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z_R-Ideals and z_R°-Ideals in Subrings of $\mathbb{R}^X$

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Abstract. Let X be a topological space and R be a subring of $\mathbb{R}^X$. By determining some special topologies on X associated with the subring R, characterizations of maximal fixed and maximal g-ideals in R of the form $M_x(R)$ are given. Moreover, the classes of z_R-ideals and z_R°-ideals are introduced in R which are topological generalizations of z-ideals and z°-ideals of $C(X)$, respectively. Various characterizations of these ideals are established. Also, coincidence of z_R-ideals with z-ideals and z_R°-ideals with z°-ideals in R are investigated. It turns out that some fundamental statements in the context of $C(X)$ are extended to the subrings of $\mathbb{R}^X$.

Keywords: $Z(R)$-topology, $Coz(R)$-topology, g-ideal, z_R-ideal, z_R°-ideal, invertible subring.

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1. Introduction

For a topological space X, $\mathbb{R}^X$ denotes the algebra of all real-valued functions and $C(X)$ (resp., $C^*(X)$) denotes the subalgebra of $\mathbb{R}^X$ consisting of all continuous functions (resp., bounded continuous functions). Moreover, we use R to denote a unital subring of $\mathbb{R}^X$. Note that topological spaces which are considered in this paper are not necessarily Tychonoff. For each $f \in \mathbb{R}^X$,

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$Z(f) = \{x \in X : f(x) = 0\}$ denotes the zero-set of f and $Coz(f)$ denotes the complement of $Z(f)$ with respect to X. We denote by $Z(R)$ the collection of all the zero-sets of elements of R, we use $Z(X)$ instead of $Z(C(X))$. We denote by $M_x(R)$ the set $\{f \in R : x \in Z(f)\}$, $M_x(C(X))$ is denoted by M_x. The subring R is called invertible, if $f \in R$ and $Z(f) = \emptyset$ implies that f is invertible in R. Moreover, R is called a lattice-ordered subring if it is a sublattice of $\mathbb{R}^X$ (i.e., $f \wedge g$ and $f \vee g$ are in R for each $f, g \in R$). It is clear that $C(X)$ is an invertible lattice-orderd subring of $\mathbb{R}^X$. However, the same statement does not hold for $C^*(X)$. A proper ideal I of R is called a growing ideal, briefly, a g-ideal, if contains no invertible element of $\mathbb{R}^X$, i.e., $Z(f) \neq \emptyset$ for each $f \in I$. It is evident that a subring R is invertible if and only if every ideal every ideal of R is a g-ideal. Clearly, M^{*p}, for each $p \in \beta X \setminus \upsilon X$, is not a g-ideal of $C^*(X)$. An ideal I of R is called fixed if $\bigcap_{f \in I} Z(f) \neq \emptyset$, otherwise, it is called free. By a maximal fixed ideal of R, we mean a fixed ideal which is maximal in the set of all fixed ideals of R. An ideal I in a commutative ring S is called a z-ideal (resp., z°-ideal) if $M_a(S) \subseteq I$ (resp., $P_a(S) \subseteq I$), for each $a \in I$, where $M_a(S)$ (resp., $P_a(S)$) denotes the intersection of all the maximal (resp., minimal prime) ideals of S containing a. It is well-known that in $C(X)$ an ideal I is a z-ideal (resp., z°-ideal) if and only if whenever $Z(f) \subseteq Z(g)$ (resp., $\operatorname{int}_X Z(f) \subseteq \operatorname{int}_X Z(g)$), $f \in I$ and $g \in C(X)$, then $g \in I$.

This paper consists of 4 sections. Section 1, as we have already noticed, is the introduction, in which we determine two special topologies on X which the subring R generate, namely, $Z(R)$-topology and $Coz(R)$-topology. Comparison and coincidence of these topologies are studied. Section 2 deals with maximal ideals in R, specially, maximal fixed and maximal g-ideals. Using the $Z(R)$-topology, characterizations of maximal fixed ideals of R, which are of the form $M_x(R)$, are given. Moreover, relations between mapping "$x \longrightarrow M_x(R)$" and the separation properties of the topological space $(X, \tau_{Z(R)})$ will be found. In section 3, we introduce the notion of z_R-ideal in a subring R as a natural topological generalization of the notion of z-ideal in $C(X)$. Various characterizations of these ideals via $Z(R)$-topology are given and relations between z_R-ideals and z-ideals in R (by their algebraic descriptions) are discussed. Section 4 deals with z_R°-ideals of R which are natural topological generalizations of z°-ideals of $C(X)$. Using $Coz(R)$-topology, coincidence of z_R°-ideals with z°-ideals of R (by their algebraic descriptions) are established.

Definition 1.1. For each subring R of $\mathbb{R}^X$, clearly, $Z(R)$ and $Coz(R)$ constitute bases for some topologies on X. The induced topologies are called $Z(R)$-topology and $Coz(R)$-topology, respectively, and are denoted by $\tau_{Z(R)}$ and $\tau_{Coz(R)}$, respectively.

In the next three statements we compare these topologies. Note that two subsets S_1, S_2 of $\mathbb{R}^X$ are called zero-set equivalent, if $Z(S_1) = Z(S_2)$.

Proposition 1.2. *Let R be a subring of $\mathbb{R}^X$, if S and $C(\mathbb{R})$ are zero-set equivalent subsets of $\mathbb{R}^\mathbb{R}$ and $gof \in R$ for each $f \in R$ and each $g \in S$, then $\tau_{Coz(R)} \subseteq \tau_{Z(R)}$ and the equality does not hold, in general.*

Proof. We are to show that $Coz(R) \subseteq \tau_{Z(R)}$. If $x \notin Z(f)$ where $f \in R$, then there is a g in S such that $f(x) \in Z(g)$ and $f^{-1}(Z(g)) \cap Z(f) = \emptyset$. Therefore, $gof \in R$, $x \in Z(gof)$ and $Z(gof) \cap Z(f) = \emptyset$ which proves the inclusion. Now, we show that the inclusion may be proper. Let (X, τ_X) be a Tychonoff space which has at least one non-open zero-set Z. Set $R = C(X)$, then $\tau_{Coz(R)} = \tau_X$, whereas $Z \notin \tau_X$ and hence, $\tau_{Coz(R)} \subsetneq \tau_{Z(R)}$. □

Proof of the following proposition is standard.

Proposition 1.3. *The following statements are equivalent.*
(a) $\tau_{Coz(R)} \subseteq \tau_{Z(R)}$.
(b) Every $Z \in Z(R)$ is clopen under $Z(R)$-topology.

The annihilator of $f \in R$ in R is defined to be the set $\{g \in R : fg = 0\}$ and is denoted by $Ann_R(f)$. A simple reasoning shows that if X is equipped with the $Coz(R)$-topology, then $Ann_R(f) = \{g \in R : Coz(g) \subseteq \mathrm{int}_X Z(f)\} = \{g \in R : \mathrm{cl}_X(Coz(g)) \subseteq Z(f)\}$.

Proposition 1.4. *The following statements are equivalent.*
(a) $\tau_{Z(R)} \subseteq \tau_{Coz(R)}$.
(b) $Z(f)$ is clopen in $(X, \tau_{Coz(R)})$ for every $f \in R$.
(c) For each $f \in R$, $Z(f) = \bigcup_{g \in Ann_R(f)} Coz(g)$.
(d) For each $f \in R$, $(Ann_R(f), f)$ is a free ideal.

Proof. The implications (a)⇒(b)⇒(c) are clear.

(c)⇒(d). This clear by the hypothesis and the fact that whenever $f \in R$ and I is an ideal of R, then $\bigcap_{h \in (I,f)} Z(h) = \bigcap_{g \in I}(Z(f) \cap Z(g))$.

(d)⇒(a). Let $f \in R$ and $x \in Z(f)$. By (d), there exists $g \in Ann_R(f)$ such that $x \notin Z(f) \cap Z(g)$. Hence, $x \notin Z(g)$ and $x \in Coz(g) \subseteq Z(f)$ and so $Z(f) \in \tau_{Coz(R)}$. □

An immediate consequence of Propositions 1.3 and 1.4 is that $\tau_{Coz(R)} = \tau_{Z(R)}$ if and only if $Z(f)$ is clopen under both $Z(R)$-topology and $Coz(R)$-topology, for each $f \in R$.

2. Characterization of Maximal Fixed Ideals in Subrings

We remind that maximal fixed ideals of $C(X)$ coincide with its fixed maximal ideals and are of the form $M_x = \{f \in C(X) : f(x) = 0\}$, where $x \in X$. This fact is generalized for some special subalgebras of $C(X)$, such as intermediate subalgebras (subalgebras of $C(X)$ containing $C^*(X)$, see [7]), $C_c(X)$ (the subalgebra of $C(X)$ consisting of all functions with countable image, see [9]) and the subalgebras of the form $\mathbb{R} + I$ where I is an ideal of $C(X)$, see [13].

We will show that the same statement does not hold for arbitrary subrings of $\mathbb{R}^X$, in general.

Remark 2.1. (a) Every maximal fixed ideal and fixed maximal ideal of R is of the form $M_x(R) = \{f \in R : f(x) = 0\}$ for some $x \in X$. However, parts (1) and (2) of Example 2.2 show that the ideals $M_x(R)$ are not necessarily maximal ideals or even maximal fixed ideals in R.

(b) Every fixed maximal ideal is both a maximal fixed ideal and a maximal g-ideal. But the converse is not necessarily true, in general, see part (1) of Example 2.2 and Example 2.3.

(c) A maximal fixed ideal need not be a maximal g-ideal, see Example 2.3.

(d) Every fixed maximal g-ideal is a maximal fixed ideal.

Example 2.2. (1) Let X be a Tychonoff space, $x \in X$ and $R = \mathbb{Z} + M_x$. Then $M_x(R) = M_x$ is not a maximal ideal in R, since $2\mathbb{Z} + M_x$ is a proper ideal of R and $M_x \subsetneq 2\mathbb{Z} + M_x$. Therefore, $M_x(R)$ is a maximal fixed ideal and a maximal g-ideal which is not a maximal ideal.

(2) Let X be a topological space with more than one point and $a \in X$. Also, let $t \in \mathbb{R}$ be a transcendental number and define $f : X \longrightarrow \mathbb{R}$ by $f(a) = 0$ and $f(x) = t$, for every $x \neq a$. Set $R = \{\sum_{i=0}^{n} m_i f^i : \ n \in \mathbb{N} \cup \{0\}, \ m_i \in \mathbb{Z}\}$. Evidently, $M_a(R) = (f)$ and $M_x(R) = \{0\}$, for every $x \neq a$. Therefore, $M_x(R)$ is not a maximal fixed ideal for any $x \neq a$.

In the next example we construct a subring R such that, for some $x \in X$, $M_x(R)$ is a maximal fixed ideal which is not a maximal g-ideal.

Example 2.3. Let $X = \mathbb{R}$, $a \in \mathbb{R} \setminus \mathbb{Q}$, $b \in \mathbb{R} \setminus \{0\}$ and t be a transcendental number. For every $\epsilon > 0$, define $f_\epsilon : X \longrightarrow \mathbb{R}$ by $f_\epsilon(x) = 0$, if $|x - a| < \epsilon$ and $f_\epsilon(x) = b$, if $|x - a| \geq \epsilon$. Also, define $f : X \longrightarrow \mathbb{R}$ by $f(x) = 0$, if $x \in \mathbb{Q}$ and $f(x) = t$, if $x \in \mathbb{R} \setminus \mathbb{Q}$. Let R be the algebra over $\mathbb{Q}$ generated by $\{f_\epsilon : \ \epsilon > 0\} \cup \{f, 1\}$. Evidently, R is a subring of $\mathbb{R}^X$, and $M_a(R)$ equales to (f_a) which is not a maximal ideal. It is easy to see that $M_a(R)$ is a maximal fixed ideal and $M_a(R) = I$, where I is the ideal generated by $\{f_\epsilon : \ \epsilon > 0\}$. Clearly, $Z(f) \cap Z(g) \neq \emptyset$, for all $g \in I$. Hence $J = (I, f)$ is a g-ideal which strictly contains I. Therefore, I is not a maximal g-ideal.

Proposition 2.4. *The following statements hold for a subring R of $\mathbb{R}^X$.*

(a) $M_x(R)$ is a maximal g-ideal if and only if whenever $Z \in Z(R)$ and $x \notin Z$, then $x \notin cl_{\tau_{Z(R)}} Z$.

(b) For each $x \in X$, $M_x(R)$ is a maximal g-ideal if and only if every $Z \in Z(R)$ is clopen under $Z(R)$-topology.

Proof. (a $\Rightarrow$). Let $f \in R$ and $x \notin Z(f)$, thus, the ideal $(M_x(R), f)$ contains an invertible element of $\mathbb{R}^X$. Hence, there are $g \in M_x(R)$ and $h \in R$ such that $Z(g + fh) = \emptyset$. Consequently, $x \in Z(g)$ and $Z(f) \cap Z(g) = \emptyset$.

(a $\Leftarrow$). Assume that $f \notin M_x(R)$. Then there is some $g \in R$ such that $x \in Z(g)$ and $Z(f) \cap Z(g) = Z(f^2 + g^2) = \emptyset$. Hence, $(M_x(R), f)$ contains an invertible element of $\mathbb{R}^X$. Also, clearly, $M_x(R)$ is a g-ideal. Thus, $M_x(R)$ is a maximal g-ideal.

(b). An easy consequence of (a). □

Corollary 2.5. *If $M_x(R)$ is a maximal ideal for each $x \in X$, then every $Z \in Z(R)$ is clopen under $Z(R)$-topology.*

Corollary 2.6. *Let R be an invertible subring. Then every $Z \in Z(R)$ is clopen under $Z(R)$-topology if and only if $M_x(R)$ is a maximal ideal for each $x \in X$.*

Proof. By our hypothesis and Proposition 2.4, this is clear. □

The following lemma is a restatement of the fact that the transcendental degree of $\mathbb{R}$ over $\mathbb{Q}$ is unountable, see [14].

Lemma 2.7. *Let $S = \mathbb{Q}[y_1, ..., y_n]$ be the ring of n-variable polynomials with rational coefficients. Then there exists an uncountable set X of transcendental numbers for which $F(a_1, \cdots, a_n) \neq 0$, for every distinct elements $a_1, \cdots, a_n$ of X and every $F \in S$.*

The following example shows that the converse of Corollary 2.5 does not hold, in general.

Example 2.8. Let S be the polynomial ring $\mathbb{Q}[y_1, ..., y_n]$, where $n \in \mathbb{N}$ and $n > 1$. By Lemma 2.7, there exists an infinite set of transcendental numbers X for which $F(a_1, \cdots, a_n) \neq 0$, for every $a_1, \cdots, a_n \in X$ and every $F \in S$. For each $a \in X$, define the function $f_a : X \longrightarrow \mathbb{R}$ by $f_a(a) = 0$ and $f_a(x) = x$ for each $x \neq a$. Now, set

$$R = \{F(f_{a_1}, ..., f_{a_n}) : \ F \in S, \ n \in \mathbb{N}, \ a_1, ..., a_n \in X\}.$$

Hence, $M_a(R) = (f_a)$, for each $a \in X$, which is not a maximal ideal. However, every $Z \in Z(R)$ is clopen under $Z(R)$-topology.

Proposition 2.9. *If R is a subalgebra of $\mathbb{R}^X$, then $M_x(R)$ is a maximal g-ideal and a maximal fixed ideal for every $x \in X$.*

Proof. It suffices to prove that every element of $Z(R)$ is closed under $Z(R)$-topology. To this aim, suppose that $a \in X$ and $a \notin Z(f)$, for some $f \in R$. Put $g = f - f(a)$. Clearly, $Z(g) \in Z(R)$, $a \in Z(g)$ and $Z(g) \cap Z(f) = \emptyset$. □

Corollary 2.10. *If R is an invertible subalgebra of $\mathbb{R}^X$, then $M_x(R)$ is a maximal ideal for each $x \in X$.*

The converse of Corollary 2.10 does not hold, in general. For example, let R denote the collection of all single variable polynomials over $\mathbb{R}$. Then, $M_r(R)$ is the maximal ideal $(x - r)$ for each $r \in \mathbb{R}$. However, $f = x^2 + 1$ is invertible in

$\mathbb{R}^{\mathbb{R}}$ which is not invertible in R. Note that the subalgebras $C_c(X)$ and $\mathbb{R}+I$, for each ideal I in $C(X)$, satisfy Corollary 2.10 and so $M_x(C_c(X))$ and $M_x(\mathbb{R}+I)$ are maximal ideals of $C_c(X)$ and $\mathbb{R}+I$, respectively, for each $x \in X$. Remark that in parts (b) and (e) of the following proposition we assume that "=" is a partial order on X.

Proposition 2.11. *For a subring R of $\mathbb{R}^X$, the following statements hold.*

(a) The mapping $x \longrightarrow M_x(R)$ is a one-one correspondence if and only if $(X, \tau_{Z(R)})$ is a T_0-space.

(b) The mapping $x \longrightarrow M_x(R)$ is an order isomorphism between X and the set of all maximal fixed ideals of R if and only if $(X, \tau_{Z(R)})$ is a T_1-space.

(c) For every two distinct elements $x, y \in X$, $M_x(R)+M_y(R)$ is not a g-ideal if and only if $(X, \tau_{Z(R)})$ is a T_2-space.

(d) The mapping $x \longrightarrow M_x(R)$ is an order embedding between X and the set of all maximal g-ideals of R if and only if $(X, \tau_{Z(R)})$ is a T_0-space and every element of $Z(R)$ is clopen under $Z(R)$-topology.

Proof. (a). Let x, y be distinct points of X, so $M_x(R) \neq M_y(R)$, say $M_x(R) \nsubseteq M_y(R)$. Hence, there exists $f \in M_x(R) \setminus M_y(R)$. Thus $x \in Z(f)$ and $y \notin Z(f)$. It is clear that the above reasoning is reversible and hence we are done.

(b $\Rightarrow$). Suppose that x and y are two distinct points of X. Since $M_x(R) \nsubseteq M_y(R)$, there exists $f \in M_x(R) \setminus M_y(R)$. Consequently, $x \in Z(f)$ and $y \notin Z(f)$.

(b $\Leftarrow$). Suppose that $x \in X$ and I is a fixed ideal in R containing $M_x(R)$. Take $y \in \cap_{f \in I} Z(f)$. Clearly, $M_x(R) \subseteq I \subseteq M_y(R)$. It suffices to show $x = y$. Suppose that $x \neq y$ and seek a contradiction. By our hypothesis, there exists $f \in R$ such that $x \in Z(f)$ and $y \notin Z(f)$. Therefore, $M_x(R) \nsubseteq M_y(R)$ and this is a contradiction. Now, by part (a), the proof is complete.

(c). For any two distinct points $x, y \in X$, clearly, $M_x(R) + M_y(R)$ is not a g-ideal if and only if there exist $f \in M_x(R)$ and $g \in M_y(R)$ such that $Z(f) \cap Z(g) = \emptyset$.

(d $\Rightarrow$). By part (a), clearly, $(X, \tau_{Z(R)})$ is a T_0-space. Now, Suppose that $f \in R$ and $x \notin Z(f)$. Since $M_x(R)$ is a maximal g-ideal, it follows that $(M_x(R), f)$ has an invertible element of $\mathbb{R}^X$ and so there exists $g \in M_x(R)$, such that $Z(g) \cap Z(f) = \emptyset$. Thus, $Z(f)$ is closed and hence is clopen under $Z(R)$-topology.

(d $\Leftarrow$). Suppose that $x \in X$, it suffices to show that $M_x(R)$ is a maximal g-ideal. Assume that I is an ideal which properly contains $M_x(R)$. Hence, there exists $f \in I$ such that $x \notin Z(f)$. By our hypothesis, there is $g \in R$ such that $x \in Z(g)$ and $Z(g) \cap Z(f) = \emptyset$. Therefore, $Z(f^2+g^2) = \emptyset$ and $f^2+g^2 \in I$, hence, I is not a g-ideal. □

It is easy to see that $M_x(R)$, for each $x \in X$, is a prime ideal of R and thus the hull-kernel topology may be defined on the family $\{M_x(R) : x \in X\}$.

By considering this space, the next statement gives a relation between $Z(R)$-topology on X and points of X.

Proposition 2.12. *Let R be a subring of $\mathbb{R}^X$ and X equipped with the $Coz(R)$-topology. Then the mapping $\Phi : X \to \{M_x(R) : x \in X\}$ defined by $x \mapsto M_x(R)$ is a homeomorphism if and only if $(X, \tau_{Z(R)})$ is a T_0-space.*

Proof. By part (a) of Theorem 2.12, Φ is a one-one correspondence if and only if $(X, \tau_{Z(R)})$ is a T_0-space. Also, if $f \in R$ and $x \in Z(f)$, then $f \in M_x(R)$ which means that basic closed sets of X equipped with the $Coz(R)$-topology are mapped to the basic closed sets in $\{M_x(R) : x \in X\}$ equipped with the hull-kernel topology by the mapping Φ and therefore, it is a homeomrohpism. □

3. z_R-Ideals and z-Ideals in Subrings

In this section we introduce z_R-ideals in a subring R and via the $Z(R)$-topology and maximal g-ideals of R, various characterizations of these ideals are given.

Definition 3.1. A subset $\mathcal{F}$ of $Z(R)$ is called z_R-filter on X, if

(a) $\emptyset \notin \mathcal{F}$.

(b) If $Z_1, Z_2 \in \mathcal{F}$, then $Z_1 \cap Z_2 \in \mathcal{F}$.

(c) If $Z_1 \in \mathcal{F}$, $Z_2 \in Z(R)$ and $Z_1 \subseteq Z_2$, then $Z_2 \in \mathcal{F}$.

Moreover, $\mathcal{F}$ is called a prime z_R-filter, if whenever $Z_1 \cup Z_2 \in \mathcal{F}$, then $Z_1 \in \mathcal{F}$ or $Z_2 \in \mathcal{F}$ for each $Z_1, Z_2 \in Z(R)$. Also, $\mathcal{F}$ is called a z_R-ultrafilter, if $\mathcal{F}$ is maximal among z_R-filters on X.

The following proposition immediately follows from Definition 3.1.

Proposition 3.2. *For any subring R, the following statements hold.*

(a) $I \subseteq R$ is a g-ideal in R if and only if $Z_R(I) = \{Z(f) : f \in I\}$ is a z_R-filter on X.

(b) $\mathcal{F}$ is a z_R-filter on X if and only if $Z_R^{-1}(\mathcal{F}) = \{f \in R : Z(f) \in \mathcal{F}\}$ is a g-ideal.

(c) $\mathcal{F}$ is a prime z_R-filter on X if and only if $Z_R^{-1}(\mathcal{F})$ is a prime g-ideal.

(d) $\mathcal{A}$ is a z_R-ultrafilter on X if and only if $Z_R^{-1}(\mathcal{A})$ is a maximal g-ideal.

(e) If M is a maximal g-ideal in R, then $Z_R(M)$ is a z_R-ultrafilter on X.

It is easy to see that for an ideal I of R we always have $I \subseteq Z_R^{-1}Z_R(I)$ and the inclusion may be proper. We call an ideal I in R a z_R-ideal, if $I = Z_R^{-1}Z_R(I)$. It follows that every z_R-ideal is semiprime and arbitrary intersections of z_R-ideals is a z_R-ideal. Also, the zero ideal, the ideals of the form $M_x(R)$, maximal g-ideals and $Z^{-1}(\mathcal{F})$, for each z_R-filter $\mathcal{F}$, are all z_R-ideals of R. For each $f \in R$, the intersection of all the maximal ideals, maximal g-ideals and maximal fixed ideals of R containing f are denoted by $M_f(R)$, $MG_f(R)$ and $MF_f(R)$, respectively. It is easy to observe that $MG_f(R)$ is a z_R-ideal for each $f \in R$.

Obviously, $MG_f \cap MG_g = MG_{fg}$, $MF_f \cap MF_g = MF_{fg}$, $MG_{f^2+g^2} = MG_{(f,g)}$ and $MF_{f^2+g^2} = MF_{(f,g)}$ for all $f, g \in R$.

Proposition 3.3. *Let $(X, \tau_{Z(R)})$ be a T_1-space. Then the following statemnets hold.*
(a) The following statements are equivalent.
(1) $g \in MF_f(R)$.
(2) $MF_g(R) \subseteq MF_f(R)$.
(3) $Z(f) \subseteq Z(g)$.
(b) $MF_f(R) = \{g \in R : \ Z(f) \subseteq Z(g)\}$.
(c) An ideal I of R is a z_R-ideal if and only if $MF_f(R) \subseteq I$ for every $f \in I$.

Proof. (a: $1 \Rightarrow 2$). Evident.

(a: $2 \Rightarrow 3$). Let $x \in Z(f)$. Then $f \in M_x(R)$ and thus $MF_g(R) \subseteq MF_f(R) \subseteq M_x(R)$. This implies $g \in M_x(R)$ and hence $x \in Z(g)$.

(a: $3 \Rightarrow 1$). If $g \notin MF_f(R)$, then there exists $x \in X$ such that $f \in M_x(R)$ and $g \notin M_x(R)$. Therefore, $x \in Z(f) \setminus Z(g)$ and so $Z(f) \subsetneq Z(g)$.

(b) and (c) obviously follow from part (a). □

Lemma 3.4. *Assume that every $Z \in Z(R)$ is clopen under $Z(R)$-topology. Then $MG_f(R) = MF_f(R)$, for every $f \in R$.*

Proof. Suppose that $f \in R$. By part (b) of Proposition 2.4, $M_x(R)$ is a maximal g-ideal for each $x \in X$. Consequently, $MG_f(R) \subseteq MF_f(R)$. Now, assume that $g \notin MG_f(R)$. Hence, there exists a maximal g-ideal M in R such that $f \in M$ and $g \notin M$. Thus, there exists $h \in M$ such that $Z(g) \cap Z(h) = \emptyset$. Since $f^2 + h^2 \in M$ and M is a g-ideal, there is a point $x \in Z(f^2 + h^2) = Z(f) \cap Z(h)$. Clearly, $g \notin M_x(R)$ and $f \in M_x(R)$. Therefore, $g \notin MF_f(R)$. □

Proposition 3.3 and Lemma 3.4 imply the next statement.

Proposition 3.5. *Let $(X, \tau_{Z(R)})$ be a T_1-space and every $Z \in Z(R)$ be a clopen set under $Z(R)$-topology. Then the following statements hold.*
(a) The following statements are equivalent.
(1) $g \in MG_f(R)$.
(2) $MG_g(R) \subseteq MG_f(R)$.
(3) $Z(f) \subseteq Z(g)$.
(b) $MG_f(R) = \{g \in R : \ Z(f) \subseteq Z(g)\}$.
(c) An ideal I of R is z_R-ideal if and only if $MG_f(R) \subseteq I$ for every $f \in I$.

The following corollary follows from Corollary 2.6 and Proposition 3.5.

Corollary 3.6. *Let R be an invertible subalgebra of $\mathbb{R}^X$. Then the following statements hold.*

(a) The following conditions are equivalent;
(1) $g \in M_f(R)$.
(2) $M_g(R) \subseteq M_f(R)$.
(3) $Z(f) \subseteq Z(g)$.
(b) $M_f(R) = \{g \in R: \ Z(f) \subseteq Z(g)\}$.
(c) An ideal I *of* R *is* z_R*-ideal if and only if* $M_f(R) \subseteq I$ *for every* $f \in I$.

It follows from Corollary 3.6 that for an invertible subalgebra R, the notion of z_R-ideal coincides with the notion of z-ideal. The next statement extend this fact and shows that this coincidence is equivalent to invertibility of R.

Theorem 3.7. *Let* R *be a subring of* $\mathbb{R}^X$. *The following statements are equivalent.*

(a) Every maximal ideal in R *is a* g*-ideal.*

(b) Every maximal g*-ideal of* R *is a maximal ideal and if* J *is a maximal ideal of* R*, then every maximal element in the set of* g*-ideals contained in* J *is a prime ideal.*

(c) Every maximal ideal in R *is a* g*-ideal.*

(d) R *is an invertible subring.*

(e) Every z*-ideal of* R *is a* z_R*-ideal.*

Moreover, if R *is a subalgebra and one of (a)-(c) holds, then every* z_R*-ideal is a* z*-ideal.*

Proof. (a) $\Rightarrow$ (b). This is clear.

(b)$\Rightarrow$(c). Suppose that M is a maximal ideal and P is a maximal element of G_M, where G_M is the set of all g-ideals contained in M. Assume that J is a maximal ideal of R containing P. Then $M \cap J = P$. As $M \cap J$ is prime and both M and J are maximal ideal, we have $M = J$. Hence, M is a maximal g-ideal.

(c)$\Rightarrow$(d). Suppose that $Z(f) = \emptyset$ for $f \in R$ and, on the contrary, f is a non-unit element of R. Clearly, there exists a maximal ideal M of R containing f. By our hypothesis, M is a g-ideal which contradicts with $Z(f) = \emptyset$.

(d)$\Rightarrow$(e). Suppose that I is a z-ideal and $Z(f) \subseteq Z(g)$ where $f \in I$ and $g \in R$. Since I is a z-ideal, it follows that $M_f \subseteq I$. It suffices to prove that $g \in M_f$. To see this, suppose that M is a maximal ideal containing f. As R is invertible, M is a g-ideal and so it is a maximal g-ideal. Obviously, M is a z_R-ideal and so $g \in M$.

(e)$\Rightarrow$(a). Suppose that M is a maximal ideal and, on the contrary, M is not a g-ideal. Thus, there exists $f \in M$ such that $Z(f) = \emptyset$. By (e), M is a z_R-ideal and since $f \in M$, it follows that $M = R$, which is a contradiction.

Now, suppose that one of (a)-(c) holds, R is a subalgebra and I is a z_R-ideal of R. By our hypothesis, $MF_f(R) = M_f(R)$ for every $f \in R$, and thus we are done. □

It is well-known that every minimal prime ideals over a z-ideal is also a z-ideal, see [10, Theorem 14.7]. The same statement holds for z_R-ideals as the following proposition shows.

Proposition 3.8. *Let I be a z_R-ideal of R and P a prime ideal in R minimal over I. Then P is a z_R-ideal.*

Proof. Assume that $Z(f) = Z(g)$ and $f \in P$. Thus, there exists $h \notin P$, such that $fh \in I$. Since $Z(fh) = Z(gh)$ and I is a z_R-ideal, it follows that $gh \in I \subseteq P$. As $h \notin P$, clearly, this implies that $g \in P$. □

An immediate consequence of Proposition 3.8 is that every minimal prime ideal in a subring R is a z_R-ideal. By the following statement, we extend some fundamental statements about z-ideals in the literature of $C(X)$ to the subrings of $\mathbb{R}^X$, namely, [10, 2.9, 5.3 and 5.5]. The proofs are left to the reader.

Proposition 3.9. *Let R be a lattice-ordered subring of $\mathbb{R}^X$ and I be a z_R-ideal in R. Then the following statements hold.*
(a) The following statements are equivalent
(1) I is a prime ideal;
(2) I contains a prime ideal;
(3) if $fg = 0$, then $f \in I$ or $g \in I$;
(4) for each $f \in R$, there is a $Z \in Z_R(I)$ on which f does not change sign.
(b) Every prime g-ideal of R is contained in a unique maximal g-ideal.
(c) If P is a prime ideal of R, then $Z_R(P)$ is a prime z_R-filter on X.
(d) If $\mathcal{P}$ is a prime z_R-filter on X, then $Z_R^{-1}(\mathcal{P})$ is a prime ideal in R.
(e) Every z_R-ideal of R is absolutely convex.
Thus, if I is an absolutely convex ideal of R, then R/I is a lattice ring.
(f) $I(f) \geq 0$ if and only if $f \geq 0$ on some $Z \in Z_R(I)$.
(g) Suppose that there exists $Z \in Z_R(I)$ such that $f(x) > 0$, for every $x \in Z$, then $I(f) > 0$. The converse is true whenever I is a maximal g-ideal.

4. z_R°-Ideals and z°-Ideals in Subrings

In this section we generalize the concept of z°-ideals of $C(X)$ to the subrings of $\mathbb{R}^X$ and introduce z_R°-ideal. Coincidence of z_R°-ideals with z°-ideals of R is discussed. Note that, for each element f of a commutative rings S, we use $P_f(S)$ to denote the intersection of all the minimal prime ideals in S containing f.

Definition 4.1. An ideal I of a subring R of $\mathbb{R}^X$ is called a z_R°-ideal, if $\mathrm{int}_X Z(f) \subseteq \mathrm{int}_X Z(g)$, where $f \in I$ and $g \in R$, implies $g \in I$.

The following statement investigates some characterizations of z_R°-ideals in subrings.

Theorem 4.2. *Let R be a subing of $\mathbb{R}^X$ and I be an ideal in R. The following statements are equivalent.*

(a) I is a z_R°-ideal.
(b) Whenever $Ann_C(f) \subseteq Ann_C(g)$ where $f \in I$ and $g \in R$, then $g \in I$.
(c) $R \cap P_f(C) \subseteq I$ for each $f \in I$.
(d) Whenever $P_g(C) \cap R \subseteq P_f(C) \cap R$, where $f \in I$ and $g \in R$, then $g \in I$.

Proof. (a⇒b). First note that by [3, Lemma 2.1] we have $Ann_C(f) \subseteq Ann_C(g)$ if and only if $\text{int}_X Z(f) \subseteq \text{int}_X Z(g)$ for each $f, g \in C(X)$. Now, let I be a z_R°-ideal in R and $Ann_C(f) \subseteq Ann_C(g)$ where $f \in I$ and $g \in R$. Thus, by our hypothesis, we have $\text{int}_X Z(f) \subseteq \text{int}_X Z(g)$ which implies that $g \in I$.

(b⇒c). By [3, Proposition 2.3], we have $P_f(C) = \{g \in C(X) : Ann_C(f) \subseteq Ann_C(g)\}$. Thus the proof is evident.

(c⇒d). Let $P_g(C) \cap R \subseteq P_f(C) \cap R$, where $f \in I$ and $g \in R$. As $f \in I$, by our hypothesis, $P_f(C) \cap R \subseteq I$ and thus $P_g(C) \cap R \subseteq I$ which implies that $g \in I$.

(d⇒a). Let $\text{int}_X Z(f) \subseteq \text{int}_X Z(g)$ where $f \in I$ and $g \in R$. Therefore, by [3, Lemma 2.1], we have $P_f(C) \subseteq P_g(C)$ and hence $P_f(C) \cap R \subseteq P_g(C) \cap R$. Thus we are done by our hypothesis. □

Lemma 4.3. *Let R be a subring of $\mathbb{R}^X$, then for each $f \in R$ we have $P_f(C) \subseteq P_f(R)$.*

Proof. Let $g \in P_f(C)$. By [3, Proposition 2.3.], we have $Ann_C(f) \subseteq Ann_C(g)$. Therefore, $Ann_R(f) = Ann_C(f) \cap R \subseteq Ann_C(g) \cap R = Ann_R(g)$. Thus, by [2, Proposition 1.5] we are done. □

Theorem 4.4. *Let R be a subring of $\mathbb{R}^X$. Then every z_R°-ideal in R is a z°-ideal if and only if $P_f(R) = P_f(C)$ for each $f \in R$.*

Proof. (⇒). Asumme on the contrary that there exists some $f \in R$ such that $P_f(R) \neq P_f(C)$. Thus, using Theorem 4.2 we have $P_f(C) \subsetneq P_f(R)$. Again by Theorem 4.2, $P_f(C) \cap R$ is a z_R°-ideal in R. Also, it is clear that this ideal is not a z°-ideal, since, $P_f(R) \not\subseteq P_f(C) \cap R$.

(⇐). Let I be a z_R°-ideal in R and $f \in I$. By Theorem 4.2, $P_f(C) \cap R \subseteq I$. Thus, by our hypothesis, $P_f(R) \subseteq I$ which means that I is a z°-ideal in R. □

From Theorem 4.2 it follows that every z°-ideal in a subring R is a z_R°-ideal. However, the converse of this fact does not hold, in general. The following example gives an example of a subring R which has a z_R°-ideal that is not a z°-ideal.

Example 4.5. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be defined by $f(x) = \begin{cases} x & x > 0 \\ 0 & x \leq 0 \end{cases}$. It is clear that $f \in C(\mathbb{R})$. Now, let $R = \{\sum_{i=0}^n r_i f^i : r_i \in \mathbb{R}, n = 0, 1, ...\}$. It is easy to see that $P_f(R) = R$, however, $P_f(C) \cap R \neq R$. Also, by Theorem 4.2, $P_f(C) \cap R$ is z_R°-ideal and it is clear that this ideal is not a z°-ideal.

The next theorem gives a sufficent conditions on X in order that z_R°-ideals in a subring R coincide with z°-ideals of R.

Theorem 4.6. *Let R be a subring of $\mathbb{R}^X$ and X be equipped with the $Coz(R)$-topology. Then an ideal I in R is a z°-ideal if and only if it is a z_R°-ideal.*

Proof. Let I be a z_R°-ideal in R and $f \in I$. As X is equipped with the $Coz(R)$-topology, we have $g \in Ann_R(f)$ if and only if $Coz(g) \subseteq \text{int}_X Z(f)$ for each $f, g \in R$. Therefore, $P_f(R) = Ann_R Ann_R(f) = \{g \in R : Coz(g) \cap \text{int}_X Z(f) = \emptyset\} = \{g \in R : Ann_R(f) \subseteq Ann_R(g)\}$. Hence, $P_f(R) \subseteq I$ which means that I is a z°-ideal in R. This completes the proof, since, as former stated, every z°-ideal in R is a z_R°-ideal. □

Note that the condition that X is equipped with the $Coz(R)$-topology is a sufficient condition for coincidence of z_R°-ideals with z°-ideals in a given subring R. The next example shows that this condition is not necessary.

Example 4.7. Let $X = \mathbb{R} \setminus \{0\}$ with the topology inherits from the usual topology on $\mathbb{R}$. Also, let $f : X \longrightarrow \mathbb{R}$ be defined by $f(x) = \begin{cases} 1 & x > 0 \\ 0 & x < 0 \end{cases}$. It is clear that $f \in C(X)$ and $f^2 = f$. Now, set $R = \{r + sf : r, s \in \mathbb{R}\}$. It is clear that R is a subring of $C(X)$. Also, by a routine reasoning, one can proves that the only ideals of R are the ideals (0), (f), $(1 - f)$ and R. Moreover, the minimal prime ideals of R are only the ideals (f) and $(1-f)$. These imply that every z_R°-ideal is a z°-ideal in R. However, clearly, X is not equipped with the $Coz(R)$-topology.

It follows from Theorem 4.6 that for an intermediate subalgebra $A(X)$ of $C(X)$, z_A°-ideals coincide with z°-ideals of $A(X)$. However, the same statement does not true for z_A-ideals and z-ideals in $A(X)$, in general, see [6, Theorem 2.2]. Moreover, Theorem 3,7 together with Theorem 4.6 imply that in the subalgebras of $C(X)$ which are of the form $\mathbb{R} + I$, where I is a free ideal in $C(X)$, $z_{\mathbb{R}+I}$-ideals concide with z-ideals of $\mathbb{R} + I$ and $z_{\mathbb{R}+I}^\circ$-ideals coincide with z°-ideals, too. Note that whenever I is a free ideal in $C(X)$, then $\mathbb{R} + I$ determines the topology of X.

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