

## Generalized Approximate Amenability of Direct Sum of Banach Algebras

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ABSTRACT. In the present paper for two  $\mathfrak{A}$ -module Banach algebras  $A$  and  $B$ , we investigate relations between  $\varphi$ - $\mathfrak{A}$ -module approximate amenability of  $A$ ,  $\psi$ - $\mathfrak{A}$ -module approximate amenability of  $B$ , and  $\varphi \oplus \psi$ - $\mathfrak{A}$ -module approximate amenability of  $A \oplus B$  ( $l^1$ -direct sum of  $A$  and  $B$ ), where  $\varphi \in \text{Hom}_{\mathfrak{A}}(A)$  and  $\psi \in \text{Hom}_{\mathfrak{A}}(B)$ .

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### 1. INTRODUCTION

The notion of approximate amenable Banach algebras was introduced and extensively studied by Ghahramani and Loy in [5]. They showed in [6] that if  $A$  and  $B$  are approximately amenable Banach algebras and one of  $A$  or  $B$  has a bounded approximate identity, then  $A \oplus B$  is approximately amenable, but in general the direct sum of two approximately amenable Banach algebras need not be approximately amenable (see [7]).

The concept of module amenable Banach algebras was introduced by Amini in [1], and the notion of module approximate amenable Banach algebras was studied by Pourmahmood and Bodaghi in [15]. Recently, some authors have

studied  $\varphi$ -derivations, and  $\varphi$ -amenability of Banach algebra  $A$ , whenever  $\varphi$  is a continuous homomorphism on  $A$  (see [8, 9, 10, 11, 12]).

The aim of the present paper is to investigate generalized approximate amenability of  $A \oplus B$ .

The organization of this paper is as follows:

Section 2 is devoted to the notations and definitions which are needed throughout the paper.

In section 3 for  $\mathfrak{A}$ -module Banach algebras  $A$  and  $B$  where each has a bounded approximate identity we show that  $A$  is  $\varphi$ - $\mathfrak{A}^\#$ -module approximately amenable and  $B$  is  $\psi$ - $\mathfrak{A}^\#$ -module approximately amenable if and only if  $A \oplus B$  is  $\varphi \oplus \psi$ - $\mathfrak{A}^\#$ -module approximately amenable.

In section 4 we show that if  $\mathfrak{A}$  has a bounded approximate identity and  $\frac{A}{J_{A, \mathfrak{A}}}$  and  $\frac{B}{J_{B, \mathfrak{A}}}$  are unital, then  $A$  is  $\varphi$ - $\mathfrak{A}$ -module approximately amenable and  $B$  is  $\psi$ - $\mathfrak{A}$ -module approximately amenable if and only if  $A \oplus B$  is  $\varphi \oplus \psi$ - $\mathfrak{A}$ -module approximately amenable.

## 2. PRELIMINARIES

Let  $\mathfrak{A}$  and  $A$  be Banach algebras such that  $A$  is a Banach  $\mathfrak{A}$ -bimodule with compatible actions given by

$$\alpha.(ab) = (\alpha.a)b, \quad (ab).\alpha = a(b.\alpha) \quad (a, b \in A, \alpha \in \mathfrak{A}).$$

Let  $X$  be a Banach  $A$ -bimodule and a Banach  $\mathfrak{A}$ -bimodule with compatible left actions defined by

$$\alpha.(a.x) = (\alpha.a).x, \quad a.(\alpha.x) = (a.\alpha).x, \quad (\alpha.x).a = \alpha.(x.a)$$

$$(a \in A, \alpha \in \mathfrak{A}, x \in X), \quad (2.1)$$

and similar for the right or two-sided actions. Then we say that  $X$  is a Banach  $A$ - $\mathfrak{A}$ -module. A Banach  $A$ - $\mathfrak{A}$ -module  $X$  is called commutative  $A$ - $\mathfrak{A}$ -module, if  $\alpha.x = x.\alpha$  ( $\alpha \in \mathfrak{A}, x \in X$ ). Note that in general,  $A$  does not satisfy the compatibility condition  $a.(\alpha.b) = (a.\alpha).b$  ( $a, b \in A, \alpha \in \mathfrak{A}$ ).

If  $X$  is a commutative Banach  $A$ - $\mathfrak{A}$ -module, then so is  $X^*$ , where the actions of  $A$  and  $\mathfrak{A}$  on  $X^*$  are defined as follows

$$\langle \alpha.f, x \rangle = \langle f, x.\alpha \rangle, \quad \langle a.f, x \rangle = \langle f, x.a \rangle \quad (a \in A, \alpha \in \mathfrak{A}, x \in X, f \in X^*),$$

and similar for the right actions.

Let  $A$  and  $B$  be Banach  $\mathfrak{A}$ -bimodules. Then a  $\mathfrak{A}$ -module morphism from  $A$  to  $B$  is a norm continuous map  $h : A \rightarrow B$  with  $h(a \pm b) = h(a) \pm h(b)$  which is multiplicative, that is

$$h(\alpha.a) = \alpha.h(a), \quad h(a.\alpha) = h(a).\alpha, \quad h(ab) = h(a)h(b) \quad (a \in A, b \in B, \alpha \in \mathfrak{A}).$$

We denote by  $\text{Hom}_{\mathfrak{A}}(A, B)$ , the space of all such morphism and denote  $\text{Hom}_{\mathfrak{A}}(A, A)$  by  $\text{Hom}_{\mathfrak{A}}(A)$ . In the case that  $\mathfrak{A} = \mathbb{C}$ , we denote  $\text{Hom}_{\mathbb{C}}(A, B)$  by  $\text{Hom}(A, B)$  and denote  $\text{Hom}_{\mathbb{C}}(A, A)$  by  $\text{Hom}(A)$ .

Let  $X$  be a Banach  $A$ -bimodule and let  $\varphi \in \text{Hom}_{\mathfrak{A}}(A)$ . A bounded map  $D : A \rightarrow X$  is called a  $\varphi$ - $\mathfrak{A}$ -module derivation if

$$D(a \pm b) = D(a) \pm D(b), \quad D(ab) = D(a).\varphi(b) + \varphi(a).D(b) \quad (a, b \in A), \quad (2.2)$$

and

$$D(\alpha.a) = \alpha.D(a), \quad D(a.\alpha) = D(a).\alpha \quad (a \in A, \alpha \in \mathfrak{A}). \quad (2.3)$$

Although  $D$  in general is not linear, but still its boundedness implies its norm continuity.

Let  $X$  be a commutative Banach  $A$ - $\mathfrak{A}$ -module. For every  $x \in X$  define  $ad_x^\varphi$  by  $ad_x^\varphi(a) = \varphi(a).x - x.\varphi(a)$  ( $a \in A$ ). It is easily seen that  $ad_x^\varphi$  is a  $\varphi$ - $\mathfrak{A}$ -module derivation. A  $\varphi$ - $\mathfrak{A}$ -module derivation  $D$  is called  $\varphi$ -inner if there is  $x \in X$  such that  $D(a) = ad_x^\varphi(a)$  ( $a \in A$ ) and is called approximately  $\varphi$ -inner if there exists a net  $(x_\alpha)_\alpha \subseteq X$  such that  $D(a) = \lim_\alpha ad_{x_\alpha}^\varphi(a)$  ( $a \in A$ ). A Banach algebra  $A$  is called  $\varphi$ - $\mathfrak{A}$ -module amenable if for any commutative Banach  $A$ - $\mathfrak{A}$ -module  $X$ , each  $\varphi$ - $\mathfrak{A}$ -module derivation  $D : A \rightarrow X^*$  is  $\varphi$ -inner, and  $A$  is called  $\varphi$ - $\mathfrak{A}$ -module approximately amenable if each  $\varphi$ - $\mathfrak{A}$ -module derivation  $D : A \rightarrow X^*$  is approximately  $\varphi$ -inner (see [1, 15]).

In the case that  $\mathfrak{A} = \mathbb{C}$ ,  $\varphi$ - $\mathfrak{A}$ -module derivations (resp.  $\varphi$ - $\mathfrak{A}$ -module amenable Banach algebras,  $\varphi$ - $\mathfrak{A}$ -module approximately amenable Banach algebras) are called  $\varphi$ -derivation (resp.  $\varphi$ -amenable,  $\varphi$ -approximately amenable) (see [9, 10]).

### 3. $\varphi \oplus \psi$ -MODULE APPROXIMATE AMENABILITY OF THE DIRECT SUM OF BANACH ALGEBRAS

We commence this section with the following remark from [1]:

*Remark 3.1.* Assume that  $A$  has a bounded approximate identity  $(e_\alpha)_\alpha$ , and let  $M_{\mathfrak{A}}(A)$  denotes the algebra of  $\mathfrak{A}$ -multipliers of  $A$ , that is  $M_{\mathfrak{A}}(A) = \{(T_1, T_2) : T_1, T_2 \in L_{\mathfrak{A}}(A) : T_1(ab) = T_1(a)b, T_2(ab) = aT_2(b) (a, b \in A)\}$ , where  $L_{\mathfrak{A}}(A)$  is the space of all  $\mathfrak{A}$ -module morphisms on  $A$ . Then  $M_{\mathfrak{A}}(A)$  is an  $A$ - $\mathfrak{A}$ -module and  $A$  embeds in  $M_{\mathfrak{A}}(A)$  via  $a \mapsto (L_a, R_a)$ , where  $L_a(b) = ab, R_a(b) = ba$  ( $a, b \in A$ ). For any element  $T = (T_1, T_2)$  of  $M_{\mathfrak{A}}(A)$  it is easy to see that  $\|T_1\| = \|T_2\|$  and if we put  $\|T\|$  equal to this common value, then  $M_{\mathfrak{A}}(A)$  becomes a Banach  $A$ - $\mathfrak{A}$ -module, and  $A$  is dense in  $M_{\mathfrak{A}}(A)$  in the strict topology.

Before proving our next proposition we note that if  $\varphi \in \text{Hom}_{\mathfrak{A}}(A)$ , then by continuity of  $\varphi$  in the strict topology, it can be extended to an  $\mathfrak{A}$ -homomorphism  $\tilde{\varphi} : M_{\mathfrak{A}}(A) \rightarrow M_{\mathfrak{A}}(A)$  defined by  $\tilde{\varphi}(L_a, R_a) = (L_{\varphi(a)}, R_{\varphi(a)})$ .

**Proposition 3.2.** *Let  $A$  be an  $\mathfrak{A}$ -module Banach algebra with a bounded approximate identity  $(e_\alpha)_\alpha$ , and let  $\varphi \in \text{Hom}_{\mathfrak{A}}(A)$ . Then  $A$  is  $\varphi$ - $\mathfrak{A}$ -module approximately amenable if and only if  $M_{\mathfrak{A}}(A)$  is  $\tilde{\varphi}$ - $\mathfrak{A}$ -module approximately amenable.*

*Proof.* Let  $M_{\mathfrak{A}}(A)$  be  $\tilde{\varphi}$ - $\mathfrak{A}$ -module approximately amenable and let  $D : A \rightarrow X^*$  be a  $\varphi$ - $\mathfrak{A}$ -module derivation for some commutative Banach  $A$ - $\mathfrak{A}$ -module  $X$ . Then by the following actions

$$T.x = \lim_{\alpha} T_1(e_\alpha).x, \quad x.T = \lim_{\alpha} x.T_2(e_\alpha) \quad (x \in X, T = (T_1, T_2) \in M_{\mathfrak{A}}(A)),$$

$X$  is a commutative Banach  $M_{\mathfrak{A}}(A)$ - $\mathfrak{A}$ -module and by continuity of  $D$  in the strict topology, it can be extended to a bounded  $\tilde{\varphi}$ - $\mathfrak{A}$ -derivation  $\tilde{D} : M_{\mathfrak{A}}(A) \rightarrow X^*$ , defined by  $\tilde{D}(L_a, R_a) = D(a)$ . From the  $\tilde{\varphi}$ - $\mathfrak{A}$ -module approximate amenability of  $M_{\mathfrak{A}}(A)$ , it follows that there exists a net  $(x_\beta^*)_\beta \subset X^*$  such that

$$\tilde{D}(T) = \lim_{\beta} (\tilde{\varphi}(T).x_\beta^* - x_\beta^*.\tilde{\varphi}(T)).$$

Hence for every  $a \in A$  we have

$$\begin{aligned} D(a) &= \tilde{D}(L_a, R_a) = \lim_{\beta} (\tilde{\varphi}(L_a, R_a).x_\beta^* - x_\beta^*.\tilde{\varphi}(L_a, R_a)) \\ &= \lim_{\beta} ((L_{\varphi(a)}, R_{\varphi(a)}).x_\beta^* - x_\beta^*.(L_{\varphi(a)}, R_{\varphi(a)})) \\ &= \lim_{\beta} (\lim_{\alpha} L_{\varphi(a)}(e_\alpha).x_\beta^* - \lim_{\alpha} x_\beta^*.R_{\varphi(a)}(e_\alpha)) \\ &= \lim_{\beta} (\varphi(a).x_\beta^* - x_\beta^*.\varphi(a)). \end{aligned}$$

This means that  $D$  is approximately  $\varphi$ -inner and so  $A$  is  $\varphi$ - $\mathfrak{A}$ -module approximately amenable.

Conversely, Suppose that  $A$  is  $\varphi$ - $\mathfrak{A}$ -module approximately amenable. Let  $X$  be a commutative Banach  $M_{\mathfrak{A}}(A)$ - $\mathfrak{A}$ -module and let  $D : M_{\mathfrak{A}}(A) \rightarrow X^*$  be a  $\tilde{\varphi}$ - $\mathfrak{A}$ -module derivation. We consider the module actions of  $A$  on  $X$  by

$$a.x = (L_a, R_a).x, \quad x.a = x.(L_a, R_a) \quad (a \in A, x \in X). \quad (3.1)$$

Thus  $X$  is a commutative Banach  $A$ - $\mathfrak{A}$ -module. Define  $\tilde{D} : A \rightarrow X^*$  by  $\tilde{D}(a) = D(L_a, R_a)$  ( $a \in A$ ). It is easy to see that  $\tilde{D}$  is a  $\varphi$ - $\mathfrak{A}$ -module derivation and from the  $\varphi$ - $\mathfrak{A}$ -module approximate amenability of  $A$ , it follows that there exists a net  $(x_\beta^*)_\beta \subset X^*$  such that

$$\tilde{D}(a) = \lim_{\beta} (\varphi(a).x_\beta^* - x_\beta^*.\varphi(a)) \quad (a \in A).$$

Then  $D(L_a, R_a) = \lim_{\beta} (\tilde{\varphi}(L_a, R_a).x_\beta^* - x_\beta^*.\tilde{\varphi}(L_a, R_a))$ . Now by the continuity of  $D$  and  $\tilde{\varphi}$ , and density of  $A$  in  $M_{\mathfrak{A}}(A)$  in the strict topology, we conclude that

$$D(T) = \lim_{\beta} (\tilde{\varphi}(T).x_\beta^* - x_\beta^*.\tilde{\varphi}(T)) \quad (T \in M_{\mathfrak{A}}(A)).$$

So  $D$  is a approximately  $\tilde{\varphi}$ -inner. Therefore  $M_{\mathfrak{A}}(A)$  is  $\tilde{\varphi}$ - $\mathfrak{A}$ -module approximately amenable.  $\square$

Let  $I$  be a closed ideal of a Banach algebra  $A$  with a bounded approximate identity  $(e_\alpha)_\alpha$ , and let  $X$  be a commutative Banach  $I$ - $\mathfrak{A}$ -module. Let  $\varphi \in \text{Hom}_{\mathfrak{A}}(A)$  be such that  $\varphi|_I \subset I$ , then  $X$  is a commutative Banach  $A$ - $\mathfrak{A}$ -module with the following actions

$$a.x = \lim_{\alpha} \varphi(e_\alpha)a.x, \quad x.a = \lim_{\alpha} x.\varphi(e_\alpha)a \quad (a \in A, x \in X). \quad (3.2)$$

**Proposition 3.3.** *Let  $I$  be a closed ideal of an  $\mathfrak{A}$ -module Banach algebra  $A$  which has a bounded approximate identity  $\{e_\alpha\}$ , and let  $I$  be  $\mathfrak{A}$ -invariant, i.e.  $\mathfrak{A}.I \subseteq I$ . Let  $\varphi \in \text{Hom}_{\mathfrak{A}}(A)$  be such that  $\varphi|_I \subset I$ . If  $A$  is  $\varphi$ - $\mathfrak{A}$ -module approximately amenable, then  $I$  is  $\varphi|_I$ - $\mathfrak{A}$ -module approximately amenable.*

*Proof.* Let  $X$  be a commutative Banach  $M_{\mathfrak{A}}(I)$ - $\mathfrak{A}$ -module, and  $D : M_{\mathfrak{A}}(I) \rightarrow X^*$  be a  $\tilde{\varphi}$ - $\mathfrak{A}$ -module derivation. By the same actions as (3.1), we can consider  $X$  as a commutative Banach  $I$ - $\mathfrak{A}$ -module. So, by (3.2),  $X$  is a commutative Banach  $A$ - $\mathfrak{A}$ -module. By definition of  $M_{\mathfrak{A}}(I)$ , there is an  $\mathfrak{A}$ -module morphism  $h : A \rightarrow M_{\mathfrak{A}}(I)$  and  $D \circ h$  is a module derivation on  $A$ , so it is approximately  $\varphi$ -inner. Hence  $D$  is approximately  $\tilde{\varphi}$ -inner. Since  $I$  has a bounded approximate identity, by Proposition 3.2,  $I$  is  $\varphi|_I$ - $\mathfrak{A}$ -module approximately amenable.  $\square$

Let  $A$  and  $B$  be  $\mathfrak{A}$ -module Banach algebras. It is well known that  $A \oplus B$ , the  $l^1$ -direct sum of  $A$  and  $B$ , is a Banach algebra with respect to the canonical multiplication defined by  $(a, b)(c, d) := (ac, bd)$ , and is a Banach  $\mathfrak{A}$ -bimodule by the following actions

$$\alpha.(a, b) := (\alpha.a, \alpha.b), \quad (a, b).\alpha := (a.\alpha, b.\alpha) \quad (\alpha \in \mathfrak{A}, a \in A, b \in B).$$

We note that if  $\varphi \in \text{Hom}_{\mathfrak{A}}(A)$  and  $\psi \in \text{Hom}_{\mathfrak{A}}(B)$ , then  $\varphi \oplus \psi : A \oplus B \rightarrow A \oplus B$  defined by  $\varphi \oplus \psi(a, b) = (\varphi(a), \psi(b))$  is an  $\mathfrak{A}$ -morphism on  $A \oplus B$ .

**Lemma 3.4.** *Let  $A$  be a unital  $\mathfrak{A}$ -module Banach algebra,  $\varphi \in \text{Hom}_{\mathfrak{A}}(A)$ , and let  $D : A \rightarrow X^*$  be a  $\varphi$ - $\mathfrak{A}$ -module derivation for some commutative Banach  $A$ - $\mathfrak{A}$ -module  $X$ . If the left (resp. right, two-sided) action of  $\varphi(A)$  on  $X^*$  is zero, then  $D$  is  $\varphi$ -inner.*

*Proof.* Let  $e_A$  be the identity of  $A$  and let the left (resp. right, two-sided) action of  $\varphi(A)$  on  $X^*$  is zero. We can easily show that  $D = ad_{-D(e)}^{\varphi}$  (resp.  $D = ad_{D(e)}^{\varphi}, D = 0$ ). So  $D$  is  $\varphi$ -inner.  $\square$

The proof of the following proposition is adopted from that of Proposition 2.7 of [5].

**Proposition 3.5.** *Let  $A$  and  $B$  be unital  $\mathfrak{A}$ -module Banach algebras with identities  $e_A$  and  $e_B$ , respectively, and let  $\varphi \in \text{Hom}_{\mathfrak{A}}(A)$  and  $\psi \in \text{Hom}_{\mathfrak{A}}(B)$  such that  $\varphi(e_A).\alpha = \alpha.\varphi(e_A)$ , and  $\psi(e_B).\alpha = \alpha.\psi(e_B)$  ( $\alpha \in \mathfrak{A}$ ). If  $A$  is  $\varphi$ - $\mathfrak{A}$ -module approximately amenable and  $B$  is  $\psi$ - $\mathfrak{A}$ -module approximately amenable, then  $A \oplus B$  is  $\varphi \oplus \psi$ - $\mathfrak{A}$ -module approximately amenable.*

*Proof.* Let  $X$  be a commutative Banach  $A \oplus B$ - $\mathfrak{A}$ -module and let  $D : A \oplus B \rightarrow X^*$  be a  $\varphi \oplus \psi$ - $\mathfrak{A}$ -module derivation. Write  $Y_1 = \varphi(e_A).X^*.\varphi(e_A)$ ,  $Y_2 = \psi(e_B).X^*.\psi(e_B)$ ,  $Y_3 = \varphi(e_A).X^*.\psi(e_B)$ ,  $Y_4 = \psi(e_B).X^*.\varphi(e_A)$ ,  $Y_5 = (1 - \varphi(e_A))(1 - \psi(e_B)).X^*.\varphi(e_A)$ ,  $Y_6 = (1 - \varphi(e_A))(1 - \psi(e_B)).X^*.\psi(e_B)$ ,  $Y_7 = \varphi(e_A).X^*.(1 - \varphi(e_A))(1 - \psi(e_B))$ ,  $Y_8 = \psi(e_B).X^*.(1 - \varphi(e_A))(1 - \psi(e_B))$ ,  $Y_9 = (1 - \varphi(e_A))(1 - \psi(e_B)).X^*.(1 - \varphi(e_A))(1 - \psi(e_B))$  and let  $\pi_j : X^* \rightarrow Y_j$  be the associated projections. Thus  $X^* = Y_1 \oplus Y_2 \oplus Y_3 \oplus Y_4 \oplus Y_5 \oplus Y_6 \oplus Y_7 \oplus Y_8 \oplus Y_9$ . Consider the derivations  $D_j = \pi_j \circ D$ , so  $D = D_1 + D_2 + D_3 + D_4 + D_5 + D_6 + D_7 + D_8 + D_9$ . From the fact that  $\varphi(e_A).\alpha = \alpha.\varphi(e_A)$  ( $\alpha \in \mathfrak{A}$ ), and  $\psi(e_B).\alpha = \alpha.\psi(e_B)$  ( $\alpha \in \mathfrak{A}$ ), one can easily check that  $Y_j$  for  $j = 1, \dots, 9$  is a commutative Banach  $A \oplus B$ - $\mathfrak{A}$ -module. Since the action of  $\varphi(A) \oplus \psi(B)$  on (at least) one side on  $Y_5$  (resp.  $Y_6, Y_7, Y_8, Y_9$ ) is zero, by Lemma 3.4, we conclude that  $D_5$  (resp.  $D_6, D_7, D_8, D_9$ ) is approximately  $\varphi \oplus \psi$ -inner.

From the  $\varphi$ - $\mathfrak{A}$ -module approximate amenability of  $A$ , it follows that the  $\varphi \oplus \psi$ - $\mathfrak{A}$ -module derivation  $A \oplus 0 \rightarrow \varphi(e_A).X^*.\varphi(e_A)$  is approximately  $\varphi \oplus \psi$ -inner and since the action of  $0 \oplus \psi(B)$  on  $\varphi(e_A).X^*.\varphi(e_A)$  is zero, we conclude that  $D_1$  is approximately  $\varphi \oplus \psi$ -inner. Similarly, the  $\varphi \oplus \psi$ - $\mathfrak{A}$ -module derivation  $D_2 : A \oplus B \rightarrow \psi(e_B).X^*.\psi(e_B)$  is approximately  $\varphi \oplus \psi$ -inner.

The right action of  $\varphi(A) \oplus 0$  on  $\varphi(e_A).X^*.\psi(e_B)$  is zero. Hence, by Lemma 3.4,  $D_3|_{A \oplus 0}$  is  $\varphi \oplus \psi$ -inner. So there exists  $\xi \in \varphi(e_A).X^*.\psi(e_B)$  such that

$$D_3|_{A \oplus 0}(a, 0) = \varphi(a).\xi - \xi.\varphi(a) = (\varphi(a), \psi(b))\varphi(e_A).\xi.\psi(e_B),$$

for every  $a \in A$  and  $b \in B$ . Similarly, there exists  $\eta \in \varphi(e_A).X^*.\psi(e_B)$  such that

$$D_3|_{0 \oplus B}(0, b) = \psi(b).\eta - \eta.\psi(b) = -\varphi(e_A).\eta.\psi(e_B)(\varphi(a), \psi(b)),$$

for every  $a \in A$  and  $b \in B$ . Hence

$$D_3(a, b) = (\varphi(a), \psi(b))\varphi(e_A).\xi.\psi(e_B) - \varphi(e_A).\eta.\psi(e_B)(\varphi(a), \psi(b)).$$

Since  $D_3(e_A, e_B) = 0$ , it follows that

$$0 = D_3(e_A, e_B) = \varphi(e_A).\xi.\psi(e_B) - \varphi(e_A).\eta.\psi(e_B).$$

Then for every  $a \in A$  and  $b \in B$ , we have

$$D_3(a, b) = (\varphi(a), \psi(b))\varphi(e_A).\xi.\psi(e_B) - \varphi(e_A).\xi.\psi(e_B)(\varphi(a), \psi(b)).$$

Thus  $D_3$  is  $\varphi \oplus \psi$ -inner. The same argument holds for the  $\varphi \oplus \psi$ - $\mathfrak{A}$ -module derivation  $D_4 : A \oplus B \rightarrow \psi(e_B).X^*.\varphi(e_A)$ . Therefore  $D$  is approximately  $\varphi \oplus \psi$ -inner, and so  $A \oplus B$  is  $\varphi \oplus \psi$ - $\mathfrak{A}$ -module approximately amenable.  $\square$

**Lemma 3.6.** *Let  $A$  and  $B$  be  $\mathfrak{A}$ -module Banach algebras,  $\varphi \in \text{Hom}_{\mathfrak{A}}(A)$  and  $\psi \in \text{Hom}_{\mathfrak{A}}(B)$ . If there is a  $h$  in  $\text{Hom}_{\mathfrak{A}}(A, B)$  such that  $h \circ \varphi = \psi \circ h$  and the range of  $h$  is a dense subset of  $B$ , then  $\varphi$ - $\mathfrak{A}$ -module approximate amenability of  $A$  implies  $\psi$ - $\mathfrak{A}$ -module approximate amenability of  $B$ .*

*Proof.* Let  $D : B \rightarrow X^*$  be a  $\psi$ - $\mathfrak{A}$ -module derivation for some commutative Banach  $B$ - $\mathfrak{A}$ -module  $X$ . Then by the following actions

$$a \bullet x = h(a).x, \quad x \bullet a = x.h(a) \quad (a \in A, x \in X),$$

$X$  is a commutative Banach  $A$ - $\mathfrak{A}$ -module. Let  $\tilde{D} = D \circ h : A \rightarrow X^*$ . One can easily prove that  $\tilde{D}$  is a  $\varphi$ - $\mathfrak{A}$ -module derivation. From the  $\varphi$ - $\mathfrak{A}$ -module approximate amenability of  $A$ , it follows that there exists a net  $(x_\alpha^*)_\alpha$  in  $X^*$  such that  $\tilde{D}(a) = \lim_\alpha (\varphi(a) \bullet x_\alpha^* - x_\alpha^* \bullet \varphi(a))$  ( $a \in A$ ). Now continuity and density of  $h(A)$  in  $B$ , imply that  $D$  is approximately  $\psi$ -inner. Therefore  $B$  is  $\psi$ - $\mathfrak{A}$ -module approximately amenable.  $\square$

**Proposition 3.7.** *Let  $A$  and  $B$  be  $\mathfrak{A}$ -module Banach algebras,  $\varphi \in \text{Hom}_{\mathfrak{A}}(A)$  and  $\psi \in \text{Hom}_{\mathfrak{A}}(B)$ . If  $A$  is not  $\varphi$ - $\mathfrak{A}$ -module approximately amenable or  $B$  is not  $\psi$ - $\mathfrak{A}$ -module approximately amenable, then  $A \oplus B$  is not  $\varphi \oplus \psi$ - $\mathfrak{A}$ -module approximately amenable.*

*Proof.* Suppose that  $A$  is not  $\varphi$ - $\mathfrak{A}$ -module approximately amenable. The projection map  $\pi : A \oplus B \rightarrow A$  determines an  $\mathfrak{A}$ -module epimorphism of  $A \oplus B$  onto  $A$  such that  $\pi \circ (\varphi \oplus \psi) = \varphi \circ \pi$ . So, if  $A \oplus B$  is  $\varphi \oplus \psi$ - $\mathfrak{A}$ -module approximately amenable, then by Lemma 3.6,  $A$  is  $\varphi$ - $\mathfrak{A}$ -module approximately amenable. This contradicts the fact that  $A$  is not  $\varphi$ - $\mathfrak{A}$ -module approximately amenable. Therefore  $A \oplus B$  is not  $\varphi \oplus \psi$ - $\mathfrak{A}$ -module approximately amenable. Similarly, we can prove the result for  $B$ .  $\square$

Let  $\mathfrak{A}$  be a non-unital Banach algebra. Then  $\mathfrak{A}^\# = \mathfrak{A} \oplus \mathbb{C}$ , the unitization of  $\mathfrak{A}$  is a unital Banach algebra which contains  $\mathfrak{A}$  as a closed ideal. Let  $A$  be a Banach  $\mathfrak{A}$ -bimodule. Then  $A$  is a Banach  $\mathfrak{A}^\#$ -module with the following module actions:

$$(\alpha, \lambda).a = \alpha.a + \lambda a, \quad a.(\alpha, \lambda) = a.\alpha + \lambda a \quad (\lambda \in \mathbb{C}, \alpha \in \mathfrak{A}, a \in A).$$

Let  $A^\# = (A \oplus \mathfrak{A}^\#, \bullet)$ , where the multiplication  $\bullet$  is defined through

$$(a, u) \bullet (b, v) = (ab + a.v + u.b, uv) \quad (a, b \in A, u, v \in \mathfrak{A}^\#).$$

Then with the actions defined by

$$u.(a, v) = (u.a, uv), \quad (a, v).u = (a.u, vu) \quad (a \in A, u, v \in \mathfrak{A}^\#),$$

$A^\#$  is a unital  $\mathfrak{A}^\#$ -module Banach algebra with the identity  $1_{A^\#} = (0, 1_{\mathfrak{A}^\#})$  (see [4]).

Before we turn to our next result we note that if for every  $\varphi \in \text{Hom}_{\mathfrak{A}^\#}(A)$ , one defines  $\varphi^\# : A^\# \rightarrow A^\#$  by  $\varphi^\#(a, u) = (\varphi(a), u)$  ( $(a, u) \in A^\#$ ), then  $\varphi^\# \in \text{Hom}_{\mathfrak{A}^\#}(A^\#)$ .

The following proposition generalizes Proposition 2.7 of [5].

**Theorem 3.8.** *Let  $A$  and  $B$  be  $\mathfrak{A}$ -module Banach algebras and each has a bounded approximate identity. Let  $\varphi \in \text{Hom}_{\mathfrak{A}^\#}(A)$  and  $\psi \in \text{Hom}_{\mathfrak{A}^\#}(B)$ . Then  $A$*

is  $\varphi\text{-}\mathfrak{A}^\#$ -module approximately amenable and  $B$  is  $\psi\text{-}\mathfrak{A}^\#$ -module approximately amenable if and only if  $A \oplus B$  is  $\varphi \oplus \psi\text{-}\mathfrak{A}^\#$ -module approximately amenable.

*Proof.* Suppose that  $A$  is  $\varphi\text{-}\mathfrak{A}^\#$ -module approximately amenable and  $B$  is  $\psi\text{-}\mathfrak{A}^\#$ -module approximately amenable. By Proposition 12 of [13],  $A^\#$  is  $\varphi^\#\text{-}\mathfrak{A}^\#$ -module approximately amenable and  $B^\#$  is  $\psi^\#\text{-}\mathfrak{A}^\#$ -module approximately amenable, so by Proposition 3.5,  $A^\# \oplus B^\#$  is  $\varphi^\# \oplus \psi^\#\text{-}\mathfrak{A}^\#$ -module approximately amenable. Since  $A \oplus B$  is a closed  $\mathfrak{A}^\#$ -invariant ideal in  $A^\# \oplus B^\#$ , the result follows from Proposition 3.3.

For the converse, suppose that  $A \oplus B$  is  $\varphi \oplus \psi\text{-}\mathfrak{A}^\#$ -module approximately amenable. Then by Proposition 3.7,  $A$  is  $\varphi\text{-}\mathfrak{A}^\#$ -module approximately amenable and  $B$  is  $\psi\text{-}\mathfrak{A}^\#$ -module approximately amenable.  $\square$

#### 4. $\varphi \oplus \psi$ -MODULE APPROXIMATE AMENABILITY AND $\varphi \oplus \psi$ -AMENABILITY OF DIRECT SUM OF BANACH ALGEBRAS

We start this section with the following definition:

**Definition 4.1.** We say the Banach algebra  $\mathfrak{A}$  acts trivially on  $A$  from the left (right) if for every  $\alpha \in \mathfrak{A}$  and  $a \in A$ ,  $\alpha.a = f(\alpha)a$  (resp.  $a.\alpha = f(\alpha)a$ ), where  $f$  is a multiplicative linear functional on  $\mathfrak{A}$ .

We assume that  $J_{A,\mathfrak{A}}$  is the closed linear span of

$$\{(a.\alpha)b - a(\alpha.b) \mid \alpha \in \mathfrak{A}, a, b \in A\},$$

in  $A$ . It follows immediately that  $J_{A,\mathfrak{A}}$  is both  $A$ -submodule and  $\mathfrak{A}$ -submodule of  $A$ . So  $\frac{A}{J_{A,\mathfrak{A}}}$  is both Banach  $A$ -module and  $\mathfrak{A}$ -module (see page 346 of [14]).

To prove our next result we need to quote the following lemma from [2].

**Lemma 4.2.** Let  $A$  be a Banach algebra and Banach  $\mathfrak{A}$ -module with compatible actions, and  $J_0$  be a closed ideal of  $A$  such that  $J_{A,\mathfrak{A}} \subseteq J_0$ . If  $\frac{A}{J_0}$  has a left or right identity  $e + J_0$ , then for each  $\alpha \in \mathfrak{A}$  and  $a \in A$  we have  $a.\alpha - \alpha.a \in J_0$ , i.e.,  $\frac{A}{J_0}$  is commutative Banach  $\mathfrak{A}$ -module.

Before we turn to our next result we note that if for every  $\varphi \in \text{Hom}_{\mathfrak{A}}(A)$ , one defines  $\bar{\varphi} : \frac{A}{J_{A,\mathfrak{A}}} \rightarrow \frac{A}{J_{A,\mathfrak{A}}}$  by  $\bar{\varphi}(a + J_{A,\mathfrak{A}}) = \varphi(a) + J_{A,\mathfrak{A}}$ , then  $\bar{\varphi} \in \text{Hom}_{\mathfrak{A}}(\frac{A}{J_{A,\mathfrak{A}}})$ .

**Theorem 4.3.** Let  $A$  and  $B$  be  $\mathfrak{A}$ -module Banach algebras and let  $\varphi \in \text{Hom}_{\mathfrak{A}}(A)$  and  $\psi \in \text{Hom}_{\mathfrak{A}}(B)$ . Then the following statements are valid:

- (i)  $A \oplus B$  is  $\varphi \oplus \psi\text{-}\mathfrak{A}$ -module amenable (resp.  $\varphi \oplus \psi\text{-}\mathfrak{A}$ -module approximately amenable) if and only if  $\frac{A}{J_{A,\mathfrak{A}}} \oplus \frac{B}{J_{B,\mathfrak{A}}}$  is  $\bar{\varphi} \oplus \bar{\psi}\text{-}\mathfrak{A}$ -module amenable (resp.  $\bar{\varphi} \oplus \bar{\psi}\text{-}\mathfrak{A}$ -module approximately amenable).
- (ii) Let  $\mathfrak{A}$  acts on  $A$  and  $B$  trivially from the left by  $f \in \text{Hom}_{\mathbb{C}}(\mathfrak{A})$ . Suppose that  $\frac{A}{J_{A,\mathfrak{A}}}$  and  $\frac{B}{J_{B,\mathfrak{A}}}$  are unital, and  $A \oplus B$  is  $\varphi \oplus \psi\text{-}\mathfrak{A}$ -module amenable (resp.  $\varphi \oplus \psi\text{-}\mathfrak{A}$ -module approximately amenable), then  $\frac{A}{J_{A,\mathfrak{A}}} \oplus \frac{B}{J_{B,\mathfrak{A}}}$  is  $\bar{\varphi} \oplus \bar{\psi}$ -amenable (resp.  $\bar{\varphi} \oplus \bar{\psi}$ -approximately amenable).

(iii) Let  $\mathfrak{A}$  have a bounded approximately identity and  $\frac{A}{J_{A,\mathfrak{A}}} \oplus \frac{B}{J_{B,\mathfrak{A}}}$  is  $\overline{\varphi} \oplus \overline{\psi}$ -amenable (resp.  $\overline{\varphi} \oplus \overline{\psi}$ -approximately amenable). Then  $A \oplus B$  is  $\varphi \oplus \psi$ - $\mathfrak{A}$ -module amenable (resp.  $\varphi \oplus \psi$ - $\mathfrak{A}$ -module approximately amenable).

*Proof.* (i) Let  $A \oplus B$  be  $\varphi \oplus \psi$ - $\mathfrak{A}$ -module amenable, and let  $D : \frac{A}{J_{A,\mathfrak{A}}} \oplus \frac{B}{J_{B,\mathfrak{A}}} \rightarrow X^*$  be  $\overline{\varphi} \oplus \overline{\psi}$ - $\mathfrak{A}$ -module derivation for some commutative Banach  $\frac{A}{J_{A,\mathfrak{A}}} \oplus \frac{B}{J_{B,\mathfrak{A}}}$ - $\mathfrak{A}$ -module  $X$ . Then  $X$  becomes a  $A \oplus B$ -bimodule through the following actions

$$(a, b).x := (a + J_{A,\mathfrak{A}}, b + J_{B,\mathfrak{A}}).x \quad (a \in A, b \in B, x \in X), \tag{4.1}$$

and

$$x.(a, b) := x.(a + J_{A,\mathfrak{A}}, b + J_{B,\mathfrak{A}}) \quad (a \in A, b \in B, x \in X). \tag{4.2}$$

Hence  $X$  is a commutative Banach  $A \oplus B$ - $\mathfrak{A}$ -module. Define  $\tilde{D} : A \oplus B \rightarrow X^*$  by

$$\tilde{D}(a, b) = D(a + J_{A,\mathfrak{A}}, b + J_{B,\mathfrak{A}}) \quad (a \in A, b \in B).$$

It is easy to check that,  $\tilde{D}$  is a  $\varphi \oplus \psi$ - $\mathfrak{A}$ -module derivation. From the  $\varphi \oplus \psi$ - $\mathfrak{A}$ -module amenability of  $A \oplus B$ , it follows that there exists  $x^* \in X^*$  such that

$$\tilde{D}(a, b) = \varphi \oplus \psi(a, b).x^* - x^*.\varphi \oplus \psi(a, b) \quad (a \in A, b \in B).$$

Thus

$$\begin{aligned} D(a + J_{A,\mathfrak{A}}, b + J_{B,\mathfrak{A}}) &= \overline{\varphi} \oplus \overline{\psi}(a + J_{A,\mathfrak{A}}, b + J_{B,\mathfrak{A}}).x^* \\ &\quad - x^*.\overline{\varphi} \oplus \overline{\psi}(a + J_{A,\mathfrak{A}}, b + J_{B,\mathfrak{A}}). \end{aligned}$$

This means that  $D$  is  $\overline{\varphi} \oplus \overline{\psi}$ -inner. Therefore  $\frac{A}{J_{A,\mathfrak{A}}} \oplus \frac{B}{J_{B,\mathfrak{A}}}$  is  $\overline{\varphi} \oplus \overline{\psi}$ - $\mathfrak{A}$ -module amenable.

Conversely, suppose that  $\frac{A}{J_{A,\mathfrak{A}}} \oplus \frac{B}{J_{B,\mathfrak{A}}}$  is  $\overline{\varphi} \oplus \overline{\psi}$ - $\mathfrak{A}$ -module amenable. Let  $D : A \oplus B \rightarrow X^*$  be a  $\varphi \oplus \psi$ - $\mathfrak{A}$ -module derivation for some commutative Banach  $A \oplus B$ - $\mathfrak{A}$ -module  $X$ . We consider the following module actions of  $\frac{A}{J_{A,\mathfrak{A}}} \oplus \frac{B}{J_{B,\mathfrak{A}}}$  on  $X$ ,

$$(a + J_{A,\mathfrak{A}}, b + J_{B,\mathfrak{A}}).x := (a, b).x, \quad x.(a + J_{A,\mathfrak{A}}, b + J_{B,\mathfrak{A}}) := x.(a, b),$$

for all  $a \in A, b \in B$  and  $x \in X$ . Using (2.1) and the commutativity of  $X$ , we have  $J_{A,\mathfrak{A}}X = J_{B,\mathfrak{A}}X = XJ_{A,\mathfrak{A}} = XJ_{B,\mathfrak{A}} = 0$ . Thus  $(J_{A,\mathfrak{A}} \oplus J_{B,\mathfrak{A}})X = X(J_{A,\mathfrak{A}} \oplus J_{B,\mathfrak{A}}) = 0$ . So  $X$  is a commutative Banach  $\frac{A}{J_{A,\mathfrak{A}}} \oplus \frac{B}{J_{B,\mathfrak{A}}}$ - $\mathfrak{A}$ -module. Define  $\tilde{D} : \frac{A}{J_{A,\mathfrak{A}}} \oplus \frac{B}{J_{B,\mathfrak{A}}} \rightarrow X^*$  by

$$\tilde{D}(a + J_{A,\mathfrak{A}}, b + J_{B,\mathfrak{A}}) = D(a, b) \quad (a \in A, b \in B).$$

Also using (2.2) and (2.3) we see that  $D$  vanishes on  $J_{A,\mathfrak{A}} \oplus J_{B,\mathfrak{A}}$ . Hence  $\tilde{D}$  is well defined. One can easily check that  $\tilde{D}$  is a  $\overline{\varphi} \oplus \overline{\psi}$ - $\mathfrak{A}$ -module derivation.

Now from the  $\bar{\varphi} \oplus \bar{\psi}$ - $\mathfrak{A}$ -module amenability of  $\frac{A}{J_{A,\mathfrak{A}}} \oplus \frac{B}{J_{B,\mathfrak{A}}}$ , it follows that there exists  $x^* \in X^*$  such that

$$\begin{aligned} \tilde{D}(a + J_{A,\mathfrak{A}}, b + J_{B,\mathfrak{A}}) &= \bar{\varphi} \oplus \bar{\psi}(a + J_{A,\mathfrak{A}}, b + J_{B,\mathfrak{A}}).x^* \\ &\quad - x^*.\bar{\varphi} \oplus \bar{\psi}(a + J_{A,\mathfrak{A}}, b + J_{B,\mathfrak{A}}) \quad (a \in A, b \in B). \end{aligned}$$

It follows that

$$D(a, b) = \varphi \oplus \psi(a, b).x^* - x^*.\varphi \oplus \psi(a, b) \quad (a \in A, b \in B).$$

Thus  $D$  is  $\varphi \oplus \psi$ -inner. So  $A \oplus B$  is  $\varphi \oplus \psi$ - $\mathfrak{A}$ -module amenable.

Similarly, we can show that  $A \oplus B$  is  $\varphi \oplus \psi$ - $\mathfrak{A}$ -module approximately amenable if and only if  $\frac{A}{J_{A,\mathfrak{A}}} \oplus \frac{B}{J_{B,\mathfrak{A}}}$  is  $\bar{\varphi} \oplus \bar{\psi}$ - $\mathfrak{A}$ -module approximately amenable.

(ii) Let  $A \oplus B$  be  $\varphi \oplus \psi$ - $\mathfrak{A}$ -module amenable and let  $D : \frac{A}{J_{A,\mathfrak{A}}} \oplus \frac{B}{J_{B,\mathfrak{A}}} \rightarrow X^*$  be a derivation for some Banach  $\frac{A}{J_{A,\mathfrak{A}}} \oplus \frac{B}{J_{B,\mathfrak{A}}}$ -bimodule  $X$ . Then  $X$  becomes a  $A \oplus B$ -bimodule through the actions as (4.1) and (4.2). Also  $X$  is an  $\mathfrak{A}$ -bimodule with  $f$ -trivial actions, that is

$$\alpha.x = x.\alpha = f(\alpha)x \quad (\alpha \in \mathfrak{A}, x \in X).$$

Then  $X$  is a commutative Banach  $A \oplus B$ - $\mathfrak{A}$ -module. Define

$$\Gamma : \frac{A \oplus B}{I} \rightarrow \frac{A}{J_{A,\mathfrak{A}}} \oplus \frac{B}{J_{B,\mathfrak{A}}}, \quad (a, b) + I \mapsto (a + J_{A,\mathfrak{A}}, b + J_{B,\mathfrak{A}}),$$

where  $I = J_{A,\mathfrak{A}} \oplus J_{B,\mathfrak{A}}$ . It is routinely checked that  $\Gamma$  defines an  $\mathfrak{A}$ -bimodule morphism. Let  $\Pi : A \oplus B \rightarrow \frac{A \oplus B}{I}$  be the quotient map, and let  $\tilde{D} := D \circ \Gamma \circ \Pi : A \oplus B \rightarrow X^*$ . For every  $(a, b), (a', b') \in A \oplus B$ , we may easily prove that

$$\tilde{D}((a, b)(a', b')) = \tilde{D}(a, b).\varphi \oplus \psi(a', b') + \varphi \oplus \psi(a, b).\tilde{D}(a', b'),$$

and for every  $(a, b) \in A \oplus B$ , and  $\alpha \in \mathfrak{A}$ , we have

$$\begin{aligned} \tilde{D}(\alpha.(a, b)) &= \tilde{D}((\alpha.a, \alpha.b)) = \tilde{D}((f(\alpha)a, f(\alpha)b)) \\ &= D((f(\alpha)a + J_{A,\mathfrak{A}}, f(\alpha)b + J_{B,\mathfrak{A}})) \\ &= D(f(\alpha)(a + J_{A,\mathfrak{A}}, b + J_{B,\mathfrak{A}})) \\ &= f(\alpha)D((a + J_{A,\mathfrak{A}}, b + J_{B,\mathfrak{A}})) \\ &= \alpha.D((a + J_{A,\mathfrak{A}}, b + J_{B,\mathfrak{A}})) \\ &= \alpha.\tilde{D}(a, b), \end{aligned}$$

and using Lemma 4.2, we have

$$\begin{aligned} \tilde{D}((a, b).\alpha) &= \tilde{D}((a.\alpha, b.\alpha)) = D((a.\alpha + J_{A,\mathfrak{A}}, b.\alpha + J_{B,\mathfrak{A}})) \\ &= D((\alpha.a + J_{A,\mathfrak{A}}, \alpha.b + J_{B,\mathfrak{A}})) \\ &= D(f(\alpha)(a + J_{A,\mathfrak{A}}, b + J_{B,\mathfrak{A}})) \\ &= f(\alpha)D((a + J_{A,\mathfrak{A}}, b + J_{B,\mathfrak{A}})) \\ &= D((a + J_{A,\mathfrak{A}}, b + J_{B,\mathfrak{A}}).\alpha) \\ &= \tilde{D}(a, b).\alpha. \end{aligned}$$

Thus  $\tilde{D}$  is a  $\varphi \oplus \psi$ - $\mathfrak{A}$ -module derivation and from the  $\varphi \oplus \psi$ - $\mathfrak{A}$ -module amenability of  $A \oplus B$ , it follows that there exists  $x^* \in X^*$  such that

$$\tilde{D}(a, b) = \varphi \oplus \psi(a, b).x^* - x^*.\varphi \oplus \psi(a, b) \quad (a \in A, b \in B).$$

It follows that

$$\begin{aligned} D(a + J_{A,\mathfrak{A}}, b + J_{B,\mathfrak{A}}) &= \bar{\varphi} \oplus \bar{\psi}(a + J_{A,\mathfrak{A}}, b + J_{B,\mathfrak{A}}).x^* \\ &\quad - x^*.\bar{\varphi} \oplus \bar{\psi}(a + J_{A,\mathfrak{A}}, b + J_{B,\mathfrak{A}}). \end{aligned}$$

So  $D$  is  $\bar{\varphi} \oplus \bar{\psi}$ -inner. Therefore  $\frac{A}{J_{A,\mathfrak{A}}} \oplus \frac{B}{J_{B,\mathfrak{A}}}$  is  $\bar{\varphi} \oplus \bar{\psi}$ -amenable.

(iii) Suppose that  $\frac{A}{J_{A,\mathfrak{A}}} \oplus \frac{B}{J_{B,\mathfrak{A}}}$  is  $\bar{\varphi} \oplus \bar{\psi}$ -amenable. Since  $\mathfrak{A}$  has a bounded approximate identity, by Proposition 2.1 of [1], we conclude that  $\frac{A}{J_{A,\mathfrak{A}}} \oplus \frac{B}{J_{B,\mathfrak{A}}}$  is  $\bar{\varphi} \oplus \bar{\psi}$ - $\mathfrak{A}$ -module amenable. So by (i),  $A \oplus B$  is  $\varphi \oplus \psi$ - $\mathfrak{A}$ -module amenable.

Similar relations can be obtained between the  $\varphi \oplus \psi$ - $\mathfrak{A}$ -module approximate amenability of  $A \oplus B$  and  $\bar{\varphi} \oplus \bar{\psi}$ -approximate amenability of  $\frac{A}{J_{A,\mathfrak{A}}} \oplus \frac{B}{J_{B,\mathfrak{A}}}$ .  $\square$

**Proposition 4.4.** *Let  $A$  be an  $\mathfrak{A}$ -module Banach algebra, where  $\mathfrak{A}$  acts on  $A$  trivially from the left by  $f \in \text{Hom}_{\mathbb{C}}(\mathfrak{A})$ . Let  $\varphi \in \text{Hom}_{\mathfrak{A}}(A)$  and  $\frac{A}{J_{A,\mathfrak{A}}}$  be unital. If  $A$  is  $\varphi$ - $\mathfrak{A}$ -module approximately amenable, then  $\frac{A}{J_{A,\mathfrak{A}}}$  is  $\bar{\varphi}$ -approximately amenable.*

*Proof.* Let  $X$  be a Banach  $\frac{A}{J_{A,\mathfrak{A}}}$ -bimodule and  $D : \frac{A}{J_{A,\mathfrak{A}}} \rightarrow X^*$  be a  $\bar{\varphi}$ -derivation. Then  $X$  becomes a  $A$ -bimodule through the following actions

$$a.x = (a + J_{A,\mathfrak{A}}).x, \quad x.a = x.(a + J_{A,\mathfrak{A}}) \quad (a \in A, x \in X),$$

and  $X$  is an  $\mathfrak{A}$ -bimodule with  $f$ -trivial actions, that is  $\alpha.x = x.\alpha = f(\alpha)x$  ( $\alpha \in \mathfrak{A}, x \in X$ ). By Lemma 4.2,  $f(\alpha)a - a.\alpha \in J_{A,\mathfrak{A}}$  ( $\alpha \in \mathfrak{A}, a \in A$ ). So,  $f(\alpha)a + J_{A,\mathfrak{A}} = a.\alpha + J_{A,\mathfrak{A}}$  ( $\alpha \in \mathfrak{A}, a \in A$ ), and the actions of  $\mathfrak{A}$  and  $A$  on  $X$  are compatible. Thus  $X$  is a commutative Banach  $A$ - $\mathfrak{A}$ -module. Let  $\tilde{D} : A \rightarrow X^*$  be defined by  $\tilde{D}(a) = D(a + J_{A,\mathfrak{A}})$  ( $a \in A$ ). A similar argument as in the proof of Theorem 3.2 of [2], shows that  $\tilde{D}$  is approximately  $\varphi$ -inner. So,  $D$  is approximately  $\bar{\varphi}$ -inner. Therefore  $\frac{A}{J_{A,\mathfrak{A}}}$  is  $\bar{\varphi}$ -approximately amenable.  $\square$

**Theorem 4.5.** *Let  $\mathfrak{A}$  have a bounded approximate identity, and let  $A$  and  $B$  be  $\mathfrak{A}$ -module Banach algebras, where  $\mathfrak{A}$  acts on  $A$  and  $B$  trivially from the left. Let  $\varphi \in \text{Hom}_{\mathfrak{A}}(A)$ ,  $\psi \in \text{Hom}_{\mathfrak{A}}(B)$ , and let  $\frac{A}{J_{A,\mathfrak{A}}}$  and  $\frac{B}{J_{B,\mathfrak{A}}}$  be unital. Then  $A$  is  $\varphi$ - $\mathfrak{A}$ -module approximately amenable and  $B$  is  $\psi$ - $\mathfrak{A}$ -module approximately amenable if and only if  $A \oplus B$  is  $\varphi \oplus \psi$ - $\mathfrak{A}$ -module approximately amenable.*

*Proof.* Suppose that  $A$  is  $\varphi$ - $\mathfrak{A}$ -module approximately amenable and  $B$  is  $\psi$ - $\mathfrak{A}$ -module approximately amenable. By Proposition 4.4,  $\frac{A}{J_{A,\mathfrak{A}}}$  and  $\frac{B}{J_{B,\mathfrak{A}}}$  are  $\bar{\varphi}$ -approximately amenable and  $\bar{\psi}$ -approximately amenable, respectively. Now by using Proposition 3.5 for  $\mathfrak{A} = \mathbb{C}$ , we conclude that  $\frac{A}{J_{A,\mathfrak{A}}} \oplus \frac{B}{J_{B,\mathfrak{A}}}$  is  $\bar{\varphi} \oplus \bar{\psi}$ -approximately amenable. So, Theorem 4.3, implies that  $A \oplus B$  is  $\varphi \oplus \psi$ - $\mathfrak{A}$ -module approximately amenable.

Conversely, suppose that  $A \oplus B$  is  $\varphi \oplus \psi$ - $\mathfrak{A}$ -module approximately amenable. Then by Proposition 3.7,  $A$  is  $\varphi$ - $\mathfrak{A}$ -module approximately amenable and  $B$  is  $\psi$ - $\mathfrak{A}$ -module approximately amenable.  $\square$

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