zR-ideals and zR ◦ -ideals in Subrings of RX

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Abstract. Let X be a topological space and R be a subring of RX. By determining some special topologies on X associated with the subring R, characterizations of maximal fixed and maximal g-ideals in R of the form Mr(R) are given. Moreover, the classes of zR-ideals and zR ◦ -ideals are introduced in R which are topological generalizations of z-ideals and z◦-ideals of C(X), respectively. Various characterizations of these ideals are established. Also, coincidence of zR-ideals with z-ideals and zR ◦ -ideals with z◦-ideals in R are investigated. It turns out that some fundamental statements in the context of C(X) are extended to the subrings of RX.

Keywords: z(R)-topology, Coz(R)-topology, g-ideal, zR-ideal, zR ◦ -ideal, invertible subring.


1. Introduction

For a topological space X, RX denotes the algebra of all real-valued functions and C(X) (resp., C*(X)) denotes the subalgebra of RX consisting of all continuous functions (resp., bounded continuous functions). Moreover, we use R to denote a unital subring of RX. Note that topological spaces which are considered in this paper are not necessarily Tychonoff. For each f ∈ RX,
\(Z(f) = \{x \in X : f(x) = 0\}\) denotes the zero-set of \(f\) and \(\text{Coz}(f)\) denotes the complement of \(Z(f)\) with respect to \(X\). We denote by \(Z(R)\) the collection of all the zero-sets of elements of \(R\), we use \(Z(X)\) instead of \(Z(C(X))\). We denote by \(M_s(R)\) the set \(\{f \in R : x \in Z(f)\}\), \(M_s(C(X))\) is denoted by \(M_s\). The subring \(R\) is called invertible, if \(f \in R\) and \(Z(f) = \emptyset\) implies that \(f\) is invertible in \(R\). Moreover, \(R\) is called a lattice-ordered subring if it is a sublattice of \(\mathbb{R}^X\) (i.e., \(f \land g\) and \(f \lor g\) are in \(R\) for each \(f, g \in R\)). It is clear that \(C(X)\) is an invertible lattice-ordered subring of \(\mathbb{R}^X\). However, the same statement does not hold for \(C^+(X)\). A proper ideal \(I\) of \(R\) is called a growing ideal, briefly, a \(g\)-ideal, if contains no invertible element of \(\mathbb{R}^X\), i.e., \(Z(f) \neq \emptyset\) for each \(f \in I\). It is evident that a subring \(R\) is invertible if and only if every ideal every ideal of \(R\) is a \(g\)-ideal. Clearly, \(M^p\), for each \(p \in \beta X \setminus vX\), is not a \(g\)-ideal of \(C^+(X)\). An ideal \(I\) of \(R\) is called fixed if \(\bigcap_{f \in I} Z(f) \neq \emptyset\), otherwise, it is called free. By a maximal fixed ideal of \(R\), we mean a fixed ideal which is maximal in the set of all fixed ideals of \(R\). An ideal \(I\) in a commutative ring \(S\) is called a \(z\)-ideal (resp., \(z^*\)-ideal) if \(M_a(S) \subseteq I\) (resp., \(P_a(S) \subseteq I\)), for each \(a \in I\), where \(M_a(S)\) (resp., \(P_a(S)\)) denotes the intersection of all the maximal (resp., minimal prime) ideals of \(S\) containing \(a\). It is well-known that in \(C(X)\) an ideal \(I\) is a \(z\)-ideal (resp., \(z^\bullet\)-ideal) if and only if whenever \(Z(f) \subseteq Z(g)\) (resp., \(\text{int}_{X}Z(f) \subseteq \text{int}_{X}Z(g)\)), \(f \in I\) and \(g \in C(X)\), then \(g \in I\).

This paper consists of 4 sections. Section 1, as we have already noticed, is the introduction, in which we determine two special topologies on \(X\) which the subring \(R\) generate, namely, \(Z(R)\)-topology and \(\text{Coz}(R)\)-topology. Comparison and coincidence of these topologies are studied. Section 2 deals with maximal ideals in \(R\), specially, maximal fixed and maximal \(g\)-ideals. Using the \(Z(R)\)-topology, characterizations of maximal fixed ideals of \(R\), which of are of the form \(M_s(R)\), are given. Moreover, relations between mapping \(\text{"}x \rightarrow M_s(R)\text{"}\) and the separation properties of the topological space \((X, \tau_{Z(R)})\) will be found. In section 3, we introduce the notion of \(z_R\)-ideal in a subring \(R\) as a natural topological generalization of the notion of \(z\)-ideal in \(C(X)\). Various characterizations of these ideals via \(Z(R)\)-topology are given and relations between \(z_R\)-ideals and \(z\)-ideals in \(R\) (by their algebraic descriptions) are discussed. Section 4 deals with \(z^*_R\)-ideals of \(R\) which are natural topological generalizations of \(z^*\)-ideals of \(C(X)\). Using \(\text{Coz}(R)\)-topology, coincidence of \(z^*_R\)-ideals with \(z^\bullet\)-ideals of \(R\) (by their algebraic descriptions) are established.

**Definition 1.1.** For each subring \(R\) of \(\mathbb{R}^X\), clearly, \(Z(R)\) and \(\text{Coz}(R)\) constitute bases for some topologies on \(X\). The induced topologies are called \(Z(R)\)-topology and \(\text{Coz}(R)\)-topology, respectively, and are denoted by \(\tau_{Z(R)}\) and \(\tau_{\text{Coz}(R)}\), respectively.

In the next three statements we compare these topologies. Note that two subsets \(S_1, S_2\) of \(\mathbb{R}^X\) are called zero-set equivalent, if \(Z(S_1) = Z(S_2)\).
**Proposition 1.2.** Let $R$ be a subring of $\mathbb{R}^X$, if $S$ and $C(R)$ are zero-set equivalent subsets of $\mathbb{R}^X$ and $gof \in R$ for each $f \in R$ and each $g \in S$, then $\tau_{\text{Coz}(R)} \subseteq \tau_{Z(R)}$ and the equality does not hold, in general.

**Proof.** We are to show that $\text{Coz}(R) \subseteq \tau_{Z(R)}$. If $x \notin Z(f)$ where $f \in R$, then there is a $g$ in $S$ such that $f(x) \in Z(g)$ and $f^{-1}(Z(g)) \cap Z(f) = \emptyset$. Therefore, $gof \in R$, $x \in Z(gof)$ and $Z(gof) \cap Z(f) = \emptyset$ which proves the inclusion. Now, we show that the exclusion may be proper. Let $(X, \tau_X)$ be a Tychonoff space which has at least one non-open zero-set $Z$. Set $R = C(X)$, then $\tau_{\text{Coz}(R)} = \tau_X$, whereas $Z \notin \tau_X$ and hence, $\tau_{\text{Coz}(R)} \subsetneq \tau_{Z(R)}$. \hfill $\Box$

Proof of the following proposition is standard.

**Proposition 1.3.** The following statements are equivalent.

(a) $\tau_{\text{Coz}(R)} \subseteq \tau_{Z(R)}$.

(b) Every $Z \in \tau(R)$ is clopen under $Z(R)$-topology.

The annihilator of $f \in R$ in $R$ is defined to be the set $\{g \in R : fg = 0\}$ and is denoted by $\text{Ann}_R(f)$. A simple reasoning shows that if $X$ is equipped with the $\text{Coz}(R)$-topology, then $\text{Ann}_R(f) = \{g \in R : \text{Coz}(g) \subseteq \text{int}_X Z(f)\} = \{g \in R : \text{cl}_X(\text{Coz}(g)) \subseteq Z(f)\}$.

**Proposition 1.4.** The following statements are equivalent.

(a) $\tau_{Z(R)} \subseteq \tau_{\text{Coz}(R)}$.

(b) $Z(f)$ is clopen in $(X, \tau_{\text{Coz}(R)})$ for every $f \in R$.

(c) For each $f \in R$, $Z(f) = \bigcup_{g \in \text{Ann}_R(f)} \text{Coz}(g)$.

(d) For each $f \in R$, $(\text{Ann}_R(f), f)$ is a free ideal.

**Proof.** The implications (a)$\Rightarrow$(b)$\Rightarrow$(c) are clear.

(c)$\Rightarrow$(d). This is clear by the hypothesis and the fact that whenever $f \in R$ and $I$ is an ideal of $R$, then $\bigcap_{h \in (f, I)} Z(h) = \bigcap_{g \in I} (Z(f) \cap Z(g))$.

(d)$\Rightarrow$(a). Let $f \in R$ and $x \in Z(f)$. By (d), there exists $g \in \text{Ann}_R(f)$ such that $x \notin Z(f) \cap Z(g)$. Hence, $x \notin Z(g)$ and $x \in \text{Coz}(g) \subseteq Z(f)$ and so $Z(f) \in \tau_{\text{Coz}(R)}$.

An immediate consequence of Propositions 1.3 and 1.4 is that $\tau_{\text{Coz}(R)} = \tau_{Z(R)}$ if and only if $Z(f)$ is clopen under both $Z(R)$-topology and $\text{Coz}(R)$-topology, for each $f \in R$.

2. **Characterization of Maximal Fixed Ideals in Subrings**

We remind that maximal fixed ideals of $C(X)$ coincide with its fixed maximal ideals and are of the form $M_x = \{f \in C(X) : f(x) = 0\}$, where $x \in X$. This fact is generalized for some special subalgebras of $C(X)$, such as intermediate subalgebras (subalgebras of $C(X)$ containing $C^*(X)$, see [7]), $C_c(X)$ (the subalgebra of $C(X)$ consisting of all functions with countable image, see [9]) and the subalgebras of the form $R + I$ where $I$ is an ideal of $C(X)$, see [13].
We will show that the same statement does not hold for arbitrary subrings of \( \mathbb{R}^X \), in general.

**Remark 2.1.** (a) Every maximal fixed ideal and fixed maximal ideal of \( R \) is of the form \( M_x(R) = \{ f \in R : f(x) = 0 \} \) for some \( x \in X \). However, parts (1) and (2) of Example 2.2 show that the ideals \( M_x(R) \) are not necessarily maximal ideals or even maximal fixed ideals in \( R \).

(b) Every maximal fixed ideal is both a maximal fixed ideal and a maximal \( g \)-ideal. But the converse is not necessarily true, in general, see part (1) of Example 2.2 and Example 2.3.

(c) A maximal fixed ideal need not be a maximal \( g \)-ideal, see Example 2.3.

(d) Every maximal fixed \( g \)-ideal is a maximal fixed ideal.

**Example 2.2.** (1) Let \( X \) be a Tychonoff space, \( x \in X \) and \( R = \mathbb{Z} + M_x \). Then \( M_x(R) = M_x \) is not a maximal ideal in \( R \), since \( 2\mathbb{Z} + M_x \) is a proper ideal of \( R \) and \( M_x \subseteq 2\mathbb{Z} + M_x \). Therefore, \( M_x(R) \) is a maximal fixed ideal and a maximal \( g \)-ideal which is not a maximal ideal.

(2) Let \( X \) be a topological space with more than one point and \( a \in X \). Also, let \( t \in \mathbb{R} \) be a transcendental number and define \( f : X \rightarrow \mathbb{R} \) by \( f(a) = 0 \) and \( f(x) = t \), for every \( x \neq a \). Set \( R = \{ \sum_{i=0}^{n} m_i f^i : n \in \mathbb{N} \cup \{0\}, m_i \in \mathbb{Z} \} \). Evidently, \( M_a(R) = (f) \) and \( M_x(R) = \{0\} \), for every \( x \neq a \). Therefore, \( M_x(R) \) is not a maximal fixed ideal for any \( x \neq a \).

In the next example we construct a subring \( R \) such that, for some \( x \in X \), \( M_x(R) \) is a maximal fixed ideal which is not a maximal \( g \)-ideal.

**Example 2.3.** Let \( X = \mathbb{R} \), \( a \in \mathbb{R} \setminus \mathbb{Q} \), \( b \in \mathbb{R} \setminus \{0\} \) and \( t \) be a transcendental number. For every \( \epsilon > 0 \), define \( f_\epsilon : X \rightarrow \mathbb{R} \) by \( f_\epsilon(x) = 0 \), if \( |x - a| < \epsilon \) and \( f_\epsilon(x) = b \), if \( |x - a| \geq \epsilon \). Also, define \( f : X \rightarrow \mathbb{R} \) by \( f(x) = 0 \), if \( x \in \mathbb{Q} \) and \( f(x) = t \), if \( x \in \mathbb{R} \setminus \mathbb{Q} \). Let \( R \) be the algebra over \( \mathbb{Q} \) generated by \( \{ f_\epsilon : \epsilon > 0 \} \cup \{ f, 1 \} \). Evidently, \( R \) is a subring of \( \mathbb{R}^X \), and \( M_a(R) \) equals to \( (f_a) \) which is not a maximal ideal. It is easy to see that \( M_a(R) \) is a maximal fixed ideal and \( M_x(R) = I \), where \( I \) is the ideal generated by \( \{ f_\epsilon : \epsilon > 0 \} \). Clearly, \( Z(f) \cap Z(g) \neq \emptyset \), for all \( g \in I \). Hence \( J = (I, f) \) is a \( g \)-ideal which strictly contains \( I \). Therefore, \( I \) is not a maximal \( g \)-ideal.

**Proposition 2.4.** The following statements hold for a subring \( R \) of \( \mathbb{R}^X \).

(a) \( M_x(R) \) is a maximal \( g \)-ideal if and only if whenever \( Z \in Z(R) \) and \( x \not\in Z \), then \( x \not\in \text{cl}_{Z(R)} Z \).

(b) For each \( x \in X \), \( M_x(R) \) is a maximal \( g \)-ideal if and only if every \( Z \in Z(R) \) is clopen under \( Z(R) \)-topology.

**Proof.** (a \( \Rightarrow \)). Let \( f \in R \) and \( x \not\in Z(f) \), thus, the ideal \( (M_x(R), f) \) contains an invertible element of \( \mathbb{R}^X \). Hence, there are \( g \in M_x(R) \) and \( h \in R \) such that \( Z(g + fh) = \emptyset \). Consequently, \( x \in Z(g) \) and \( Z(f) \cap Z(g) = \emptyset \).
(a $\iff$). Assume that $f \notin M_x(R)$. Then there is some $g \in R$ such that $x \in Z(g)$ and $Z(f) \cap Z(g) = Z(f^2 + g^2) = \emptyset$. Hence, $(M_x(R), f)$ contains an invertible element of $\mathbb{R}^X$. Also, clearly, $M_x(R)$ is a $g$-ideal. Thus, $M_x(R)$ is a maximal $g$-ideal.

(b). An easy consequence of (a). \hfill \Box

**Corollary 2.5.** If $M_x(R)$ is a maximal ideal for each $x \in X$, then every $Z \in Z(R)$ is clopen under $Z(R)$-topology.

**Corollary 2.6.** Let $R$ be an invertible subring. Then every $Z \in Z(R)$ is clopen under $Z(R)$-topology if and only if $M_x(R)$ is a maximal ideal for each $x \in X$.

**Proof.** By our hypothesis and Proposition 2.4, this is clear. \hfill \Box

The following lemma is a restatement of the fact that the transcendental degree of $\mathbb{R}$ over $\mathbb{Q}$ is uncountable, see [14].

**Lemma 2.7.** Let $S = \mathbb{Q}[y_1, \ldots, y_n]$ be the ring of $n$-variable polynomials with rational coefficients. Then there exists an uncountable set $X$ of transcendental numbers for which $F(a_1, \cdots, a_n) \neq 0$, for every distinct elements $a_1, \cdots, a_n$ of $X$ and every $F \in S$.

The following example shows that the converse of Corollary 2.5 does not hold, in general.

**Example 2.8.** Let $S$ be the polynomial ring $\mathbb{Q}[y_1, \ldots, y_n]$, where $n \in \mathbb{N}$ and $n > 1$. By Lemma 2.7, there exists an infinite set of transcendental numbers $X$ for which $F(a_1, \cdots, a_n) \neq 0$, for every $a_1, \cdots, a_n \in X$ and every $F \in S$. For each $a \in X$, define the function $f_a : X \rightarrow \mathbb{R}$ by $f_a(a) = 0$ and $f_a(x) = x$ for each $x \neq a$. Now, set

$$R = \{F(f_{a_1}, \ldots, f_{a_n}) : F \in S, \; n \in \mathbb{N}, \; a_1, \ldots, a_n \in X\}.$$  

Hence, $M_x(R) = (f_a)$, for each $a \in X$, which is not a maximal ideal. However, every $Z \in Z(R)$ is clopen under $Z(R)$-topology.

**Proposition 2.9.** If $R$ is a subalgebra of $\mathbb{R}^X$, then $M_x(R)$ is a maximal $g$-ideal and a maximal fixed ideal for every $x \in X$.

**Proof.** It suffices to prove that every element of $Z(R)$ is closed under $Z(R)$-topology. To this aim, suppose that $a \in X$ and $a \notin Z(f)$, for some $f \in R$. Put $g = f - f(a)$. Clearly, $Z(g) \in Z(R)$, $a \in Z(g)$ and $Z(g) \cap Z(f) = \emptyset$. \hfill \Box

**Corollary 2.10.** If $R$ is an invertible subalgebra of $\mathbb{R}^X$, then $M_x(R)$ is a maximal ideal for each $x \in X$.

The converse of Corollary 2.10 does not hold, in general. For example, let $R$ denote the collection of all single variable polynomials over $\mathbb{R}$. Then, $M_x(R)$ is the maximal ideal $(x - r)$ for each $r \in \mathbb{R}$. However, $f = x^2 + 1$ is invertible in
\( \mathbb{R}^2 \) which is not invertible in \( R \). Note that the subalgebras \( C_c(X) \) and \( R + I \), for each ideal \( I \) in \( C(X) \), satisfy Corollary 2.10 and so \( M_x(C_c(X)) \) and \( M_x(R + I) \) are maximal ideals of \( C_c(X) \) and \( R + I \), respectively, for each \( x \in X \). Remark that in parts (b) and (e) of the following proposition we assume that “\( = \)” is a partial order on \( X \).

**Proposition 2.11.** For a subring \( R \) of \( \mathbb{R}^X \), the following statements hold.

(a) The mapping \( x \mapsto M_x(R) \) is a one-one correspondence if and only if \( (X, \tau_{Z(R)}) \) is a \( T_0 \)-space.

(b) The mapping \( x \mapsto M_x(R) \) is an order isomorphism between \( X \) and the set of all maximal fixed ideals of \( R \) if and only if \( (X, \tau_{Z(R)}) \) is a \( T_1 \)-space.

(c) For every two distinct elements \( x, y \in X \), \( M_x(R) + M_y(R) \) is not a \( g \)-ideal if and only if \( (X, \tau_{Z(R)}) \) is a \( T_2 \)-space.

(d) The mapping \( x \mapsto M_x(R) \) is an order embedding between \( X \) and the set of all maximal \( g \)-ideals of \( R \) if and only if \( (X, \tau_{Z(R)}) \) is a \( T_0 \)-space and every element of \( Z(R) \) is clopen under \( Z(R) \)-topology.

**Proof.** (a). Let \( x, y \) be distinct points of \( X \), so \( M_x(R) \neq M_y(R) \), say \( M_x(R) \not\subseteq M_y(R) \). Hence, there exists \( f \in M_x(R) \setminus M_y(R) \). Thus \( x \in Z(f) \) and \( y \notin Z(f) \). It is clear that the above reasoning is reversible and hence we are done.

(b \( \Rightarrow \)). Suppose that \( x \) and \( y \) are two distinct points of \( X \). Since \( M_x(R) \not\subseteq M_y(R) \), there exists \( f \in M_x(R) \setminus M_y(R) \). Consequently, \( x \in Z(f) \) and \( y \notin Z(f) \).

(b \( \Leftarrow \)). Suppose that \( x \in X \) and \( I \) is a fixed ideal in \( R \) containing \( M_x(R) \). Take \( y \in \bigcap_{f \in I} Z(f) \). Clearly, \( M_x(R) \subseteq I \subseteq M_y(R) \). It suffices to show \( x = y \). Suppose that \( x \neq y \) and seek a contradiction. By our hypothesis, there exists \( f \in R \) such that \( x \in Z(f) \) and \( y \notin Z(f) \). Therefore, \( M_x(R) \not\subseteq M_y(R) \) and this is a contradiction. Now, by part (a), the proof is complete.

(c). For any two distinct points \( x, y \in X \), clearly, \( M_x(R) + M_y(R) \) is not a \( g \)-ideal if and only if there exist \( f \in M_x(R) \) and \( g \in M_y(R) \) such that \( Z(f) \cap Z(g) \neq \emptyset \).

(d \( \Rightarrow \)). By part (a), clearly, \( (X, \tau_{Z(R)}) \) is a \( T_0 \)-space. Now, Suppose that \( f \in R \) and \( x \notin Z(f) \). Since \( M_x(R) \) is a maximal \( g \)-ideal, it follows that \( (M_x(R), f) \) has an invertible element of \( \mathbb{R}^X \) and so there exists \( g \in M_x(R) \), such that \( Z(g) \cap Z(f) = \emptyset \). Thus, \( Z(f) \) is closed and hence is clopen under \( Z(R) \)-topology.

(d \( \Leftarrow \)). Suppose that \( x \in X \), it suffices to show that \( M_x(R) \) is a maximal \( g \)-ideal. Assume that \( I \) is an ideal which properly contains \( M_x(R) \). Hence, there exists \( f \in I \) such that \( x \notin Z(f) \). By our hypothesis, there is \( g \in R \) such that \( x \in Z(g) \) and \( Z(g) \cap Z(f) = \emptyset \). Therefore, \( Z(f^2 + g^2) = \emptyset \) and \( f^2 + g^2 \in I \), hence, \( I \) is not a \( g \)-ideal.

It is easy to see that \( M_x(R) \), for each \( x \in X \), is a prime ideal of \( R \) and thus the hull-kernel topology may be defined on the family \( \{M_x(R) : x \in X\} \).
By considering this space, the next statement gives a relation between $Z(R)$-topology on $X$ and points of $X$.

**Proposition 2.12.** Let $R$ be a subring of $\mathbb{R}^X$ and $X$ equipped with the $\text{Coz}(R)$-topology. Then the mapping $\Phi : X \to \{ M_x(R) : x \in X \}$ defined by $x \mapsto M_x(R)$ is a homeomorphism if and only if $(X, \tau_{Z(R)})$ is a $T_0$-space.

**Proof.** By part (a) of Theorem 2.12, $\Phi$ is a one-one correspondence if and only if $(X, \tau_{Z(R)})$ is a $T_0$-space. Also, if $f \in R$ and $x \in Z(f)$, then $f \in M_x(R)$ which means that basic closed sets of $X$ equipped with the $\text{Coz}(R)$-topology are mapped to the basic closed sets in $\{ M_x(R) : x \in X \}$ equipped with the hull-kernel topology by the mapping $\Phi$ and therefore, it is a homeomorphism. □

### 3. $zR$-Ideals and $z$-Ideals in Subrings

In this section, we introduce $zR$-ideals in a subring $R$ and via the $Z(R)$-topology and maximal $g$-ideals of $R$, various characterizations of these ideals are given.

**Definition 3.1.** A subset $F$ of $Z(R)$ is called a $zR$-filter on $X$, if

(a) $\emptyset \notin F$.

(b) If $Z_1, Z_2 \in F$, then $Z_1 \cap Z_2 \in F$.

(c) If $Z_1 \in F$, $Z_2 \in Z(R)$ and $Z_1 \subseteq Z_2$, then $Z_2 \in F$.

Moreover, $F$ is called a prime $zR$-filter, if whenever $Z_1 \cup Z_2 \in F$, then $Z_1 \in F$ or $Z_2 \in F$ for each $Z_1, Z_2 \in Z(R)$. Also, $F$ is called a $zR$-ultrafilter, if $F$ is maximal among $zR$-filters on $X$.

The following proposition immediately follows from Definition 3.1.

**Proposition 3.2.** For any subring $R$, the following statements hold.

(a) $I \subseteq R$ is a $g$-ideal in $R$ if and only if $Z_R(I) = \{ Z(f) : f \in I \}$ is a $zR$-filter on $X$.

(b) $F$ is a $zR$-filter on $X$ if and only if $Z^{-1}_R(F) = \{ f \in R : Z(f) \in F \}$ is a $g$-ideal.

(c) $F$ is a prime $zR$-filter on $X$ if and only if $Z^{-1}_R(F)$ is a prime $g$-ideal.

(d) $A$ is a $zR$-ultrafilter on $X$ if and only if $Z^{-1}_R(A)$ is a maximal $g$-ideal.

(e) If $M$ is a maximal $g$-ideal in $R$, then $Z_R(M)$ is a $zR$-ultrafilter on $X$.

It is easy to see that for an ideal $I$ of $R$ we always have $I \subseteq Z^{-1}_R Z_R(I)$ and the inclusion may be proper. We call an ideal $I$ in $R$ a $zR$-ideal, if $I = Z^{-1}_R Z_R(I)$.

It follows that every $zR$-ideal is semiprime and arbitrary intersections of $zR$-ideals is a $zR$-ideal. Also, the zero ideal, the ideals of the form $M_x(R)$, maximal $g$-ideals and $Z^{-1}(F)$, for each $zR$-filter $F$, are all $zR$-ideals of $R$. For each $f \in R$, the intersection of all the maximal ideals, maximal $g$-ideals and maximal fixed ideals of $R$ containing $f$ are denoted by $M_f(R)$, $MG_f(R)$ and $MF_f(R)$, respectively. It is easy to observe that $MG_f(R)$ is a $zR$-ideal for each $f \in R$.
Obviously, $MG_f \cap MG_g = MG_{fg}$, $MF_f \cap MF_g = MF_{fg}$, $MG_f^2 + g^2 = MG_{(f,g)}$ and $MF_f^2 + g^2 = MF_{(f,g)}$ for all $f, g \in R$.

**Proposition 3.3.** Let $(X, \tau_{Z(R)})$ be a $T_1$-space. Then the following statements hold.

(a) The following statements are equivalent.

1. $g \in MF_f(R)$.
2. $MF_g(R) \subseteq MF_f(R)$.
3. $Z(f) \subseteq Z(g)$.
4. $MF_f(R) = \{g \in R : Z(f) \subseteq Z(g)\}$.

(b) An ideal $I$ of $R$ is a $z_R$-ideal if and only if $MF_f(R) \subseteq I$ for every $f \in I$.

Proof. (a: 1 $\Rightarrow$ 2). Evident.

(a: 2 $\Rightarrow$ 3). Let $x \in Z(f)$. Then $f \in M_x(R)$ and thus $MF_g(R) \subseteq MF_f(R) \subseteq M_x(R)$. This implies $g \in M_x(R)$ and hence $x \in Z(g)$.

(a: 3 $\Rightarrow$ 1). If $g \notin MF_f(R)$, then there exists $x \in X$ such that $f \in M_x(R)$ and $g \notin M_x(R)$. Therefore, $x \in Z(f) \setminus Z(g)$ and so $Z(f) \not\subseteq Z(g)$.

(b) and (c) obviously follow from part (a). □

**Lemma 3.4.** Assume that every $Z \in Z(R)$ is clopen under $Z(R)$-topology. Then $MG_f(R) = MF_f(R)$, for every $f \in R$.

Proof. Suppose that $f \in R$. By part (b) of Proposition 2.4, $M_x(R)$ is a maximal $g$-ideal for each $x \in X$. Consequently, $MG_f(R) \subseteq MF_f(R)$. Now, assume that $g \notin MF_f(R)$. Hence, there exists a maximal $g$-ideal $M$ in $R$ such that $f \in M$ and $g \notin M$. Thus, there exists $h \in M$ such that $Z(g) \cap Z(h) = \emptyset$. Since $f^2 + h^2 \in M$ and $M$ is a $g$-ideal, there is a point $x \in Z(f^2 + h^2) = Z(f) \cap Z(h)$. Clearly, $g \notin M_x(R)$ and $f \in M_x(R)$. Therefore, $g \notin MF_f(R)$. □

Proposition 3.3 and Lemma 3.4 imply the next statement.

**Proposition 3.5.** Let $(X, \tau_{Z(R)})$ be a $T_1$-space and every $Z \in Z(R)$ be a clopen set under $Z(R)$-topology. Then the following statements hold.

(a) The following statements are equivalent.

1. $g \in MG_f(R)$.
2. $MG_g(R) \subseteq MG_f(R)$.
3. $Z(f) \subseteq Z(g)$.
4. $MG_f(R) = \{g \in R : Z(f) \subseteq Z(g)\}$.

(b) An ideal $I$ of $R$ is a $z_R$-ideal if and only if $MG_f(R) \subseteq I$ for every $f \in I$.

The following corollary follows from Corollary 2.6 and Proposition 3.5.

**Corollary 3.6.** Let $R$ be an invertible subalgebra of $\mathbb{R}^X$. Then the following statements hold.
zR-ideals and z◦R-ideals in subrings of RX

(a) The following conditions are equivalent:
(1) \( g \in M_f(R) \).
(2) \( M_g(R) \subseteq M_f(R) \).
(3) \( Z(f) \subseteq Z(g) \).

(b) \( M_f(R) = \{ g \in R : Z(f) \subseteq Z(g) \} \).

(c) An ideal \( I \) of \( R \) is \( z_R \)-ideal if and only if \( M_f(R) \subseteq I \) for every \( f \in I \).

It follows from Corollary 3.6 that for an invertible subalgebra \( R \), the notion of \( z_R \)-ideal coincides with the notion of \( z \)-ideal. The next statement extends this fact and shows that this coincidence is equivalent to invertibility of \( R \).

**Theorem 3.7.** Let \( R \) be a subring of \( \mathbb{R}^X \). The following statements are equivalent.

(a) Every maximal ideal in \( R \) is a \( g \)-ideal.
(b) Every maximal \( g \)-ideal of \( R \) is a maximal ideal and if \( J \) is a maximal ideal of \( R \), then every maximal element in the set of \( g \)-ideals contained in \( J \) is a prime ideal.
(c) Every maximal ideal in \( R \) is a \( g \)-ideal.
(d) \( R \) is an invertible subring.
(e) Every \( z \)-ideal of \( R \) is a \( z_R \)-ideal.

Moreover, if \( R \) is a subalgebra and one of (a)-(c) holds, then every \( z_R \)-ideal is a \( z \)-ideal.

**Proof.** (a) \( \Rightarrow \) (b). This is clear.

(b) \( \Rightarrow \) (c). Suppose that \( M \) is a maximal ideal and \( P \) is a maximal element of \( G_M \), where \( G_M \) is the set of all \( g \)-ideals contained in \( M \). Assume that \( J \) is a maximal ideal of \( R \) containing \( P \). Then \( M \cap J = P \). As \( M \cap J \) is prime and both \( M \) and \( J \) are maximal ideal, we have \( M = J \). Hence, \( M \) is a maximal \( g \)-ideal.

(c) \( \Rightarrow \) (d). Suppose that \( Z(f) = \emptyset \) for \( f \in R \) and, on the contrary, \( f \) is a non-unit element of \( R \). Clearly, there exists a maximal ideal \( M \) of \( R \) containing \( f \). By our hypothesis, \( M \) is a \( g \)-ideal which contradicts with \( Z(f) = \emptyset \).

(d) \( \Rightarrow \) (e). Suppose that \( I \) is a \( z \)-ideal and \( Z(f) \subseteq Z(g) \) where \( f \in I \) and \( g \in R \). Since \( I \) is a \( z \)-ideal, it follows that \( M_f \subseteq I \). It suffices to prove that \( g \in M_f \). To see this, suppose that \( M \) is a maximal ideal containing \( f \). As \( R \) is invertible, \( M \) is a \( g \)-ideal and so it is a maximal \( g \)-ideal. Obviously, \( M \) is a \( z_R \)-ideal and so \( g \in M \).

(e) \( \Rightarrow \) (a). Suppose that \( M \) is a maximal ideal and, on the contrary, \( M \) is not a \( g \)-ideal. Thus, there exists \( f \in M \) such that \( Z(f) = \emptyset \). By (e), \( M \) is a \( z_R \)-ideal and since \( f \in M \), it follows that \( M = R \), which is a contradiction.

Now, suppose that one of (a)-(c) holds, \( R \) is a subalgebra and \( I \) is a \( z_R \)-ideal of \( R \). By our hypothesis, \( Mf_f(R) = M_f(R) \) for every \( f \in R \), and thus we are done.

\( \square \)
It is well-known that every minimal prime ideals over a \( z \)-ideal is also a \( z \)-ideal, see [10, Theorem 14.7]. The same statement holds for \( z_R \)-ideals as the following proposition shows.

**Proposition 3.8.** Let \( I \) be a \( z_R \)-ideal of \( R \) and \( P \) a prime ideal in \( R \) minimal over \( I \). Then \( P \) is a \( z_R \)-ideal.

**Proof.** Assume that \( Z(f) = Z(g) \) and \( f \in P \). Thus, there exists \( h \notin P \), such that \( fh \in I \). Since \( Z(fh) = Z(gh) \) and \( I \) is a \( z_R \)-ideal, it follows that \( gh \in I \subseteq P \). As \( h \notin P \), clearly, this implies that \( g \in P \). \( \square \)

An immediate consequence of Proposition 3.8 is that every minimal prime ideal in a subring \( R \) is a \( z_R \)-ideal. By the following statement, we extend some fundamental statements about \( z \)-ideals in the literature of \( C(X) \) to the subrings of \( \mathbb{R}^X \), namely, [10, 2.9, 5.3 and 5.5]. The proofs are left to the reader.

**Proposition 3.9.** Let \( R \) be a lattice-ordered subring of \( \mathbb{R}^X \) and \( I \) be a \( z_R \)-ideal in \( R \). Then the following statements hold.

(a) The following statements are equivalent

1. \( I \) is a prime ideal;
2. \( I \) contains a prime ideal;
3. if \( fg = 0 \), then \( f \in I \) or \( g \in I \);
4. for each \( f \in R \), there is a \( Z \in Z_R(I) \) on which \( f \) does not change sign.

(b) Every prime \( g \)-ideal of \( R \) is contained in a unique maximal \( g \)-ideal.

(c) If \( P \) is a prime ideal of \( R \), then \( Z_R(P) \) is a prime \( z_R \)-filter on \( X \).

(d) If \( P \) is a prime \( z_R \)-filter on \( X \), then \( Z_R^{-1}(P) \) is a prime ideal in \( R \).

(e) Every \( z_R \)-ideal of \( R \) is absolutely convex.

Thus, if \( I \) is an absolutely convex ideal of \( R \), then \( R/I \) is a lattice ring.

(f) \( I(f) \geq 0 \) if and only if \( f \geq 0 \) on some \( Z \in Z_R(I) \).

(g) Suppose that there exists \( Z \in Z_R(I) \) such that \( f(x) > 0 \), for every \( x \in Z \), then \( I(f) > 0 \). The converse is true whenever \( I \) is a maximal \( g \)-ideal.

4. \( z_R^0 \)-Ideals and \( z^0 \)-Ideals in Subrings

In this section we generalize the concept of \( z^0 \)-ideals of \( C(X) \) to the subrings of \( \mathbb{R}^X \) and introduce \( z_R^0 \)-ideal. Coincidence of \( z_R^0 \)-ideals with \( z^0 \)-ideals of \( R \) is discussed. Note that, for each element \( f \) of a commutative rings \( S \), we use \( P_f(S) \) to denote the intersection of all the minimal prime ideals in \( S \) containing \( f \).

**Definition 4.1.** An ideal \( I \) of a subring \( R \) of \( \mathbb{R}^X \) is called a \( z_R^0 \)-ideal, if \( \operatorname{int}_X Z(f) \subseteq \operatorname{int}_X Z(g) \), where \( f \in I \) and \( g \in R \), implies \( g \in I \).

The following statement investigates some characterizations of \( z_R^0 \)-ideals in subrings.

**Theorem 4.2.** Let \( R \) be a subring of \( \mathbb{R}^X \) and \( I \) be an ideal in \( R \). The following statements are equivalent.
(a) $I$ is a $z^0_R$-ideal.
(b) Whenever $\text{Ann}_C(f) \subseteq \text{Ann}_C(g)$ where $f \in I$ and $g \in R$, then $g \in I$.
(c) $R \cap P_f(C) \subseteq I$ for each $f \in I$.
(d) Whenever $P_g(C) \cap R \subseteq P_f(C) \cap R$, where $f \in I$ and $g \in R$, then $g \in I$.

Proof. (a$\Rightarrow$b). First note that by [3, Lemma 2.1] we have $\text{Ann}_C(f) \subseteq \text{Ann}_C(g)$ if and only if $\text{int}_X Z(f) \subseteq \text{int}_X Z(g)$ for each $f, g \in C(X)$. Now, let $I$ be a $z^0_R$-ideal in $R$ and $\text{Ann}_C(f) \subseteq \text{Ann}_C(g)$ where $f \in I$ and $g \in R$. Thus, by our hypothesis, we have $\text{int}_X Z(f) \subseteq \text{int}_X Z(g)$ which implies that $g \in I$.

(b$\Rightarrow$c). By [3, Proposition 2.3], we have $P_f(C) = \{g \in C(X) : \text{Ann}_C(f) \subseteq \text{Ann}_C(g)\}$. Thus the proof is evident.

(c$\Rightarrow$d). Let $P_g(C) \cap R \subseteq P_f(C) \cap R$, where $f \in I$ and $g \in R$. As $f \in I$, by our hypothesis, $P_f(C) \cap R \subseteq I$ and thus $P_g(C) \cap R \subseteq I$ which implies that $g \in I$.

(d$\Rightarrow$a). Let $\text{int}_X Z(f) \subseteq \text{int}_X Z(g)$ where $f \in I$ and $g \in R$. Therefore, by [3, Lemma 2.1], we have $P_f(C) \subseteq P_g(C)$ and hence $P_f(C) \cap R \subseteq P_g(C) \cap R$. Thus we are done by our hypothesis.

\begin{lemma}
Let $R$ be a subring of $\mathbb{R}^X$, then for each $f \in R$ we have $P_f(C) \subseteq P_f(R)$.
\end{lemma}

Proof. Let $g \in P_f(C)$. By [3, Proposition 2.3], we have $\text{Ann}_C(f) \subseteq \text{Ann}_C(g)$. Therefore, $\text{Ann}_R(f) = \text{Ann}_C(f) \cap R \subseteq \text{Ann}_C(g) \cap R = \text{Ann}_R(g)$. Thus, by [2, Proposition 1.5] we are done.

\begin{theorem}
Let $R$ be a subring of $\mathbb{R}^X$. Then every $z^0_R$-ideal in $R$ is a $z^0$-ideal if and only if $P_f(R) = P_f(C)$ for each $f \in R$.
\end{theorem}

Proof. ($\Rightarrow$). Assume on the contrary that there exists some $f \in R$ such that $P_f(R) \neq P_f(C)$. Thus, using Theorem 4.2 we have $P_f(C) \subseteq P_f(R)$. Again by Theorem 4.2, $P_f(C) \cap R$ is a $z^0_R$-ideal in $R$. Also, it is clear that this ideal is not a $z^0$-ideal, since, $P_f(R) \nsubseteq P_f(C) \cap R$.

($\Leftarrow$). Let $I$ be a $z^0_R$-ideal in $R$ and $f \in I$. By Theorem 4.2, $P_f(C) \cap R \subseteq I$. Thus, by our hypothesis, $P_f(R) \subseteq I$ which means that $I$ is a $z^0$-ideal in $R$.

From Theorem 4.2 it follows that every $z^0$-ideal in a subring $R$ is a $z^0_R$-ideal. However, the converse of this fact does not hold, in general. The following example gives an example of a subring $R$ which has a $z^0_R$-ideal that is not a $z^0$-ideal.

\begin{example}
Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \begin{cases} x & x > 0 \\ 0 & x \leq 0 \end{cases}$. It is clear that $f \in C(\mathbb{R})$. Now, let $R = \{\sum_{i=0}^n r_i f^i : r_i \in \mathbb{R}, n = 0, 1, \ldots\}$. It is easy to see that $P_f(R) = R$, however, $P_f(C) \cap R \neq R$. Also, by Theorem 4.2, $P_f(C) \cap R$ is $z^0_R$-ideal and it is clear that this ideal is not a $z^0$-ideal.
\end{example}
The next theorem gives a sufficient conditions on $X$ in order that $z^0_R$-ideals in a subring $R$ coincide with $z^0$-ideals of $R$.

**Theorem 4.6.** Let $R$ be a subring of $R^X$ and $X$ be equipped with the $\text{Coz}(R)$-topology. Then an ideal $I$ in $R$ is a $z^0$-ideal if and only if it is a $z^0_R$-ideal.

**Proof.** Let $I$ be a $z^0_R$-ideal in $R$ and $f \in I$. As $X$ is equipped with the $\text{Coz}(R)$-topology, we have $g \in \text{Ann}_R(f)$ if and only if $\text{Coz}(g) \subseteq \text{int}_X Z(f)$ for each $f, g \in R$. Therefore, $P_f(R) = \text{Ann}_R \text{Ann}_R(f) = \{g \in R : \text{Coz}(g) \cap \text{int}_X Z(f) = \emptyset\} = \{g \in R : \text{Ann}_R(f) \subseteq \text{Ann}_R(g)\}$. Hence, $P_f(R) \subseteq I$ which means that $I$ is a $z^0$-ideal in $R$. This completes the proof, since, as former stated, every $z^0$-ideal in $R$ is a $z^0_R$-ideal. \hfill \Box

Note that the condition that $X$ is equipped with the $\text{Coz}(R)$-topology is a sufficient condition for coincidence of $z^0_R$-ideals with $z^0$-ideals in a given subring $R$. The next example shows that this condition is not necessary.

**Example 4.7.** Let $X = \mathbb{R} \setminus \{0\}$ with the topology inherits from the usual topology on $\mathbb{R}$. Also, let $f : X \to \mathbb{R}$ be defined by $f(x) = \begin{cases} 1 & x > 0 \\ 0 & x < 0 \end{cases}$. It is clear that $f \in C(X)$ and $f^2 = f$. Now, set $R = \{r + sf : r, s \in \mathbb{R}\}$. It is clear that $R$ is a subring of $C(X)$. Also, by a routine reasoning, one can proves that the only ideals of $R$ are the ideals $(0)$, $(f)$, $(1 - f)$ and $R$. Moreover, the minimal prime ideals of $R$ are only the ideals $(f)$ and $(1 - f)$. These imply that every $z^0_R$-ideal is a $z^0$-ideal in $R$. However, clearly, $X$ is not equipped with the $\text{Coz}(R)$-topology.

It follows from Theorem 4.6 that for an intermediate subalgebra $A(X)$ of $C(X)$, $z^0_A$-ideals coincide with $z^0$-ideals of $A(X)$. However, the same statement does not true for $z_A$-ideals and $z$-ideals in $A(X)$, in general, see [6, Theorem 2.2]. Moreover, Theorem 3.7 together with Theorem 4.6 imply that in the subalgebras of $C(X)$ which are of the form $\mathbb{R} + I$, where $I$ is a free ideal in $C(X)$, $z_{\mathbb{R}+I}$-ideals coincide with $z$-ideals of $\mathbb{R} + I$ and $z^0_{\mathbb{R}+I}$-ideals coincide with $z^0$-ideals, too. Note that whenever $I$ is a free ideal in $C(X)$, then $\mathbb{R} + I$ determines the topology of $X$.

**Acknowledgments**

The authors would like to thank to the referee for careful reading the paper.

**References**