X \times X \rightarrow Y$ be a 2-adjonable mapping. Then there exists a mapping $g : Y \times Y \rightarrow X$ such that (3.1) holds. For every $x, y, z, w \in X$, every $s, t \in Y$, every $\alpha, \beta \in \mathbb{C}$, every $a, b \in A$, we have

$$\begin{aligned} (f(\alpha ax + y, \beta bz + w), s | t) &= (\alpha ax + y, \beta bz + w | g(s, t)) \\ &= \alpha \bar{\beta} ab^* (x, z | g(s, t)) + \alpha a (x, w | g(s, t)) + \bar{\beta} b^* (y, z | g(s, t)) + (y, w | g(s, t)) \\ &= \alpha \bar{\beta} ab^* (f(x, z), s | t) + \alpha a (f(x, w), s | t) + \bar{\beta} b^* (f(y, z), s | t) + (f(y, w), s | t) \\ &= (\alpha \bar{\beta} ab^* f(x, z) + \alpha a f(x, w) + \bar{\beta} b^* f(y, z) + f(y, w), s | t). \end{aligned}$$

It follows from lemma 3.4 that f is A -2 linear. \square

4. SUPERSTABILITY OF 2-ADJOINABLE MAPPINGS

In this section, X and Y denote 2-pre Hilbert A -modules and $\dim(X) > 1$, $\dim(Y) > 1$ and $\phi : X^2 \times Y^2 \rightarrow [0, \infty)$ is a function. We start our work with following definition.

Definition 4.1. A (not necessarily A -2 linear) mapping $f : X \times X \rightarrow Y$ is called

ϕ -perturbation of an 2-adjointable mapping if there exists a mapping (not necessarily A -2 linear) $g : Y \times Y \rightarrow X$ such that

$$\|(f(x, y), s | t) - (x, y | g(s, t))\| \leq \phi(x, y, s, t) \quad (4.1)$$

for all $x, y \in X$ and $s, t \in Y$.

Theorem 4.2. Let $f : X \times X \rightarrow Y$ be a ϕ -perturbation of a 2-adjointable mapping with corresponding mapping $g : Y \times Y \rightarrow X$. Suppose for some sequence c_n of non-zero complex numbers the following conditions hold:

$$\lim_{n \rightarrow \infty} |c_n|^{-1} \phi(c_n x, y, s, t) = 0 \quad (x, y \in X, s, t \in Y) \quad (4.2)$$

$$\lim_{n \rightarrow \infty} |c_n|^{-1} \phi(x, y, c_n s, t) = 0 \quad (x, y \in X, s, t \in Y) \quad (4.3)$$

Then f is 2-adjointable and hence f is A-2 linear.

Proof. Let $\lambda \in \mathbb{C}$ be an arbitrary number. Putting λx instead x in (4.1), we get

$$\|(f(\lambda x, y), s | t) - (\lambda x, y | g(s, t))\| \leq \phi(\lambda x, y, s, t)$$

multiplication of (4.1) by $|\lambda|$, we have

$$\|(\lambda f(x, y), s | t) - \lambda(x, y | g(s, t))\| \leq |\lambda| \phi(x, y, s, t)$$

Thus,

$$\|(f(\lambda x, y), s | t) - (\lambda f(x, y), s | t)\| \leq \phi(\lambda x, y, s, t) + |\lambda| \phi(x, y, s, t) \quad (4.4)$$

Replacing $c_n s$ by s in (4.4), we get

$$\|f(\lambda x, y), s | t) - (\lambda f(x, y), s | t)\| \leq |c_n|^{-1} \phi(\lambda x, y, c_n s, t) + |\lambda| |c_n|^{-1} \phi(x, y, c_n s, t)$$

hence, as $n \rightarrow \infty$, applying (4.3) we obtain

$$(f(\lambda x, y), s | t) - (\lambda f(x, y), s | t) = 0 \quad (\lambda \in \mathbb{C}, x, y \in X, s, t \in Y).$$

It follows from proposition 3.4 that

$$f(\lambda x, y) = \lambda f(x, y) \quad (\lambda \in \mathbb{C}, x, y \in X) \quad (4.5)$$

Now, we take $c_n x$ instead x in (4.1) to get

$$\|(f(c_n x, y), s | t) - (c_n x, y | g(s, t))\| \leq \phi(c_n x, y, s, t).$$

It follows from (4.5) that

$$\|(f(x, y), s | t) - (x, y | g(s, t))\| \leq |c_n|^{-1} \phi(c_n x, y, s, t)$$

hence, as $n \rightarrow \infty$, applying (4.2) we get

$$(f(x, y), s | t) = (x, y | g(s, t)) \quad (x, y \in X, s, t \in Y).$$

Therefore f is 2-adjointable and by Lemma 3.5, f is A-2 linear. \square

In the following, we let $c_n = a^n$ that $a > 1$. we get the following results.

Corollary 4.3. *If $f : X \times X \rightarrow Y$ is a ϕ -perturbation of a 2-adjointable mapping, where*

$\phi(x, y, s, t) = \epsilon \|x|y\|_X^p \|s|t\|_Y^q$ ($\epsilon \geq 0$, $0 < p < 1$, $0 < q < 1$), then f is 2-adjointable and hence f is A-2-linear.

Corollary 4.4. *If $f : X \times X \rightarrow Y$ is a ϕ -perturbation of a 2-adjointable mapping, where*

$\phi(x, y, s, t) = \epsilon_1 \|x|y\|_X^p + \epsilon_2 \|s|t\|_Y^q$ ($\epsilon_1 \geq 0$, $\epsilon_2 \geq 0$, $0 < p < 1$, $0 < q < 1$). Then f is 2-adjointable and hence f is A-2 linear.

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