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Spectra of Some New Graph Operations and Some New Classes of Integral Graphs

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Abstract. In this paper, we define duplication corona, duplication neighborhood corona and duplication edge corona of two graphs. We compute their adjacency spectrum, Laplacian spectrum and signless Laplacian spectrum. As an application, our results enable us to construct infinitely many pairs of cospectral graphs and also integral graphs.

Keywords: Duplication corona, Duplication edge corona, Duplication neighborhood corona, Cospectral graphs, Integral graphs.

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1. Introduction

Throughout the paper by a graph we mean an undirected graph without loops and multiple edges. Let G be a graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G)$. The adjacency matrix of G, denoted by $A(G)$, is the $n \times n$ matrix $[a_{ij}]$, where $a_{ij} = 1$ if the vertices v_i and v_j are adjacent in G and 0 otherwise. The Laplacian matrix of the graph G, denoted by $L(G)$, is defined as $D(G) - A(G)$, where $D(G)$ is the diagonal degree matrix of G. The signless Laplacian matrix of the graph G, denoted by $Q(G)$, is defined

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as $D(G) + A(G)$. We denote the eigenvalues of $A(G)$, $L(G)$ and $Q(G)$, respectively, by $\lambda_1(G) \geq \lambda_2(G) \geq \ldots \geq \lambda_n(G)$, $\mu_1(G) = 0 \leq \mu_2(G) \leq \ldots \leq \mu_n(G)$ and $\gamma_1(G) \geq \gamma_2(G) \geq \ldots \geq \gamma_n(G)$. The collection of eigenvalues of $A(G)$ (respectively, $L(G)$, $Q(G)$) together with their multiplicities is called the adjacency spectrum (respectively, Laplacian spectrum, signless Laplacian spectrum) of G. Studies on these spectra of graphs can be found in [6, 7, 8, 19] and references therein. Two graphs are said to be adjacency cospectral (respectively, Laplacian cospectral, signless Laplacian cospectral) if they have the same adjacency spectrum (respectively, Laplacian spectrum, signless Laplacian spectrum).

In literature, many graph operations such as disjoint union, NEPS, corona, edge corona, neighborhood corona, common neighborhood graphs, etc., have been introduced and their spectral properties have been studied, see [1, 2, 4, 8, 9, 11, 12, 15, 17, 18, 22]. Recently, several variants of corona product of two graphs have been introduced and their spectra are computed. In [16], Liu and Lu introduced subdivision-vertex and subdivision-edge neighbourhood corona of two graphs and provided a complete description of their spectra. In [15], Lan and Zhou introduced R-vertex corona, R-edge corona, R-vertex neighborhood corona and R-edge neighborhood corona, and studied their spectra.

Motivated by these works, in this paper, we introduce duplication corona, duplication edge corona and duplication neighborhood corona of two graphs. In Section 3, we give the adjacency spectrum, Laplacian spectrum and signless Laplacian spectrum of duplication corona. In Sections 4 and 5, we give the adjacency spectrum, Laplacian spectrum and signless Laplacian spectrum of duplication neighborhood corona and duplication edge corona of two graphs G and H. In Section 6, using the results obtained in Sections 3, 4 and 5, we give some methods to construct infinitely many pairs of cospectral graphs and also integral graphs.

2. Preliminaries

In this section, we give some definitions and lemmas which are useful to prove our main results.

Let G be a graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G) = \{e_1, e_2, \ldots, e_m\}$. The duplication graph $Du(G)$ of G is a bipartite graph with vertex partition sets $U = \{u_1, u_2, \ldots, u_n\}$ and $V = \{v_1, v_2, \ldots, v_n\}$, where u_iv_j is an edge if and only if v_iv_j is an edge in G, see [13]. Now we define three new graph operations based on duplication graph $Du(G)$ as follows:

Definition 2.1. The duplication corona $G \boxminus H$ of two graphs G and H is the graph obtained by taking one copy of $Du(G)$ and $|V|$ copies of H, and then joining the vertex v_i of $Du(G)$ to every vertex in the ith copy of H.

Definition 2.2. The duplication neighborhood corona $G \boxtimes H$ of two graphs G and H is the graph obtained by taking one copy of $Du(G)$ and $|V|$ copies of H, and then joining the neighbors of the vertex v_i of $Du(G)$ to every vertex in the ith copy of H.

Definition 2.3. The duplication edge corona $G \boxplus H$ of two graphs G and H is the graph obtained by taking one copy of $Du(G)$ and $|E(G)|$ copies of H, and then joining a pair of vertices v_i and v_j of $Du(G)$ to every vertex in the kth copy of H whenever $v_i v_j = e_k \in E(G)$.

Let $A = (a_{ij})$ be an $n \times m$ matrix and $B = (b_{ij})$ be an $p \times q$ matrix. Then the Kronecker product [8] of A and B, denoted by $A \otimes B$, is the np by mq matrix obtained by replacing each entry a_{ij} of A by $a_{ij}B$. It is well-known that $(A \otimes B)(C \otimes D) = AC \otimes BD$ whenever the products AC and BD exist. The M-coronal [5, 18] of a square matrix M of order n, denoted by $\Gamma_M(x)$, is defined as follows:

$$\Gamma_M(x) = e^T(xI_n - M)^{-1}e,$$

where e is the column vector of size n whose all entries are 1. If M is a square matrix of order n such that sum of entries in each row is a constant 'r', then it is easy to see that $\Gamma_M(x) = n/(x-r)$. Further for a complete bipartite graph $K_{p,q}$, we have

$$\Gamma_{A(K_{p,q})}(x) = \frac{(p+q)x + 2pq}{x^2 - pq},$$

see [5]. The following lemma is useful to prove our main results.

Lemma 2.4 ([8])**.** *If M, N, P and Q are matrices with M being a non-singular matrix, then*

$$\begin{vmatrix} M & N \\ P & Q \end{vmatrix} = |M||Q - PM^{-1}N|.$$

3. Spectra of Duplication Corona

Let M be a square matrix. We denote the characteristic polynomial of M by

$$f(M, x) := det(xI - M).$$

In this section, we compute the adjacency spectrum, Laplacian spectrum and signless Laplacian spectrum of duplication corona of two graphs G_1 and G_2 in some cases. We denote by e and I_n, the column vector of size m whose all entries are 1 and the identity matrix of order n, respectively.

Theorem 3.1. *Let G_1 and G_2 be two graphs on n and m vertices, respectively. Then*

$$f(A(G_1 \boxminus G_2), x) = \prod_{i=1}^{m}(x-\lambda_i(G_2))^n \prod_{i=1}^{n}(x-\Gamma_{A(G_2)}(x))x - \lambda_i^2(G_1).$$

Proof. With suitable labeling of the vertices of $G_1 \boxminus G_2$, its adjacency matrix $A(G_1 \boxminus G_2)$ can be formulated as follows:

$$A(G_1 \boxminus G_2) = \begin{pmatrix} I_n \otimes A(G_2) & 0 & I_n \otimes e \\ 0 & 0 & A(G_1) \\ I_n \otimes e^T & A(G_1) & 0 \end{pmatrix}.$$

By Lemma 2.4, we have

$$\begin{aligned} f(A(G_1 \boxminus G_2), x) &= det \begin{pmatrix} I_n \otimes (xI_m - A(G_2)) & 0 & -I_n \otimes e \\ 0 & xI_n & -A(G_1) \\ -I_n \otimes e^T & -A(G_1) & xI_n \end{pmatrix} \\ &= \prod_{i=1}^{m}(x-\lambda_i(G_2))^n \; det\ S, \qquad (3.1) \end{aligned}$$

where

$$S = \begin{pmatrix} xI_n & -A(G_1) \\ -A(G_1) & (x-\Gamma_{A(G_2)}(x))I_n \end{pmatrix}.$$

Using Lemma 2.4, we obtain

$$\begin{aligned} det\ S &= x^n det((x-\Gamma_{A(G_2)}(x))I_n - A^2(G_1)/x) \\ &= \prod_{i=1}^{n}(x-\Gamma_{A(G_2)}(x))x - \lambda_i^2(G_1). \qquad (3.2) \end{aligned}$$

From (3.1) and (3.2), the result follows. $\square$

As $\Gamma_M(x) = \dfrac{n}{x-r}$, where M is the square matrix of order n with each of its row sum a constant 'r' and $\Gamma_{K_{p,q}}(x) = \dfrac{(p+q)x+2pq}{x^2-pq}$, proofs of the following two corollaries follow immediately from the above theorem.

Corollary 3.2. *Let G_1 be an arbitrary graph and G_2 be an r-regular graph on n and m vertices, respectively. Then the adjacency spectrum of $G_1 \boxminus G_2$ consists of*

a. $\lambda_i(G_2)$ *with multiplicity* n *for* $i = 2, 3, \ldots, m$ *and*
b. *the three roots of the polynomial*

$$x^3 - rx^2 - (\lambda_i^2(G_1) + m)x + r\lambda_i^2(G_1)$$

for $i = 1, 2, \ldots, n$.

Corollary 3.3. *Let* G_1 *be an arbitrary graph on n vertices. Then the adjacency spectrum of* $G_1 \boxminus K_{p,q}$ *consists of*

(a) 0 *with multiplicity* $n(p + q - 2)$ *and*
(b) *the four roots of the polynomial*

$$x^4 - \left(\lambda_i^2(G_1) + pq + p + q\right) x^2 - 2\,pqx + \lambda_i^2(G_1)pq$$

for $i = 1, 2, \ldots, n$.

Theorem 3.4. *Let* G_1 *be an* r_1*-regular on* n *vertices and* G_2 *be an arbitrary graph on* m *vertices. Then the Laplacian spectrum of* $G_1 \boxminus G_2$ *consists of*

a. $\mu_i(G_2) + 1$ *with multiplicity* n *for* $i = 2, 3, \ldots, m$ *and*
b. *the three roots of the polynomial*
$x^3 - (m + 2\,r_1 + 1)\,x^2 + \left(-\mu_i(G_1)^2 + 2\,\mu_i(G_1)r_1 + mr_1 + 2\,r_1\right) x + \mu_i(G_1)^2 - 2\,\mu_i(G_1)r_1$ *for* $i = 1, 2, \ldots, n$.

Proof. With suitable labeling of the vertices of $G_1 \boxminus G_2$, its Laplacian matrix $L(G_1 \boxminus G_2)$ can be formulated as follows:

$$L(G_1 \boxminus G_2) = \begin{pmatrix} I_n \otimes (I_m + L(G_2)) & 0 & -I_n \otimes e \\ 0 & r_1 I_n & -A(G_1) \\ -I_n \otimes e^T & -A(G_1) & (r_1 + m)I_n \end{pmatrix}.$$

By Lemma 2.4, we have

$$\begin{aligned} f(L(G_1 \boxminus G_2), x) &= det \begin{pmatrix} I_n \otimes ((x-1)I_m - L(G_2)) & 0 & I_n \otimes e \\ 0 & (x - r_1)I_n & A(G_1) \\ I_n \otimes e^T & A(G_1) & (x - r_1 - m)I_n \end{pmatrix} \\ &= \prod_{i=1}^{m} (x - \mu_i(G_2) - 1)^n \; det\; S, \end{aligned} \tag{3.3}$$

where

$$S = \begin{pmatrix} (x - r_1)I_n & A(G_1) \\ A(G_1) & (x - \Gamma_{L(G_2)}(x-1) - r_1 - m)I_n \end{pmatrix}.$$

Using Lemma 2.4, we obtain

$$\begin{aligned} det\ S &= (x-r_1)^n det((x-\Gamma_{_{L(G_2)}}(x-1)-r_1-m)I_n - A^2(G_1)/(x-r_1)) \\ &= \prod_{i=1}^{n}(x-m/(x-1)-r_1-m)(x-r_1)-(\mu_i(G_1)-r_1)^2. \end{aligned} \tag{3.4}$$

From (3.3) and (3.4), the desired result follows. □

Let $t(G)$ denote the number of spanning trees of G. It is well known [8] that for a connected graph G on n vertices, $t(G)$ is given by

$$t(G) = \frac{\mu_2(G)\cdots\mu_n(G)}{n}. \tag{3.5}$$

Corollary 3.5. *Let G_1 be an r_1-regular graph on n vertices and G_2 be an arbitrary graph on m vertices. Then the number of spanning trees of $G_1 \boxminus G_2$ is given by*

$$t(G_1 \boxminus G_2) = r_1 t(G_1)\prod_{i=2}^{n}(2r_1-\mu_i(G_1))\prod_{i=2}^{m}(\mu_i(G_2)+1)^n.$$

Proof. Proof follows directly from the above theorem and (3.5). □

Theorem 3.6. *Let G_1 be an r_1-regular graph on n vertices and G_2 be an r_2-regular graph on m vertices. Then the signless Laplacian spectrum of $G_1 \boxminus G_2$ consists of*

a. $\gamma_i(G_2)+1$ *with multiplicity n for $i = 2,3,\dots,m$ and*
b. *the three roots of the polynomial*
$x^3-(2r_1+2r_2+m+1)x^2+(4r_1r_2+2r_1\gamma_i(G_1)+r_1m+2r_2m-\gamma_i^2(G_1)+2r_1)x-4r_1r_2\gamma_i(G_1)-2r_1r_2m+2r_2\gamma_i^2(G_1)-2\gamma_i(G_1)r_1+\gamma_i^2(G_1)$
for $i = 1,2,\dots,n$.

Proof. With suitable labeling of the vertices of $G_1 \boxminus G_2$, its signless Laplacian matrix $Q(G_1 \boxminus G_2)$ can be formulated as follows:

$$Q(G_1 \boxminus G_2) = \begin{pmatrix} I_n \otimes (I_m + Q(G_2)) & 0 & I_n \otimes e \\ 0 & r_1 I_n & A(G_1) \\ I_n \otimes e^T & A(G_1) & (r_1+m)I_n \end{pmatrix}.$$

Rest of the proof is similar to the proof of Theorem 3.4. □

4. Spectra of Duplication Neighborhood Corona

We compute the adjacency spectrum, Laplacian spectrum and signless Laplacian spectrum of duplication neighborhood corona of two graphs G_1 and G_2 in some cases.

Theorem 4.1. *Let G_1 and G_2 be two graphs on n and m vertices, respectively. Then*

$$f(A(G_1 \boxtimes G_2), x) = \prod_{i=1}^{m}(x - \lambda_i(G_2))^n \prod_{i=1}^{n}(x - \Gamma_{A(G_2)}(x)\lambda_i^2(G_1))x - \lambda_i^2(G_1).$$

Proof. By a proper labeling of the vertices of $G_1 \boxtimes G_2$, its adjacency matrix $A(G_1 \boxtimes G_2)$ can be written as follows:

$$A(G_1 \boxtimes G_2) = \begin{pmatrix} I_n \otimes A(G_2) & 0 & A(G_1) \otimes e \\ 0 & 0 & A(G_1) \\ A(G_1) \otimes e^T & A(G_1) & 0 \end{pmatrix}.$$

By Lemma 2.4, we have

$$\begin{aligned} f(A(G_1 \boxtimes G_2), x) &= det \begin{pmatrix} I_n \otimes (xI_m - A(G_2)) & 0 & -A(G_1) \otimes e \\ 0 & xI_n & -A(G_1) \\ -A(G_1) \otimes e^T & -A(G_1) & xI_n \end{pmatrix} \\ &= \prod_{i=1}^{m}(x - \lambda_i(G_2))^n \; det\ S, \end{aligned} \tag{4.1}$$

where

$$S = \begin{pmatrix} xI_n & -A(G_1) \\ -A(G_1) & xI_n - \Gamma_{A(G_2)}(x)A^2(G) \end{pmatrix}.$$

Using Lemma 2.4, we see that

$$\begin{aligned} det\ S &= x^n det(xI_n - \Gamma_{A(G_2)}(x)A^2(G_1) - A^2(G_1)/x) \\ &= \prod_{i=1}^{n}(xI_n - \Gamma_{A(G_2)}(x)\lambda_i^2(G_1))x - \lambda_i^2(G_1). \end{aligned} \tag{4.2}$$

From (4.1) and (4.2), the result follows. $\square$

Proofs of the following two corollaries follow immediately by the above theorem.

Corollary 4.2. *Let G_1 be an arbitrary graph and G_2 be an r-regular graph on n and m vertices, respectively. Then the adjacency spectrum of $G_1 \boxtimes G_2$ consists of*

a. $\lambda_i(G_2)$ *with multiplicity n for $i = 2, 3, \ldots, m$ and*
b. the three roots of the polynomial

$$x^3 - rx^2 - (\lambda_i^2(G_1)m + \lambda_i^2(G_1))x + \lambda_i^2(G_1)r$$

for $i = 1, 2, \ldots, n$.

Corollary 4.3. *Let G_1 be an arbitrary graph on n vertices. Then the adjacency spectrum of $G_1 \boxtimes K_{p,q}$ consists of*

(a) 0 *with multiplicity $n(p + q - 2)$ and*
(b) the four roots of the polynomial

$$x^4 - (\lambda_i^2(G_1)p + \lambda_i^2(G_1)q + \lambda_i^2(G_1) + pq)x^2 - 2\lambda_i^2(G_1)pqx + \lambda_i^2(G_1)pq$$

for $i = 1, 2, \ldots, n$.

Theorem 4.4. *Let G_1 be an r_1-regular graph on n vertices and G_2 be an arbitrary graph on m vertices. Then the Laplacian spectrum of $G_1 \boxtimes G_2$ consists of*

a. $\mu_i(G_2) + r_1$ *with multiplicity n for $i = 1, 2, \ldots, m$ and*
b. the three roots of the polynomial $x^2 - (mr_1 + 2r_1)x - \mu_i^2(G_1)m + 2\mu_i(G_1)mr_1 - \mu_i^2(G_1) + 2\mu_i(G_1)r_1$ *for* $i = 1, 2, \ldots, n$.

Proof. With suitable labeling of the vertices of $G_1 \boxtimes G_2$, its Laplacian matrix $L(G_1 \boxtimes G_2)$ can be formulated as follows:

$$L(G_1 \boxtimes G_2) = \begin{pmatrix} I_n \otimes (r_1 I_m + L(G_2)) & 0 & -A(G_1) \otimes e \\ 0 & r_1 I_n & -A(G_1) \\ -A(G_1) \otimes e^T & -A(G_1) & r_1(m+1)I_n \end{pmatrix}.$$

By Lemma 2.4, we have

$$\begin{aligned} f(L(G_1 \boxtimes G_2), x) &= det \begin{pmatrix} I_n \otimes ((x - r_1)I_m - L(G_2)) & 0 & A(G_1) \otimes e \\ 0 & (x - r_1)I_n & A(G_1) \\ A(G_1) \otimes e^T & A(G_1) & (x - r_1 - r_1 m)I_n \end{pmatrix} \\ &= \prod_{i=1}^{m} (x - \mu_i(G_2) - r_1)^n \; det\, S, \qquad (4.3) \end{aligned}$$

where

$$S = \begin{pmatrix} (x-r_1)I_n & A(G_1) \\ \\ A(G_1) & (x-r_1-mr_1)I_n - \Gamma_{L(G_2)}(x-r_1)A^2(G_1) \end{pmatrix}.$$

Using Lemma 2.4, we obtain

$$\begin{aligned} det\ S &= (x-r_1)^n det((x-r_1-mr_1)I_n - \Gamma_{L(G_2)}(x-r_1)A^2(G_1) - A^2(G_1)/(x-r_1)) \\ &= \prod_{i=1}^{n}(x-r_1-mr_1-\frac{m}{x-r_1}(\mu_i(G_1)-r_1)^2)(x-r_1)-(\mu_i(G_1)-r_1)^2. \quad (4.4) \end{aligned}$$

From (4.3) and (4.4), the desired result follows. □

Corollary 4.5. *Let G_1 be an r_1-regular graph on n vertices and G_2 be an arbitrary graph on m vertices. Then the number of spanning trees of $G_1 \boxtimes G_2$ is given by*

$$t(G_1 \boxtimes G_2) = r_1 t(G_1)\prod_{i=2}^{n}(m+1)(2r_1-\mu_i(G_1))\prod_{i=1}^{m}(\mu_i(G_2)+r_1)^n.$$

Proof. Proof follows directly from the above theorem and (3.5). □

Theorem 4.6. *Let G_1 be an r_1-regular on n vertices and G_2 be an r_2-regular graph on m vertices. Then the signless Laplacian spectrum of $G_1 \boxtimes G_2$ consists of*

a. $\gamma_i(G_2)+r_1$ with multiplicity n for $i = 2, 3, \ldots, m$ and

b. the three roots of the polynomial
$x^3-(r_1m+3r_1+2r_2)x^2+(-\gamma_i^2(G_1)m+2\gamma_i(G_1)r_1m+r_1^2m+2r_1r_2m-\gamma_i^2(G_1)+2\gamma_i(G_1)r_1+2r_1^2+4r_1r_2)x+\gamma_i^2(G_1)r_1m-2\gamma_i(G_1)r_1^2m-2r_1^2r_2m+\gamma_i^2(G_1)r_1+2\gamma_i^2(G_1)r_2-2\gamma_i(G_1)r_1^2-4\gamma_i(G_1)r_1r_2$ *for $i = 1, 2, \ldots, n$.*

Proof. With suitable labeling of the vertices of $G_1 \boxtimes G_2$, its signless Laplacian matrix $Q(G_1 \boxtimes G_2)$ can be formulated as follows:

$$Q(G_1 \boxtimes G_2) = \begin{pmatrix} I_n \otimes (r_1 I_m + Q(G_2)) & 0 & A(G_1)\otimes e \\ \\ 0 & r_1 I_n & A(G_1) \\ \\ A(G_1)\otimes e^T & A(G_1) & r_1(m+1)I_n \end{pmatrix}.$$

Rest of the proof is similar to the proof of Theorem 4.4. □

5. Spectra of Duplication Edge Corona

In this section, we compute the adjacency spectrum, Laplacian spectrum and signless Laplacian spectrum of duplication edge corona of two graphs G_1 and G_2 in some cases. We denote by e, I_{m_1} and B, the column vector of size n_2 whose all entries are 1, the identity matrix of order m_1 and the incidence matrix of G_1, respectively. In the following theorems and corollaries we assume that $r_1 \geq 2$.

Theorem 5.1. *Let G_1 be an r_1-regular graph with n_1 vertices, m_1 edges and G_2 be a graph on n_2 vertices. Then*

$$f(A(G_1 \boxplus G_2), x) = \prod_{i=1}^{n_2}(x - \lambda_i(G_2))^{m_1} \prod_{i=1}^{n_1}(x - \Gamma_{A(G_2)}(x)(\lambda_i(G_1) + r_1))x - \lambda_i^2(G_1).$$

Proof. With suitable labeling of the vertices of $G_1 \boxplus G_2$, its adjacency matrix $A(G_1 \boxplus G_2)$ can be formulated as follows:

$$A(G_1 \boxplus G_2) = \begin{pmatrix} I_{m_1} \otimes A(G_2) & 0 & B \otimes e \\ 0 & 0 & A(G_1) \\ B^T \otimes e^T & A(G_1) & 0 \end{pmatrix}.$$

By Lemma 2.4, we have

$$\begin{aligned} f(A(G_1 \boxplus G_2), x) &= det \begin{pmatrix} I_{m_1} \otimes (xI_{n_2} - A(G_2)) & 0 & -B \otimes e \\ 0 & xI_{n_1} & -A(G_1) \\ -B^T \otimes e^T & -A(G_1) & xI_{n_1} \end{pmatrix} \\ &= \prod_{i=1}^{n_2}(x - \lambda_i(G_2))^{m_1} \; det\ S, \end{aligned} \tag{5.1}$$

where

$$S = \begin{pmatrix} xI_{n_1} & -A(G_1) \\ -A(G_1) & xI_{n_1} - \Gamma_{A(G_2)}(x)(A(G_1) + r_1 I_{n_1}) \end{pmatrix}.$$

Using Lemma 2.4, we see that

$$\begin{aligned} det\ S &= x^{n_1} det(xI_{n_1} - \Gamma_{A(G_2)}(x)(A(G_1) + r_1 I_{n_1}) - A^2(G_1)/x) \\ &= \prod_{i=1}^{n_1}(x - \Gamma_{A(G_2)}(x)(\lambda_i(G_1) + r_1))x - \lambda_i^2(G_1). \end{aligned} \tag{5.2}$$

From (5.1) and (5.2), the result follows. □

Proofs of the following two corollaries follow immediately by the above theorem.

Corollary 5.2. *Let G_1 be an r_1-regular graph with n_1 vertices, m_1 edges and G_2 be an r_2-regular graph on n_2 vertices. Then the adjacency spectrum of $G_1 \boxminus G_2$ consists of*

a. $\lambda_i(G_2)$ *with multiplicity* m_1 *for* $i = 2, 3, \ldots, n_2$,
b. r_2 *with multiplicity* $m_1 - n_1$ *and*
c. the three roots of the polynomial

$$x^3 - r_2x^2 - (\lambda_i^2(G_1) + \lambda_i(G_1)m + r_1m)x + \lambda_i^2(G_1)r_2$$

for $i = 1, 2, \ldots, n_1$.

Corollary 5.3. *Let* G_1 *be an* r_1*-regular graph with* n_1 *vertices and* m_1 *edges. Then the adjacency spectrum of* $G_1 \boxplus K_{p,q}$ *consists of*

(a) 0 *with multiplicity* $m_1(p + q - 2)$,
(b) $\pm\sqrt{pq}$ *with multiplicity* $m_1 - n_1$ *and*
(c) the four roots of the polynomial $x^4 - (\lambda_i^2(G_1) + \lambda_i(G_1)p + \lambda_i(G_1)q + r_1p + r_1q + pq)x^2 + (-2\lambda_i(G_1)pq - 2r_1pq)x + \lambda_i^2(G_1)pq$ *for* $i = 1, 2, \ldots, n_1$.

Theorem 5.4. *Let* G_1 *be an* r_1*-regular with* n_1 *vertices and* m_1 *edges and* G_2 *be an arbitrary graph on* n_2 *vertices. Then the Laplacian spectrum of* $G_1 \boxplus G_2$ *consists of*

a. $\mu_i(G_2)+2$ *with multiplicity* m_1 *for* $i = 2, 3, \ldots, n_2$, 2*with multiplicity* m_1-n_1 *and"'*
b. the three roots of the polynomial $x^3 - (n_2r_1 + 2r_1 + 2)x^2 + (n_2r_1^2 - \mu_i^2(G_1) + \mu_i(G_1)n_2 + 2\mu_i(G_1)r_1 + 4r_1)x - \mu_i(G_1)n_2r_1 + 2\mu_i^2(G_1) - 4\mu_i(G_1)r_1$ *for* $i = 1, 2, \ldots, n_1$.

Proof. With suitable labeling of the vertices of $G_1 \boxplus G_2$, its Laplacian matrix $L(G_1 \boxplus G_2)$ can be formulated as follows:

$$L(G_1 \boxplus G_2) = \begin{pmatrix} I_{m_1} \otimes (2I_{n_2} + L(G_2)) & 0 & -B \otimes e \\ 0 & r_1I_{n_1} & -A(G_1) \\ -B^T \otimes e^T & -A(G_1) & r_1(n_2+1)I_{n_1} \end{pmatrix}.$$

By Lemma 2.4, we have

$$\begin{aligned} f(L(G_1 \boxplus G_2), x) &= det \begin{pmatrix} I_{m_1} \otimes ((x-2)I_{n_2} - L(G_2)) & 0 & B \otimes e \\ 0 & (x-r_1)I_{n_1} & A(G_1) \\ B^T \otimes e^T & A(G_1) & (x - r_1 - r_1n_2)I_{n_1} \end{pmatrix} \\ &= \prod_{i=1}^{n_2}(x - \mu_i(G_2) - 2)^{m_1}\ det\ S, \end{aligned} \tag{5.3}$$

where

$$S = \begin{pmatrix} (x-r_1)I_{n_1} & A(G_1) \\ A(G_1) & (x - r_1 - n_2r_1)I_{n_1} - \Gamma_{L(G_2)}(x-2)(A(G_1) + r_1I_{n_1}) \end{pmatrix}.$$

Using Lemma 2.4, we obtain

$$\begin{aligned} det\ S &= (x-r_1)^{n_1} det((x-r_1-n_2r_1)I_{n_1} - \Gamma_{L(G_2)}(x-2)(A(G_1)+r_1I_{n_1}) - A^2(G_1)/(x-r_1)) \\ &= \prod_{i=1}^{n_1}(x-r_1-n_2r_1+\frac{n_2}{x-2}(\mu_i(G_1)-2r_1))(x-r_1)-(\mu_i(G_1)-r_1)^2. \end{aligned} \tag{5.4}$$

From (5.3) and (5.4), the required result follows. □

Corollary 5.5. *Let G_1 be an r_1-regular graph on n vertices and G_2 be an arbitrary graph on m vertices. Then the number of spanning trees of $G_1 \boxplus G_2$ is given by*

$$t(G_1 \boxplus G_2) = 2^{1-n} r_1 t(G_1) \prod_{i=2}^{n}(mr_1 - 2\mu_i(G_1) + 4r_1) \prod_{i=1}^{m}(\mu_i(G_2)+2)^{nr_1/2}.$$

Proof. Proof follows directly from the above theorem and (3.5). □

Theorem 5.6. *Let G_1 be an r_1-regular with n_1 vertices and m_1 edges and G_2 be an r_2-regular graph on n_2 vertices. Then the signless Laplacian spectrum of $G_1 \boxplus G_2$ consists of*

a. $\gamma_i(G_2)+2$ *with multiplicity* m_1 *for* $i = 2, 3, \ldots, n_2$, $2r_2+2$ *with multiplicity* $m_1 - n_1$ *and"*

b. the three roots of the polynomial $x^3 - (r_1n_2 + 2\,r_1 + 2\,r_2 + 2)\,x^2 + ({r_1}^2n_2 + 2\,r_1r_2n_2 - {\gamma_i(G_1)}^2 + 2\,\gamma_i(G_1)r_1 + \gamma_i(G_1)n_2 + 4\,r_1r_2 + 2\,r_1n_2 + 4\,r_1)x - 2\,{r_1}^2r_2n_2 + 2\,{\gamma_i(G_1)}^2r_2 - 4\,\gamma_i(G_1)r_1r_2 - \gamma_i(G_1)r_1n_2 - 2\,{r_1}^2n_2 + 2\,{\gamma_i(G_1)}^2 - 4\,\gamma_i(G_1)r_1$ *for* $i = 1, 2, \ldots n_1$.

Proof. With suitable labeling of the vertices of $G_1 \boxplus G_2$, its signless Laplacian matrix $Q(G_1 \boxplus G_2)$ can be formulated as follows:

$$Q(G_1 \boxplus G_2) = \begin{pmatrix} I_{m_1} \otimes (2I_{n_2} + Q(G_2)) & 0 & B \otimes e \\ 0 & r_1 I_{n_1} & A(G_1) \\ B^T \otimes e^T & A(G_1) & r_1(n_2+1)I_{n_1} \end{pmatrix}.$$

Rest of the proof is similar to the proof of Theorem 5.4. □

6. Applications

Let G be a graph. If all the eigenvalues of $A(G)$ are integers then the graph G is said to be an integral graph [10]. For example, the graphs K_n, $K_{m,n}$ (mn a perfect square), C_6, the cocktail parity graph $CP(n) = \overline{nK_2}$, are all integral graphs. The notion of integral graphs was first introduced by Harary and Schwenk in 1974 [10]. In general, the problem of characterizing integral graphs seems to be very difficult. More details about integral graphs can be found in [3, 10, 14, 20, 21] and references therein. In this section, using the results obtained in the previous sections, we give some methods to construct infinite family of integral graphs starting with an integral graph. At the end of the section, we also give some methods to construct infinitely many pairs of cospectral graphs.

From Corollaries 3.2, 4.2 and 5.2, it follows that

a. If G is an integral graph of order n, then $G \boxminus mK_1$ is integral if and only if $\lambda_i^2(G) + m$ is a perfect square for $i = 1, 2, \ldots, n$.
b. If $G \boxminus mK_1$ is an integral graph, then $(K_2 \otimes G) \boxminus mK_1$ is integral, where $\otimes$ denotes the direct product of two graphs.
c. If G is an integral graph of order n, then $G \boxtimes (m^2 - 1)K_1$ is an integral graph.
d. If G is an integral r-regular graph of order n, then $G \boxplus mK_1$ is integral if and only if $\lambda_i^2(G) + m(\lambda_i(G) + r)$ is a perfect square for $i = 1, 2, \cdots, n$.

In particular, we have the following:

i. $K_n \boxminus (m^2 - 1)K_1$ is integral if and only if and $n^2 - 2n + m^2$ is a perfect square.
ii. $K_{p,q} \boxminus (m^2)K_1$ is integral if and only if $pq + m^2$ is a perfect square.
iii. $K_{p,q} \boxtimes (m^2 - 1)K_1$ is integral if and only if pq is a perfect square.
iv. $K_n \boxplus mK_1$ is integral if and only if and $(n-1)(n+2m-1)$ and $m(n-2)+1$ are perfect squares.
v. $K_{n,n} \boxplus mK_1$ is integral if and only if mn and n^2+2mn are perfect squares.

The above observations enable us to construct some new classes of integral graphs.

Example 6.1. a. The graph $K_{2n^2} \boxminus (4n^2 - 1)K_1$ is integral for all $n = 1, 2, \ldots$
b. The graph $K_{m^2,(n^2-1)} \boxminus m^2K_1$ is integral for $m = 1, 2, \ldots, n = 2, 3, \ldots$
c. The graph $K_n \boxtimes (m^2 - 1)K_1$ is integral for all n and m.
d. The graph $\overline{nK_2} \boxtimes (m^2 - 1)K_1$ is integral for all n and m.
e. The graph $K_{p^2,q^2} \boxtimes (m^2 - 1)K_1$ is integral for all m, p and q.
f. The graph $K_{n+1} \boxplus (4n)K_1$ is integral for all $n = 1, 2, \ldots$
g. The graph $K_{n,n} \boxplus 4nK_1$ is integral for all $n = 1, 2, \ldots$

Now we give some methods to construct infinite family of cospectral graphs. From Theorems 3.1 and 4.1, one can easily notice the following.

a. If G_1 and G_2 are adjacency cospectral graphs and H is an arbitrary graph, then
   i. $G_1 \boxminus H$ and $G_2 \boxminus H$ are adjacency cospectral.
   ii. $G_1 \boxtimes H$ and $G_2 \boxtimes H$ are adjacency cospectral.
b. If G is an arbitrary graph and H_1, H_2 are adjacency cospectral graphs with $\Gamma_{A(H_1)}(x) = \Gamma_{A(H_2)}(x)$, then
   i. $G \boxminus H_1$ and $G \boxminus H_2$ are adjacency cospectral.

ii. $G \boxtimes H_1$ and $G \boxtimes H_2$ are adjacency cospectral.

Similarly, using Theorems 3.4, 3.6, 4.4 and 4.6, one can construct Laplacian cospectral and signless Laplacian cospectral graphs. Also from Theorem 5.1, we have the following results:

a. If G_1 and G_2 are adjacency regular cospectral graphs and H is an arbitrary graph, then $G_1 \boxplus H$ and $G_2 \boxplus H$ are adjacency cospectral.

b. If G is an arbitrary regular graph and H_1, H_2 are adjacency cospectral graphs with $\Gamma_{A(H_1)}(x) = \Gamma_{A(H_2)}(x)$, then $G \boxplus H_1$ and $G \boxplus H_2$ are adjacency cospectral.

Similarly, using Theorems 5.4 and 5.6, one can construct Laplacian cospectral and signless Laplacian cospectral graphs.

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