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Multipliers of pg-Bessel Sequences in Banach Spaces

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Abstract. In this paper, we introduce $(p, q)g-$Bessel multipliers in Banach spaces and we show that under some conditions a $(p, q)g-$Bessel multiplier is invertible. Also, we show the continuous dependency of $(p, q)g-$Bessel multipliers on their parameters.

Keywords: p-Frames, g-Frames, pg-Frames, qg-Riesz bases, $(p, q)g$-Bessel multiplier.

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1. Introduction

Frames have been introduced by J. Duffin and A.C. Schaeffer in [9], in connection with non-harmonic Fourier series. A frame for a Hilbert space is a possibly redundant set of vectors which yields, in a stable way, a representation for each vector in the space. Frames have many nice properties which make them very useful in the characterization of function space, signal processing and many other fields, see the book [7]. The concept of frames was extended to

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Banach spaces by K. Gröchenig in [11] to develop atomic decompositions from the paper [10]. See also [3], [8].

Definition 1.1. Let X be a Banach space. A countable family $\{g_i\}_{i\in I} \subset X^*$ is a p-frame for X $(1 < p < \infty)$, $1 < p < \infty$, if there exist constants $A, B > 0$ such that

$$A\|f\|_X \leq (\sum_{i\in I} |g_i(f)|^p)^{\frac{1}{p}} \leq B\|f\|_X, \quad f \in X.$$

G-frame as a natural generalization of frame in Hilbert spaces, introduced by Sun [18] in 2006. G-frame covers many previous extensions of a frame.

Definition 1.2. Let $\mathcal{H}$ be a Hilbert space and $\{\mathcal{H}_i\}_{i\in I}$ be a sequence of Hilbert spaces. We call a sequence $\{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ a g-frame for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i\in I}$ if there exist two positive constants A and B such that

$$A\|f\|^2 \leq \sum_{i\in I} \|\Lambda_i f\|^2 \leq B\|f\|^2, \quad f \in \mathcal{H}.$$

We call A and B the lower and upper $g-$frame bounds, respectively. We call $\{\Lambda_i\}_{i\in J}$ a tight g-frame if $A = B$ and Parseval g-frame if $A = B = 1$.

If $\{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ is a g-frame then $\|\Lambda_i\| \leq \sqrt{B}$ for all $i \in I$. Bessel multipliers for Hilbert spaces are investigated by Peter Balazs [4, 5]. We use the following notations for sequence spaces.

(1) $c_0 = \{\{a_n\}_{n=1}^{\infty} \subseteq \mathbb{C} : \lim_{n\to\infty} a_n = 0\}$;
(2) $l^p = \{\{a_n\}_{n=1}^{\infty} \subseteq \mathbb{C} : \|a\|_p = (\sum_{n\in\mathbb{N}} |a_n|^p)^{\frac{1}{p}} < \infty\}, 0 < p < \infty$;
(3) $l^\infty = \{\{a_n\}_{n=1}^{\infty} \subseteq \mathbb{C} : \|a\|_\infty = \sup_{n\in\mathbb{N}} |a_n| < \infty\}$.

Definition 1.3. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces. Let $\{f_i\}_{i=1}^{\infty} \subseteq \mathcal{H}_1$ and $\{g_i\}_{i=1}^{\infty} \subseteq \mathcal{H}_2$ be Bessel sequences. Fix $m = \{m_i\}_{i=1}^{\infty} \in l^\infty$. The operator

$$\mathbf{M}_{m,\{f_i\},\{g_i\}} : \mathcal{H}_1 \to \mathcal{H}_2, \quad \mathbf{M}_{m,\{f_i\},\{g_i\}}(f) = \sum_{i=1}^{\infty} m_i \langle f, f_i\rangle g_i$$

is called the Bessel multiplier of the Bessel sequences $\{f_i\}_{i=1}^{\infty}$ and $\{g_i\}_{n=1}^{\infty}$.

For more results about multipliers in Hilbert spaces we can cite the papers [6, 15, 16, 17].
Multipliers for p-Bessel sequences in Banach spaces were introduced in [13]. Rahimi investigated g-Bessel multipliers in [12]. In this note, by mixing the concepts of multipliers for p-Bessel sequences and g-Bessel multipliers, we will define multipliers for the pg-Bessel sequences ($pg-$frames) and we will investigate some of their properties.
In our opinion, it is possible that the results of this paper can be applied in Quantum Information Theory. A beautiful presentation of the connections between frames and POVM is the paper [14].

2. **Review of pg-Frames and qg-Riesz Bases**

In [1], pg-frames and qg-Riesz bases for Banach spaces have been introduced. In this section, we recall some properties of pg-frames and $qg-$Riesz bases from [1]. Throughout this section, I is a subset of $\mathbb{N}$, X is a Banach space with dual X^* and also $\{Y_i\}_{i\in I}$ is a sequence of Banach spaces.

Definition 2.1. We call a sequence $\mathbf{\Lambda} = \{\Lambda_i \in B(X, Y_i) : i \in I\}$ a $pg-$frame for X with respect to $\{Y_i : i \in I\}$ $(1 < p < \infty)$, if there exist $A, B > 0$ such that

$$A\|x\|_X \leq \left(\sum_{i\in I} \|\Lambda_i x\|^p\right)^{\frac{1}{p}} \leq B\|x\|_X, \quad \forall x \in X. \tag{2.1}$$

A, B is called the pg-frame bounds of $\{\Lambda_i\}_{i\in I}$.
If only the second inequality in (2.1) is satisfied, $\{\Lambda_i\}_{i\in I}$ is called a pg-Bessel sequence for X with respect to $\{Y_i : i \in I\}$ with bound B.

Definition 2.2. Let $\{Y_i\}_{i\in I}$ be a sequence of Banach spaces. We define

$$\left(\sum_{i\in I} \bigoplus Y_i\right)_{l_p} = \left\{\{x_i\}_{i\in I} | x_i \in Y_i, \sum_{i\in I} \|x_i\|^p < +\infty\right\}.$$

Therefore $\left(\sum_{i\in I} \bigoplus Y_i\right)_{l_p}$ is a Banach space with the norm

$$\|\{x_i\}_{i\in I}\|_p = \left(\sum_{i\in I} \|x_i\|^p\right)^{\frac{1}{p}}.$$

Let $1 < p, q < \infty$ be conjugate exponents, i.e., $\frac{1}{p} + \frac{1}{q} = 1$. If $x^* = \{x_i^*\}_{i\in I} \in (\sum_{i\in I} \bigoplus Y_i^*)_{l_q}$, then one can show that the formula

$$\langle x, x^*\rangle = \sum_{i\in I} \langle x_i, x_i^*\rangle, \quad x = \{x_i\}_{i\in I} \in (\sum_{i\in I} \bigoplus Y_i)_{l_p}$$

defines a continuous functional on $(\sum_{i\in I} \bigoplus Y_i)_{l_p}$, whose norm is equal to $\|x^*\|_q$.

Lemma 2.3. [2] *Let $1 < p, q < \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$, then*

$$\left(\sum_{i\in J} \bigoplus Y_i\right)^*_{l_p} = \left(\sum_{i\in J} \bigoplus Y_i^*\right)_{l_q};$$

where the equality holds under the duality

$$\langle x, x^*\rangle = \sum_{i\in J} \langle x_i, x_i^*\rangle.$$

Definition 2.4. Let $\mathbf{\Lambda} = \{\Lambda_i \in B(X, Y_i) : i \in I\}$ be a pg-Bessel sequence for X with respect to $\{Y_i\}$. We define the operators

$$U_\Lambda : X \to \left(\sum_{i\in I} \bigoplus Y_i\right)_{l_p}, \quad U_\Lambda(x) = \{\Lambda_i x\}_{i\in I} \tag{2.2}$$

and

$$T_\Lambda : \left(\sum_{i\in I} \bigoplus Y_i^*\right)_{l_q} \to X^* \quad T_\Lambda\{g_i\}_{i\in I} = \sum_{i\in I} \Lambda_i^* g_i. \tag{2.3}$$

U_Λ and T_Λ are called the analysis and synthesis operators of $\Lambda = \{\Lambda_i\}_{i\in I}$, respectively.

The following proposition, characterizes the *pg*-Bessel sequence by the operator T_Λ defined in (2.3).

Proposition 2.5. [1] *$\mathbf{\Lambda} = \{\Lambda_i \in B(X, Y_i) : i \in I\}$ is a pg-Bessel sequence for X with respect to $\{Y_i\}_{i\in I}$, if and only if the operator T_Λ defined in* (2.3) *is a well defined and bounded operator. In this case, $\sum_{i\in I} \Lambda_i^* g_i$ converges unconditionally for any $\{g_i\}_{i\in I} \in \left(\sum_{i\in I} \bigoplus Y_i^*\right)_{l_q}$.*

Lemma 2.6. [1] *If $\mathbf{\Lambda} = \{\Lambda_i \in B(X, Y_i) : i \in I\}$ is a pg-Bessel sequence for X with respect to $\{Y_i\}_{i\in I}$, then*

(i) $U_\Lambda^* = T$,
(ii) *If $\mathbf{\Lambda} = \{\Lambda_i \in B(X, Y_i) : i \in I\}$ is a pg-frame for reflexive Banach space X and Y_i is reflexive, for all $i \in I$ then $T_\Lambda^* = U_\Lambda$.*

Theorem 2.7. [1] *$\mathbf{\Lambda} = \{\Lambda_i \in B(X, Y_i) : i \in I\}$ is a pg-frame for X with respect to $\{Y_i\}_{i\in I}$ if and only if T_Λ defined in* (2.3) *is a bounded and onto operator.*

Definition 2.8. Let $1 < q < \infty$. A family $\mathbf{\Lambda} = \{\Lambda_i \in B(X, Y_i) : i \in I\}$ is called a $qg-$Riesz basis for X^* with respect to $\{Y_i\}_{i\in I}$, if

(i) $\{f : \Lambda_i f = 0, i \in I\} = \{0\}$ (i.e., $\{\Lambda_i\}_{i\in J}$ is g-complete);
(ii) There are positive constants A, B such that for any finite subset $I_1 \subseteq I$

$$A\left(\sum_{i\in I_1} \|g_i\|^q\right)^{\frac{1}{q}} \le \|\sum_{i\in I_1} \Lambda_i^* g_i\| \le B\left(\sum_{i\in I_1} \|g_i\|^q\right)^{\frac{1}{q}}, \quad g_i \in Y_i^*.$$

The assumptions of the Definition 2.8 imply that $\sum_{i\in J} \Lambda_i^* g_i$ converges unconditionally for all $\{g_i\}_{i\in I} \in (\sum_{i\in I} \bigoplus Y_i^*)_{l_q}$, and

$$A\left(\sum_{i\in I} \|g_i\|^q\right)^{\frac{1}{q}} \le \|\sum_{i\in I} \Lambda_i^* g_i\| \le B\left(\sum_{i\in I} \|g_i\|^q\right)^{\frac{1}{q}}.$$

In [1], it is proved that if $\mathbf{\Lambda} = \{\Lambda_i \in B(X, Y_i) : i \in I\}$ is a qg-Riesz basis for X^* with respect to $\{Y_i\}_{i\in I}$, then $\mathbf{\Lambda}$ is a pg-frame for X with respect to $\{Y_i\}_{i\in I}$. Therefore $\mathbf{\Lambda} = \{\Lambda_i \in B(X, Y_i) : i \in I\}$ is a qg-Riesz basis for X^* if and only if the operator T_Λ defined in (2.3) is an invertible operator from $\left(\sum_{i\in J} \bigoplus Y_i^*\right)_{l_q}$ onto X^*.

Theorem 2.9. [1] *Let $\{Y_i\}_{i\in I}$ be a sequence of reflexive Banach spaces. Let $\mathbf{\Lambda} = \{\Lambda_i \in B(X, Y_i) : i \in I\}$ be a pg-frame for X with respect to $\{Y_i\}_{i\in I}$. Then the following statements are equivalent:*

(i) *$\{\Lambda_i\}_{i\in I}$ is a qg-Riesz basis for X^*;*
(ii) *If $\{g_i\}_{i\in I} \in \left(\sum_{i\in I} \bigoplus Y_i^*\right)_{l_q}$ and $\sum_{i\in I} \Lambda_i^* g_i = 0$ then $g_i = 0$, for all $i \in I$;*
(iii) *$\mathcal{R}_U = (\sum_{i\in I} \bigoplus Y_i)_{l_p}$.*

3. Multipliers for pg-Bessel Sequences

In this section, we assume that X_1 and X_2 are reflexive Banach spaces and $\{Y_i\}_{i=1}^{\infty}$ is a family of reflexive Banach spaces. Also, we consider $p, q > 1$ are real numbers such that $\frac{1}{p} + \frac{1}{q} = 1$.

Proposition 3.1. *Let X be a Banach space and let $\mathbf{\Lambda} = \{\Lambda_i \in B(X, Y_i)\}_{i=1}^{\infty}$ be a pg-Bessel sequence for X with respect to $\{Y_i\}_{i=1}^{\infty}$ with the bound B.*

(1) *If $\mathbf{\Theta} = \{\Theta_i \in B(X, Y_i)\}_{i=1}^{\infty}$ is a sequence of bounded operators such that $(\sum_{i=1}^{\infty} \|\Lambda_i - \Theta_i\|^p)^{\frac{1}{p}} < K < \infty$, then Θ is a pg-Bessel sequence for X with bound $B + K$.*
(2) *Let $\mathbf{\Theta^{(n)}} = \{\Theta_i^{(n)} \in B(X, Y_i)\}_{i=1}^{\infty}$ be a sequence of bounded operators such that for all $\varepsilon > 0$ there exists $N > 0$ with*
$$\left(\sum_{i=1}^{\infty} \|\Lambda_i - \Theta_i^{(n)}\|^p\right)^{\frac{1}{p}} < \varepsilon, \qquad n \geq N,$$
then $\mathbf{\Theta^{(n)}}$ is a pg-Bessel sequence and for all $n \geq N$,
$$\|U_{\Theta^{(n)}} - U_{\Lambda}\| \leq \varepsilon, \qquad \|T_{\Theta^{(n)}} - T_{\Lambda}\| \leq \varepsilon.$$

Proof. (1) If $\{g_i\}_{i=1}^{\infty} \in (\sum_{i=1}^{\infty} \bigoplus Y_i^*)_{l_q}$ we have
$$\begin{aligned}
\|T_{\Lambda}\{g_i\}_{i=1}^{\infty} - T_{\Theta}\{g_i\}_{i=1}^{\infty}\| =& \|\sum_{i=1}^{\infty} (\Lambda_i^* - \Theta_i^*) g_i\| \\
=& \sup_{\|f\|\leq 1} \|\sum_{i=1}^{\infty} g_i(\Lambda_i f - \Theta_i f)\| \\
\leq& \sup_{\|f\|\leq 1} \sum_{i=1}^{\infty} \|g_i\| \|\Lambda_i f - \Theta_i f\| \\
\leq& \left(\sum_{i=1}^{\infty} \|g_i\|^q\right)^{\frac{1}{q}} \sup_{\|f\|\leq 1} \left(\sum_{i=1}^{\infty} \|\Lambda_i f - \Theta_i f\|^p\right)^{\frac{1}{p}} \\
\leq& K \|\{g_i\}_{i=1}^{\infty}\|_q,
\end{aligned}$$
and so
$$\begin{aligned}
\|T_{\Theta}\{g_i\}_{i=1}^{\infty}\| \leq& \|T_{\Theta}\{g_i\}_{i=1}^{\infty} - T_{\Lambda}\{g_i\}_{i=1}^{\infty}\| + \|T_{\Lambda}\{g_i\}_{i=1}^{\infty}\| \\
\leq& (B + K) \|\{g_i\}_{i=1}^{\infty}\|_q.
\end{aligned}$$

Consequently, Proposition 2.5 implies that $\{\Theta_i\}_{i=1}^{\infty}$ is a pg-Bessel sequence with the bound $B+K$.
(2) It follows from (1) that $\{\Theta_i^{(n)}\}_{i=1}^{\infty}$ is a pg-Bessel sequence and $\|T_{\Theta^{(n)}}-T_{\Lambda}\| \leq \varepsilon$ for all $n \geq N$. But for $f \in X$ and $n \geq N$ we have

$$\|U_{\Lambda}f - U_{\Theta^{(n)}}f\|_p = \left(\sum_{i=1}^{\infty} \|\Lambda_i f - \Theta_i^{(n)} f\|^p\right)^{\frac{1}{p}} \leq \left(\sum_{i=1}^{\infty} \|\Lambda_i - \Theta_i^{(n)}\|^p\right)^{\frac{1}{p}} \|f\|$$

hence $\|U_{\Theta^{(n)}} - U_{\Lambda}\| \leq \varepsilon$. □

We say that $\mathbf{\Theta^{(n)}} = \{\Theta_i^{(n)} \in B(X,Y_i)\}_{i=1}^{\infty}$ converges to $\mathbf{\Lambda} = \{\Lambda_i \in B(X,Y_i)\}_{i=1}^{\infty}$ in l_p-sense, if the condition of Proposition 3.1 (2) is fullfilled.

Proposition 3.2. *Let $\mathbf{\Lambda} = \{\Lambda_i \in B(X_2,Y_i)\}_{i=1}^{\infty}$ be a pg-Bessel sequence for X_2 with bound B_{Λ} and $\mathbf{\Theta} = \{\Theta_i \in B(X_1^*,Y_i^*)\}_{i=1}^{\infty}$ be a qg-Bessel sequence for X_1^* with bound B_{Θ}. If $m \in l^{\infty}$, then the operator*

$$\mathbf{M}_{m,\Lambda,\Theta} : X_1^* \to X_2^*, \qquad \mathbf{M}_{m,\Lambda,\Theta}(g) = \sum_{i=1}^{\infty} m_i \Lambda_i^* \Theta_i g$$

is well defined, the sum converges unconditionally for all $g \in X_1^*$ and

$$\|\mathbf{M}_{m,\Lambda,\Theta}\| \leq B_{\Lambda} B_{\Theta} \|m\|_{\infty}.$$

Proof. Let $g \in X_1^*$, then $\{m_i\Theta_i g\}_{i=1}^{\infty} \in \left(\sum_{i=1}^{\infty} \bigoplus Y_i^*\right)_{l_q}$, and Proposition 2.5 implies that $\sum_{i=1}^{\infty} m_i \Lambda_i^* \Theta_i g$ converges unconditionally and $\mathbf{M}_{m,\Lambda,\Theta}$ is well defined. Also we have

$$\begin{aligned}
\|\sum_{i=1}^{\infty} m_i \Lambda_i^* \Theta_i g\| =& \sup_{\|x\|\leq 1} |\langle x, \sum_{i=1}^{\infty} m_i \Lambda_i^* \Theta_i g\rangle| \\
=& \sup_{\|x\|\leq 1} |\sum_{i=1}^{\infty} m_i (\Theta_i g)(\Lambda_i x)| \\
\leq& \sup_{\|x\|\leq 1} \sum_{i=1}^{\infty} |m_i||(\Theta_i g)(\Lambda_i x)| \\
\leq& \|m\|_{\infty} \sup_{\|x\|\leq 1} \sum_{i=1}^{\infty} \|\Theta_i g\|\|\Lambda_i x\| \\
\leq& \|m\|_{\infty} \left(\sum_{i=1}^{\infty} \|\Theta_i g\|^q\right)^{\frac{1}{q}} \sup_{\|x\|\leq 1} \left(\sum_{i=1}^{\infty} \|\Lambda_i x\|^p\right)^{\frac{1}{p}} \\
\leq& \|m\|_{\infty} \cdot B_{\Theta} \|g\| \cdot \sup_{\|x\|\leq 1} (B_{\Lambda} \|x\|) \\
\leq& \|m\|_{\infty} \cdot B_{\Theta} \cdot B_{\Lambda} \|g\|.
\end{aligned}$$

Therefore $\mathbf{M}_{m,\Lambda,\Theta}$ is bounded and $\|\mathbf{M}_{m,\Lambda,\Theta}\| \leq B_{\Lambda} B_{\Theta} \|m\|_{\infty}$. □

Definition 3.3. Let $\mathbf{\Lambda} = \{\Lambda_i \in B(X_2, Y_i)\}_{i=1}^{\infty}$ be a pg-Bessel sequence for X_2 with bound B_Λ and $\mathbf{\Theta} = \{\Theta_i \in B(X_1^*, Y_i^*)\}_{i=1}^{\infty}$ be a qg-Bessel sequence for X_1^* with bound B_Θ. Let $m = \{m_i\}_{i=1}^{\infty} \in l^\infty$. The operator

$$\mathbf{M}_{m,\Lambda,\Theta} : X_1^* \to X_2^*, \qquad \mathbf{M}_{m,\Lambda,\Theta}(g) = \sum_{i=1}^{\infty} m_i \Lambda_i^* \Theta_i g \tag{3.1}$$

is called the $(p,q)g-$Bessel multiplier of $\mathbf{\Lambda}$, $\mathbf{\Theta}$ and m. The sequence m is called the symbol of $\mathbf{M}$.

Proposition 3.4. *Let $\mathbf{\Lambda} = \{\Lambda_i \in B(X_2, Y_i)\}_{i=1}^{\infty}$ be a qg-Riesz basis for X_2^* and $\mathbf{\Theta} = \{\Theta_i \in B(X_1^*, Y_i^*)\}_{i=1}^{\infty}$ be a qg-Bessel sequence for X_1^* with all members non-zero. Then the mapping*

$$m \to \mathbf{M}_{m,\Lambda,\Theta}$$

is injective from l^∞ into $B(X_1^*, X_2^*)$.

Proof. If $\mathbf{M}_{m,\Lambda,\Theta} = 0$, then $\sum_{i=1}^{\infty} m_i \Lambda_i^* \Theta_i g = 0$ for all $g \in X_1^*$. Then Theorem 2.9 implies that $m_i \Theta_i g = 0$ for all $i \in \mathbb{N}$ and for all $g \in X_1^*$. Since $\Theta_i \neq 0$ for each $i \in \mathbb{N}$, we get $m_i = 0$. □

Theorem 3.5. *Let $\mathbf{\Lambda} = \{\Lambda_i \in B(X_2, Y_i)\}_{i=1}^{\infty}$ be a qg-Riesz basis for X_2^* with respect to $\{Y_i\}_{i=1}^{\infty}$, then there exist a sequence $\{\widetilde{\Lambda}_i \in B(X_2^*, Y_i^*)\}_{i=1}^{\infty}$ which is a pg-Riesz basis for X_2 with respect to $\{Y_i^*\}_{i=1}^{\infty}$ such that*

$$x^* = \sum_{i=1}^{\infty} \Lambda_i^* \widetilde{\Lambda}_i x^*, \qquad x^* \in X_2^*$$

and $\widetilde{\Lambda}_k \Lambda_i^* = \delta_{k,i} I$.

Proof. Since $\mathbf{\Lambda} = \{\Lambda_i \in B(X_2, Y_i)\}_{i=1}^{\infty}$ is a pg-frame for X_2, Theorem 2.7 implies that for every $x^* \in X_2^*$ there exists a unique $\{g_i\}_{i=1}^{\infty} \in \left(\sum_{i=1}^{\infty} \bigoplus Y_i^*\right)_{l_q}$ such that $x^* = \sum_{i=1}^{\infty} \Lambda_i^* g_i$. Let us define the operator

$$\widetilde{\Lambda}_i : X_2^* \to Y_i^*, \qquad \widetilde{\Lambda}_i(x^*) = g_i.$$

By Theorem 2.9, $\widetilde{\Lambda}_i$ is well defined. Let A_Λ, B_Λ be the qg-Riesz basis bounds for $\mathbf{\Lambda} = \{\Lambda_i \in B(X_2, Y_i)\}_{i=1}^{\infty}$. Then for any $\{g_i\}_{i=1}^{\infty} \in \left(\sum_{i=1}^{\infty} \bigoplus Y_i^*\right)_{l_q}$ we have

$$A_\Lambda \left(\sum_{i=1}^{\infty} \|g_i\|^q \right)^{\frac{l}{q}} \leq \| \sum_{i=1}^{\infty} \Lambda_i^* g_i \| \leq B_\Lambda \left(\sum_{i=1}^{\infty} \|g_i\|^q \right)^{\frac{l}{q}}.$$

Therefore

$$\frac{1}{B_\Lambda} \| \sum_{i=1}^{\infty} \Lambda_i^* g_i \| \leq \left(\sum_{i=1}^{\infty} \|g_i\|^q \right)^{\frac{l}{q}} \leq \frac{1}{A_\Lambda} \| \sum_{i=1}^{\infty} \Lambda_i^* g_i \|,$$

for all $\{g_i\}_{i=1}^{\infty} \in (\sum_{i=1}^{\infty} \bigoplus Y_i^*)_{l_q}$. Hence we get

$$\frac{1}{B_\Lambda}\|x^*\| \leq \left(\sum_{i=1}^{\infty} \|\widetilde{\Lambda}_i(x^*)\|^q\right)^{\frac{l}{q}} \leq \frac{1}{A_\Lambda}\|x^*\|, \quad x^* \in X_2^*.$$

This implies that $\{\widetilde{\Lambda}_i \in B(X_2^*, Y_i^*)\}_{i=1}^{\infty}$ is a qg-frame for X_2^* with respect to $\{Y_i^*\}_{i=1}^{\infty}$ with bounds $\frac{1}{A_\Lambda}$ and $\frac{1}{B_\Lambda}$ and

$$x^* = \sum_{i=1}^{\infty} \Lambda_i^* \widetilde{\Lambda}_i x^*, \quad x^* \in X_2^*$$

and $\widetilde{\Lambda}_k \Lambda_i^* = \delta_{k,i} I$. At the other hand the synthesis operator is invertible and $U_{\widetilde{\Lambda}} = T_\Lambda^{-1}$, therefore $U_{\widetilde{\Lambda}}$ is invertible. So by Lemma 2.6, $U_{\widetilde{\Lambda}}^* = T_{\widetilde{\Lambda}}$ is invertible and therefore $\{\widetilde{\Lambda}_i\}_{i\in\mathbb{N}}$ is a pg-Riesz basis for X_2. □

By a duality argument it can be shown for a reflexive space that $\Lambda_i \tilde{\Lambda}_k^* = \delta_{i,k} I$ on X_2.

Corollary 3.6. *Let $\mathbf{\Theta} = \{\Theta_i \in B(X_1^*, Y_i^*)\}_{i=1}^{\infty}$ be a pg-Riesz basis for X_1 with respect to $\{Y_i^*\}_{i=1}^{\infty}$ with bounds A_Θ, B_Θ, then there exists a sequence $\{\widetilde{\Theta}_i \in B(X_1, Y_i)\}_{i=1}^{\infty}$ which is a qg-Riesz basis for X_1^* with respect to $\{Y_i\}_{i=1}^{\infty}$ with bounds $\frac{1}{B_\Theta}, \frac{1}{A_\Theta}$ and*

$$x = \sum_{i=1}^{\infty} \Theta_i^* \widetilde{\Theta}_i x, \qquad x \in X_1,$$

and $\widetilde{\Theta}_k \Theta_i^* = \delta_{k,i} I$.

Proposition 3.7. *Let $\mathbf{\Lambda} = \{\Lambda_i \in B(X_2, Y_i)\}_{i=1}^{\infty}$ be a qg-Riesz basis for X_2^* with respect to $\{Y_i\}_{i=1}^{\infty}$ with bound A_Λ, B_Λ and $\mathbf{\Theta} = \{\Theta_i \in B(X_1^*, Y_i^*)\}_{i=1}^{\infty}$ be a pg-Riesz basis for X_1 with respect to $\{Y_i^*\}_{i=1}^{\infty}$ with bounds A_Θ, B_Θ. If $m \in l^\infty$, then*

$$A_\Lambda A_\Theta \|m\|_\infty \leq \|\mathbf{M}_{m,\Lambda,\Theta}\| \leq B_\Lambda B_\Theta \|m\|_\infty.$$

Proof. By Proposition 3.2, it is enough to show that we have the lower bound. Corollary 3.6 implies that there exists a sequence

$$\{\widetilde{\Theta}_i \in B(X_1, Y_i)\}_{i=1}^{\infty}$$

which is a qg-Riesz basis for X_1^* (therefore a pg-frame for X_1) with respect to $\{Y_i\}_{i=}^{\infty}$ with bounds $\frac{1}{B_\Theta}, \frac{1}{A_\Theta}$ and

$$x = \sum_{i=1}^{\infty} \Theta_i^* \widetilde{\Theta}_i x, \qquad x \in X_1,$$

and $\widetilde{\Theta}_k\Theta_i^* = \delta_{k,i}I$. Let us fix $0 \neq y_k^* \in Y_k^*$ for each $k \in \mathbb{N}$, then we have

$$\begin{aligned}
\|\mathbf{M}_{m,\Lambda,\Theta}\| = \sup_{0\neq g\in X_1^*} \frac{\|\mathbf{M}_{m,\Lambda,\Theta} g\|}{\|g\|} &= \sup_{0\neq g\in X_1^*} \frac{\|\sum_{i=1}^{\infty} m_i\Lambda_i^*\Theta_i g\|}{\|g\|} \\
&\geq \sup_{k\in\mathbb{N}} \frac{\|\sum_{i=1}^{\infty} m_i\Lambda_i^*\Theta_i(\widetilde{\Theta}_k)^* y_k^*\|}{\|(\widetilde{\Theta}_k)^* y_k^*\|} \\
&= \sup_{k\in\mathbb{N}} \frac{\|m_k\Lambda_k^* y_k^*\|}{\|(\widetilde{\Theta}_k)^* y_k^*\|} \\
&= \sup_{k\in\mathbb{N}} |m_k| \frac{\|\Lambda_k^* y_k^*\|}{\|(\widetilde{\Theta}_k)^* y_k^*\|} \\
&\geq A_\Lambda A_\Theta \|m\|_\infty.
\end{aligned}$$

□

Theorem 3.8. *Let* $\mathbf{\Lambda} = \{\Lambda_i \in B(X_2, Y_i)\}_{i=1}^{\infty}$ *be a qg-Riesz basis for* X_2^* *with respect to* $\{Y_i\}_{i=1}^{\infty}$ *and* $\mathbf{\Theta} = \{\Theta_i \in B(X_1^*, Y_i^*)\}_{i=1}^{\infty}$ *be a pg-Riesz basis for* X_1 *with respect to* $\{Y_i^*\}_{i=1}^{\infty}$. *If* $m = \{m_i\}_{i=1}^{\infty}$ *satisfies* $0 < \inf_{i\in\mathbb{N}}|m_i| \leq \sup_{i\in\mathbb{N}}|m_i| < +\infty$, *then* $\mathbf{M}_{m,\Lambda,\Theta}$ *is invertible with inverse* $\mathbf{M}_{\frac{1}{m},\widetilde{\Theta},\widetilde{\Lambda}}$.

Proof. Let us consider $\{\widetilde{\Lambda}_i \in B(X_2^*, Y_i^*)\}_{i=1}^{\infty}$ and $\{\widetilde{\Theta}_i \in B(X_1, Y_i)\}_{i=1}^{\infty}$ which appear in Proposition 3.5 and Corollary 3.6, respectively. We prove that

$$(\mathbf{M}_{m,\Lambda,\Theta})^{-1} = \mathbf{M}_{\frac{1}{m},\widetilde{\Theta},\widetilde{\Lambda}}.$$

Let $g \in X_1^*$, then

$$\begin{aligned}
\mathbf{M}_{\frac{1}{m},\widetilde{\Theta},\widetilde{\Lambda}} \circ \mathbf{M}_{m,\Lambda,\Theta}(g) =& \mathbf{M}_{\frac{1}{m},\widetilde{\Theta},\widetilde{\Lambda}}\big(\sum_{i=1}^{\infty} m_i\Lambda_i^*\Theta_i g\big) \\
=& \sum_{k=1}^{\infty} \frac{1}{m_k}(\widetilde{\Theta}_k)^*\widetilde{\Lambda}_k\big(\sum_{i=1}^{\infty} m_i\Lambda_i^*\Theta_i g\big) \\
=& \sum_{k=1}^{\infty} \frac{1}{m_k}(\widetilde{\Theta}_k)^*\big(\sum_{i=1}^{\infty} m_i\widetilde{\Lambda}_k\Lambda_i^*\Theta_i g\big) \\
=& \sum_{k=1}^{\infty} \frac{1}{m_k}(\widetilde{\Theta}_k)^*(m_k\Theta_k g) \\
=& g.
\end{aligned}$$

Let us consider $f \in X_2^*$, then

$$\begin{aligned}
\mathbf{M}_{m,\Lambda,\Theta} \circ \mathbf{M}_{\frac{1}{m},\widetilde{\Theta},\widetilde{\Lambda}} f =& \mathbf{M}_{m,\Lambda,\Theta}\big(\sum_{k=1}^{\infty} \frac{1}{m_k} (\widetilde{\Theta}_k)^* \widetilde{\Lambda}_k f\big) \\
=& \sum_{i=1}^{\infty} m_i \Lambda_i^* \Theta_i \big(\sum_{k=1}^{\infty} \frac{1}{m_k} (\widetilde{\Theta}_k)^* \widetilde{\Lambda}_k f\big) \\
=& \sum_{i=1}^{\infty} m_i \Lambda_i^* \big(\sum_{k=1}^{\infty} \frac{1}{m_k} \Theta_i (\widetilde{\Theta}_k)^* \widetilde{\Lambda}_k f\big) \\
=& \sum_{i=1}^{\infty} m_i \Lambda_i^* (\frac{1}{m_i} \widetilde{\Lambda}_i f) \\
=& f.
\end{aligned}$$

□

In the next results, we show that the $(p,q)g$-Bessel multiplier $\mathbf{M} = \mathbf{M}_{m,\Lambda,\Theta}$ depends continuously on its parameters, $m = \{m_i\}_{i=1}^{\infty}$, $\mathbf{\Lambda} = \{\Lambda_i\}_{i=1}^{\infty}$ and $\mathbf{\Theta} = \{\Theta_i\}_{i=1}^{\infty}$.

Theorem 3.9. *Let $\mathbf{\Lambda} = \{\Lambda_i \in B(X_2, Y_i)\}_{i=1}^{\infty}$ be a pg-Bessel sequence for X_2 with bound B_Λ and $\mathbf{\Theta} = \{\Theta_i \in B(X_1^*, Y_i^*)\}_{i=1}^{\infty}$ be a qg-Bessel sequence for X_1^* with bound B_Θ. Let $p_1, q_1 > 1$ such that $\frac{1}{p_1} + \frac{1}{q_1} = 1$ and $m \in l^\infty$. Let $\mathbf{\Lambda^{(n)}} = \{\Lambda_i^{(n)} \in B(X_2, Y_i)\}_{i=1}^{\infty}$ be a pg-Bessel sequence for X_2 with bound $B_{\Lambda^{(n)}}$ and $\mathbf{\Theta^{(n)}} = \{\Theta_i^{(n)} \in B(X_1^*, Y_i^*)\}_{i=1}^{\infty}$ be a qg-Bessel sequence for X_1^* with bound $B_{\Theta^{(n)}}$ for all $n \in \mathbb{N}$. Then*

(1) *If $\|m^{(n)} - m\|_{p_1} \to 0$, then $\|\mathbf{M}_{m^{(n)},\Lambda,\Theta} - \mathbf{M}_{m,\Lambda,\Theta}\| \to 0$, as $n \to \infty$.*

(2) *If $m \in l^{p_1}$ and $\{\Theta_i^{(n)}\}_{i=1}^{\infty}$ converges to $\{\Theta_i\}_{i=1}^{\infty}$ in l^{q_1}-sense, then*

$$\|\mathbf{M}_{m,\Lambda,\Theta^{(n)}} - \mathbf{M}_{m,\Lambda,\Theta}\| \to 0, \quad n \to \infty.$$

(3) *If $m \in l^{p_1}$ and $\{\Lambda_i^{(n)}\}_{i=1}^{\infty}$ converges to $\{\Lambda_i\}_{i=1}^{\infty}$ in l^{q_1}-sense, then*

$$\|\mathbf{M}_{m,\Lambda^{(n)},\Theta} - \mathbf{M}_{m,\Lambda,\Theta}\| \to 0, \quad n \to \infty.$$

(4) *Let*

$$B_1 = \sup_{n\in\mathbb{N}} B_{\Lambda^{(n)}} < +\infty, \quad B_2 = \sup_{n\in\mathbb{N}} B_{\Theta^{(n)}} < +\infty.$$

If $\|m^{(n)} - m\|_{l^{p_1}} \to 0$ and $\{\Theta_i^{(n)}\}_{i=1}^{\infty}$ and $\{\Lambda_i^{(n)}\}_{i=1}^{\infty}$ converge to $\{\Theta_i\}_{i=1}^{\infty}$ and $\{\Lambda_i\}_{i=1}^{\infty}$ in l^{q_1}-sense, respectively, then

$$\|\mathbf{M}_{m^{(n)},\Lambda^{(n)},\Theta^{(n)}} - \mathbf{M}_{m,\Lambda,\Theta}\| \to 0, \quad n \to \infty.$$

Proof. (1) Using proof of the Proposition 3.2 we have

$$\begin{aligned}\|\mathbf{M}_{m^{(n)},\Lambda,\Theta}-\mathbf{M}_{m,\Lambda,\Theta}\| =&\|\mathbf{M}_{m^{(n)}-m,\Lambda,\Theta}\|\\ \leq& B_\Lambda B_\Theta\|m^{(n)}-m\|_\infty\\ \leq& B_\Lambda B_\Theta\|m^{(n)}-m\|_{p_1}\to 0.\end{aligned}$$

(2) For $g\in X_1^*$, we have

$$\begin{aligned}\|\mathbf{M}_{m,\Lambda,\Theta^{(n)}}g-\mathbf{M}_{m,\Lambda,\Theta}g\| =&\|\sum_{i=1}^{\infty}m_i\Lambda_i^*(\Theta_i^{(n)}-\Theta_i)g\|\\ \leq&\sum_{i=1}^{\infty}|m_i|\|\Lambda_i^*\|\|(\Theta_i^{(n)}-\Theta_i)g\|\\ \leq&\sum_{i=1}^{\infty}B_\Lambda|m_i|\|(\Theta_i^{(n)}-\Theta_i)g\|\\ \leq& B_\Lambda\|m\|_{p_1}\left(\sum_{i=1}^{\infty}\|(\Theta_i^{(n)}-\Theta_i)g\|^{q_1}\right)^{\frac{1}{q_1}}.\end{aligned}$$

So

$$\|\mathbf{M}_{m,\Lambda,\Theta^{(n)}}-\mathbf{M}_{m,\Lambda,\Theta}\|\leq B_\Lambda\|m\|_{p_1}\left(\sum_{i=1}^{\infty}\|(\Theta_i^{(n)}-\Theta_i)\|^{q_1}\right)^{\frac{1}{q_1}}\to 0.$$

(3) It is similar to the proof of (2).
(4) We have

$$\|\mathbf{M}_{m^{(n)},\Lambda^{(n)},\Theta^{(n)}}-\mathbf{M}_{m,\Lambda^{(n)},\Theta^{(n)}}\|\leq B_1B_2\|m^{(n)}-m\|_{p_1}, \tag{3.2}$$

$$\|\mathbf{M}_{m,\Lambda^{(n)},\Theta^{(n)}}-\mathbf{M}_{m,\Lambda,\Theta^{(n)}}\|\leq B_2\|m\|_{p_1}\left(\sum_{i=1}^{\infty}\|(\Lambda_i^{(n)}-\Lambda_i)\|^{q_1}\right)^{\frac{1}{q_1}}, \tag{3.3}$$

$$\|\mathbf{M}_{m,\Lambda,\Theta^{(n)}}-\mathbf{M}_{m,\Lambda,\Theta}\|\leq B_\Lambda\|m\|_{p}\left(\sum_{i=1}^{\infty}\|(\Theta_i^{(n)}-\Theta_i)\|^{q_1}\right)^{\frac{1}{q_1}}. \tag{3.4}$$

Since

$$\begin{aligned}\|\mathbf{M}_{m^{(n)},\Lambda^{(n)},\Theta^{(n)}}-\mathbf{M}_{m,\Lambda,\Theta}\| \leq&\|\mathbf{M}_{m^{(n)},\Lambda^{(n)},\Theta^{(n)}}-\mathbf{M}_{m,\Lambda^{(n)},\Theta^{(n)}}\|\\ &+\|\mathbf{M}_{m,\Lambda^{(n)},\Theta^{(n)}}-\mathbf{M}_{m,\Lambda,\Theta^{(n)}}\|\\ &+\|\mathbf{M}_{m,\Lambda,\Theta^{(n)}}-\mathbf{M}_{m,\Lambda,\Theta}\|,\end{aligned}$$

(3.2), (3.3), (3.4) imply that

$$\|\mathbf{M}_{\Lambda^{(n)},\Theta^{(n)},m^{(n)}}-\mathbf{M}_{\Lambda,\Theta,m}\|\to 0,\quad n\to\infty.$$

□

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