

Some Algebraic and Combinatorial Properties of the Complete T -Partite Graphs

Seyyede Masoome Seyyedi, Farhad Rahmati*

Department of Mathematics and Computer Science, Amirkabir University of Technology, P. O. Box 15875-4413, Tehran, Iran.

E-mail: mseyyedi@aut.ac.ir

E-mail: frahmati@aut.ac.ir

ABSTRACT. In this paper, we characterize the shellable complete t -partite graphs. It is also shown that for these types of graphs the concepts vertex decomposable, shellable and sequentially Cohen-Macaulay are equivalent. Furthermore, we give a combinatorial condition for the Cohen-Macaulay complete t -partite graphs.

Keywords: Cohen-Macaulay, shellable, Vertex decomposable, Edge ideal.

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1. INTRODUCTION

Let G be a finite simple graph on n vertices. Let V_G and E_G denote, respectively, the vertex set and the edge set of G . An independent set in G is a subset of V_G which none of elements are adjacent and the independence complex Δ_G of a graph G is defined by

$$\Delta_G = \{A \subseteq V_G : A \text{ is an independent set in } G\}.$$

We denote by $MIS(G)$ the set of all maximal independent sets in G (the set of facets of Δ_G); see [16]. In this paper, we obtain a lower bound for $|MIS(G)|$. Recently, researchers studied the algebraic properties of a commutative ring by

*Corresponding Author

its associated combinatorial structure, like for instance zero divisor graph; see [1, 8, 13]. Now let $R = k[x_1, \dots, x_n]$ be the polynomial ring over a field k in n variable x_1, \dots, x_n . Identifying $i \in V_G$ with the variable x_i in R , the edge ideal $I(G)$ of G will be defined as the monomial ideal generated by all of monomials $x_i x_j$ such that $\{x_i, x_j\} \in E_G$.

In recent years, researchers tried to identify Cohen-Macaulay graphs in terms of their combinatorial properties. Estrada and Villarreal in [3] showed that Cohen-Macaulayness and shellability of a bipartite graph G are the same. Herzog and Hibi in [6] proved that a bipartite graph G is Cohen-Macaulay if and only if $|V_1| = |V_2|$ and there is an order on vertices of V_1 and V_2 as x_1, \dots, x_n and y_1, \dots, y_n , respectively, such that:

- i) $x_i \sim y_i$ for $i = 1, \dots, n$,
- ii) if $x_i \sim y_j$, then $i \leq j$,
- iii) if $x_i \sim y_j$ and $x_j \sim y_k$, then $x_i \sim y_k$.

Here, we present a necessary and sufficient condition under which shellability of a complete t -partite graph is equivalent to Cohen-Macaulayness.

It is known that any vertex decomposable graph is shellable (hence sequentially Cohen-Macaulay), but the converse is not valid in general. So, it is interesting to know a family of graphs in which the property of shellability, vertex decomposability and sequentially Cohen-Macaulayness are the same. Francisco and Van Tuyl in [4] showed that n -cycles for $n = 3, 5$ belong to this family of graphs. F. Mohammadi and D. Kiani in [10] proved that in θ_{n_1, \dots, n_k} for $\{n_1, \dots, n_k\} \neq \{2, 5\}$, vertex decomposability, shellability and sequentially Cohen-Macaulayness are coincide. Van Tuyl in [14] showed that in bipartite graphs, three concepts are equivalent. In this paper, attempts have been particularly made to introduce another member of this family, that is, complete t -partite graphs.

Herzog and Hibi, in [6], showed that a bipartite graph without isolated vertices G is unmixed if and only if there exists a bipartition $V_1 = \{x_1, \dots, x_g\}$ and $V_2 = \{y_1, \dots, y_g\}$ of V_G such that:

- i) $\{x_i, y_i\} \in E_G$ for all i , and
- ii) if $\{x_i, y_j\}$ and $\{x_j, y_k\}$ are in E_G and i, j, k are distinct, then $\{x_i, y_k\} \in E_G$.

In the current paper, among other results, we provide a condition for identifying all the unmixed complete t -partite graphs.

2. MAIN RESULTS

A graph G is t -partite if its vertex set can be partitioned into disjoint independent subsets V_1, \dots, V_t . Moreover, in this paper, we consider t as the smallest number that has this property. The graph G is called *complete t -partite graph* if its vertex set can be partitioned into disjoint independent subsets V_1, \dots, V_t such that for all u and v in different partition sets, $uv \in E_G$. A

k -coloring of a graph G is a labeling $f : V(G) \rightarrow S$ where $|S| = k$. The labels are considered as colors and the set of vertices of one given color form a color class. A k -coloring is said to be proper if adjacent vertices have different labels. A graph is k -colorable if it has a proper k -coloring. The chromatic number of graph G , $\chi(G)$, is the least k such that G is k -colorable [11].

The following proposition gives a lower bound for cardinality of the set of all maximal independent sets in G , $|MIS(G)|$, in terms of chromatic number.

Proposition 2.1. *Let G be a graph. If $\chi(G) = t$, then $|MIS(G)| \geq t$.*

Proof. Since $\chi(G) = t$, there exist t color classes for G . Hence, the graph G can be considered as a t -partite graph. Suppose that V_i is the set of elements of i -th color class. Then V_i is an independent set of G and there exists a maximal independent set F_i in G such that $V_i \subseteq F_i$ for all $1 \leq i \leq t$. Thus, there exists at least t maximal independent set for G . \square

Remark 2.2. Using the definition of a complete t -partite graph, it follows that $|MIS(G)| = t$ for any complete t -partite graph G .

Definition 2.3. A simplicial complex Δ is called *shellable* if the facets (maximal faces) of Δ can be ordered F_1, \dots, F_s such that for all $1 \leq i < j \leq s$, there exist some $v \in F_j \setminus F_i$ and some $l \in \{1, \dots, j-1\}$ with $F_j \setminus F_l = \{v\}$. We call F_1, \dots, F_s to be a *shelling* of a shellable complex, Δ , when the facets are ordered as in the definition.

Now by the next theorem, all shellable complete t -partite graphs can be classified.

Theorem 2.4. *Let G be a complete t -partite graph. G is shellable if and only if G is t -colorable such that exactly one of color classes has arbitrary elements and other classes have only one element.*

Proof. \Leftarrow) Assume that $V_G = \{x_1, \dots, x_n\}$ is the set of vertices. To prove that G is shellable, we have to find a shelling F_1, \dots, F_t for Δ_G . Since proper t -vertex coloring gives a partition of V_G into t color classes, we suppose that the set of elements of i -th color class is V_i . By assumption, $V_1 = \{x_1, \dots, x_m\}$ where $m = n - t + 1$ and $V_i = \{x_{m+i-1}\}$ for all $2 \leq i \leq t$. We know that each V_i is an independent set. Now, if $x_{m+i-1} \in V_1$ for $2 \leq i \leq t$, then we can replace V_1 by $V_1 \cup V_i$ and obtain $(t-1)$ -partition for G that is a contradiction. Therefore, V_1 is a maximal independent set of G and hence a facet of Δ_G . By the same argument, each V_i is a maximal independent set. We put $F_i = V_i$. Thus, we find an ordering on the facets of Δ_G as follows:

$$F_1 = \{x_1, \dots, x_m\}, F_2 = \{x_{m+1}\}, \dots, F_t = \{x_{m+t-1}\}.$$

Since $F_i \setminus F_1 = \{x_{m+i-1}\}$ for all $2 \leq i \leq t$, F_1, \dots, F_t is a shelling of Δ_G .

\Rightarrow) Suppose that V_1, \dots, V_t is a partition of V_G . According to Remark 2.2,

we have $|MIS(G)| = t$, so Δ_G has exactly t facets. On the other hand, G is shellable and we can consider F_1, \dots, F_t as a shelling of Δ_G . Since V_i 's are maximal independent sets, we obtain $\{V_1, \dots, V_t\} = \{F_1, \dots, F_t\}$. Without loss of generality, put $F_i = V_i$ for all $1 \leq i \leq t$. There exists $x_2 \in F_2 \setminus F_1$ such that $F_2 \setminus F_1 = \{x_2\}$ because F_1, \dots, F_t is a shelling. Thus $F_2 = (F_2 \setminus F_1) \cup (F_2 \cap F_1) = \{x_2\}$.

Now, suppose by induction that $F_1 = \{x_1, \dots, x_m\}$, $F_2 = \{x_{m+1}\}$, \dots , $F_i = \{x_{m+i-1}\}$. Since F_1, \dots, F_t is a shelling of Δ_G , there exists $x_{i+1} \in F_{i+1} \setminus F_1$ and $l \in \{1, \dots, i\}$ such that $F_{i+1} \setminus F_l = \{x_{i+1}\}$, then $F_{i+1} = (F_{i+1} \setminus F_l) \cup (F_{i+1} \cap F_l) = \{x_{i+1}\}$. Hence, one of the color classes has arbitrary elements and the other classes have exactly one element. \square

Definition 2.5. A simplicial complex Δ is recursively defined to be *vertex decomposable* if it is either a simplex, or else has some vertex v so that

- i) both $\Delta \setminus v$ and $link_{\Delta}^v$ are vertex decomposable and
- ii) no face of $link_{\Delta}^v$ is a facet of $\Delta \setminus v$, where

$$link_{\Delta}^F = \{G : G \cap F = \emptyset, G \cup F \in \Delta\}.$$

A graph G is called vertex decomposable if the simplicial complex Δ_G is vertex decomposable.

The following theorem is one of the main results of this paper which characterizes all vertex decomposable complete t -partite graphs.

Theorem 2.6. *Let G be a complete t -partite graph. Then, G is vertex decomposable if and only if G is t -colorable such that exactly one of the color classes has arbitrary elements and the other classes have only one element.*

Proof. \Rightarrow) Assume that for any proper t -vertex coloring of G , there exists at least two classes with at least two elements. By Theorem 2.4, G is not shellable and hence is not vertex decomposable, by ([17], Corollary 7).

\Leftarrow) By assumption, it follows that G is a chordal graph, so G is vertex decomposable by ([17], Corollary 7). \square

Definition 2.7. A subset $C \subset V_G$ is a *minimal vertex cover* of G if:

- i) every edge of G is incident with one vertex in C , and
- ii) there is no proper subset of C with the first property.

If C satisfies only condition (i), it is called a vertex cover of G . A graph G is said to be *unmixed* if all the minimal vertex covers of G have the same number of elements.

By the next theorem, we present a combinatorial characterization of all the unmixed complete t -partite graphs.

Theorem 2.8. *Let G be a complete t -partite graph. G is unmixed if and only if G is t -colorable such that all color classes have the same number of elements.*

Proof. Since any minimal vertex cover of a complete t -partite graph G contains all the elements of $(t - 1)$ classes, then each selected $(t - 1)$ of color classes has the same number of elements if and only if all the classes have the same cardinality. \square

Definition 2.9. ([15], Definition 3.3.8) A pure d -dimensional complex Δ is called *strongly connected* if each pair of facets F, G can be connected by a sequence of facets $F = F_0, F_1, \dots, F_s = G$ such that $\dim(F_i \cap F_{i-1}) = d - 1$ for $1 \leq i \leq s$.

Lemma 2.10. ([2], Proposition 11.7) *Every Cohen-Macaulay complex is strongly connected.*

Lemma 2.11. ([15], Corollary 3.3.7) *A Cohen-Macaulay simplicial complex Δ is pure, that is, all its maximal faces have the same dimension.*

The following theorem is an effective combinatorial criterion for Cohen-Macaulayness of the complete t -partite graphs.

Theorem 2.12. *Let G be a complete t -partite graph. G is Cohen-Macaulay graph if and only if G is t -colorable such that all color classes have exactly one element.*

Proof. \Leftarrow) By assumption, we have $G = K_t$, so G is a chordal graph. By [7], G is Cohen-Macaulay if and only if G is unmixed. Thus, the assertion follows from Theorem 2.8.

\Rightarrow) Suppose that for any proper t -vertex coloring of G , there exists at least two classes with at least two elements. Since G is Cohen-Macaulay, Δ_G is pure, by Lemma 2.11. Therefore, all the color classes have the same number of elements. Assume that their size is $d + 1$, hence all facets of Δ_G are of dimension d and then $\dim(\Delta_G) = d$. We will show that it is not possible that $d + 1 \geq 2$. For any two facets F and E , ($F \neq E$), we have $F \cap E = \emptyset$, then $\dim(F \cap E) = -1 \neq d - 1$, because of $d + 1 \geq 2$. Therefore, Δ_G is not strongly connected, a contradiction to Lemma 2.10. It follows that all the color classes have exactly one element. \square

Now, we give a special condition for complete t -partite graphs under which shellability is equal to Cohen-Macaulayness.

Corollary 2.13. *Let G be a complete t -partite graph. The property of being Cohen-Macaulay for G is equivalent to being shellable if and only if G is t -colorable such that all color classes have exactly one element.*

Definition 2.14. Let k be a field and $R = k[x_1, \dots, x_n]$ be the polynomial ring over k . A graded R -module M is called *sequentially Cohen-Macaulay* (over k) if there exists finite filtration of graded R -modules

$$0 = M_0 \subset M_1 \subset M_2 \subset \dots \subset M_r = M$$

such that each M_i/M_{i-1} is Cohen-Macaulay, and the Krull dimensions of the quotients are increasing:

$$\dim(M_1/M_0) < \dim(M_2/M_1) < \dots < \dim(M_r/M_{r-1}).$$

We call a graph G sequentially Cohen-Macaulay over the field k if $R/I(G)$ is sequentially Cohen-Macaulay.

Suppose that I is a homogeneous ideal of R . The ideal generated by all homogeneous degree d elements of I is denoted by (I_d) . The concept of componentwise linear was introduced by Herzog and Hibi [5]. A homogeneous ideal I is componentwise linear if (I_d) has a linear resolution for all d . Let I be a square-free monomial ideal of R and $I_{[d]}$ be the ideal generated by the square-free monomials of degree d of I . Herzog and Hibi ([5], Proposition 1.5) have shown that the square-free monomial ideal I is componentwise linear if and only if $I_{[d]}$ has a linear resolution for all d . In [5], it is also shown that:

Theorem 2.15. *Let I be a square-free monomial ideal in a polynomial ring. Then I^\vee is componentwise linear if and only if R/I is sequentially Cohen-Macaulay.*

In [12], Stanley showed that shellability implies the sequentially Cohen-Macaulayness.

Theorem 2.16. *Let Δ be a simplicial complex, and suppose that R/I_Δ is the associated Stanley Reisner ring. If Δ is shellable, then R/I_Δ is sequentially Cohen-Macaulay.*

In the following theorem, we show that being sequentially Cohen-Macaulay of the complete t -partite graph is really a combinatorial property.

Theorem 2.17. *Let G be a complete t -partite graph. G is sequentially Cohen-Macaulay if and only if G is t -colorable such that exactly one of color classes has arbitrary elements and other classes have only one element.*

Proof. \Leftarrow) It follows from Theorems 2.4 and 2.16.

\Rightarrow) Assume that $V_G = \{x_1, \dots, x_n\}$ and V_1, \dots, V_t is a partition of V_G where V_i is the set of elements in i -th color class. We proceed by contradiction. One may consider the following cases:

Case(1) : Suppose that there exist at least two parts V_i and V_j with $|V_i| \geq 2$ and $|V_j| \geq 3$ and r is the maximum cardinality of parts of G which is at least 3. Let $J = I(G)_{[d]}^\vee$ where $d = n - r + 1$. Using Theorem 2.15, to show that G is not sequentially Cohen-Macaulay, it suffices to prove that J does not have a linear resolution.

We use simplicial homology to compute the Betti numbers of J . A square-free vector is a vector that its entries are in $\{0, 1\}$. For a monomial ideal I and

a degree $b \in \mathbb{N}^n$, define

$$K^b(I) = \{\text{square free vectors } c \in \{0, 1\}^n \text{ such that } \frac{x^b}{x^c} \in I\}$$

to be the upper Koszul simplicial complex of I in degree b ([9], Definition 1.33). For a vector $b \in \mathbb{N}^n$, the Betti numbers of I in degree b can be expressed as $\beta_{i,b}(I) = \dim_k H_{i-1}^\sim(K^b(I), k)$ ([9], Theorem 1.34). The sum of $\beta_{i,b}(I)$ over all square-free vectors b of degree j is equal to $\beta_{i,j}(I)$.

To prove that J does not have a linear resolution, we will show that $\beta_{1,n}(J) \neq 0$. We associate to the monomial $m = x_1 \dots x_n$ a unique square-free vector $b = (1, \dots, 1)$. We have a chain complex

$$\dots \rightarrow C_2(K^b(J)) \xrightarrow{\partial_2} C_1(K^b(J)) \xrightarrow{\partial_1} C_0(K^b(J)) \xrightarrow{\partial_0} C_{-1}(K^b(J)) \rightarrow 0.$$

The s -dimensional faces $[x_{i_0}, \dots, x_{i_s}]$ of $K^b(J)$ are the basis of $C_s(K^b(J))$ and

$$\partial_s([x_{i_0}, \dots, x_{i_s}]) = \sum_{t=0}^s (-1)^t [x_{i_0}, \dots, \hat{x}_{i_t}, \dots, x_{i_s}].$$

The above chain complex is introduced in ([4], Proposition 4.1). To obtain $\beta_{1,n}(J)$, we need to compute $\dim_k H_0^\sim(K^b(M), k) = \dim_k (\ker \partial_0 / \text{im } \partial_1)$. If we can find an element in $\ker \partial_0$ that is not in $\text{im } \partial_1$, we have shown that $\beta_{1,n}(J) > 0$. We suppose that x_1, x_2, x_3 belong to the part with maximum cardinality. Put $f = [x_1]$. Then, $\partial_0(f) = 0$ and hence f is in the \ker of ∂_0 . We claim that f is not in the image of ∂_1 . To prove, assume that $\partial_1([x_l, x_s]) = [x_1]$. Then, $[x_s] - [x_l] = [x_1]$, hence $[x_s] - [x_l] - [x_1] = 0$ that is a contradiction because $[x_l], [x_s], [x_1]$ are linear independent.

Case(2) : Assume that for any $1 \leq i \leq t$, we have $|V_i| = 2$. The proof of this case is similar to that of case 1. It suffices to consider $J = I(G)_{[d]}^\vee$ where $d = n - 2$.

Case(3) : Suppose that there exist at least two parts V_i and V_j with $|V_i| = |V_j| = 2$ and at least one part V_r with $|V_r| = 1$. One can apply the same argument to case 2. \square

Corollary 2.18. *Let G be a complete t -partite graph. The followings are equivalent:*

- (1) G is shellable.
- (2) G is vertex decomposable.
- (3) G is sequentially Cohen-Macaulay.

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