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On Open Packing Number of Graphs

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Abstract. In a graph $G = (V, E)$, a subset $S \subseteq V(G)$ is said to be an *open packing set* of G if no two vertices of S have a common neighbour in G. The maximum cardinality of an open packing set is called the *open packing number* of G and is denoted by $\rho^o(G)$. This parameter has been studied in [5], [6], [7] and [8]. In this paper, we characterize the graphs G with $\rho^o(G) = n - 2$, $\rho^o(G) = n - \omega(G)$ and $\rho^o(G) = n - \Delta(G)$, where n, $\omega(G)$ and $\Delta(G)$ denote the order, clique number and the maximum degree of G. Also, we discuss the open packing number for split graphs.

Keywords: Packing number, Open packing number.

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1. Introduction

By a graph $G = (V, E)$, we mean a finite, undirected graph with neither loops nor multiple edges. For graph theoretic terminology we refer to Chartrand and Lesniak [3].

The *open neighbourhood* of a vertex v is $N(v)=\{u \in V\colon\ uv \in E\}$ while its *closed neighbourhood* is $N[v]=N(v) \cup \{v\}$. For a set $S \subseteq V$, the subgraph induced by S is denoted by $\langle S\rangle$. A *clique* in a graph G is a complete subgraph of G. The maximum order of a clique in G is called the *clique number* and is

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denoted by $\omega(G)$. The *corona* of two disjoint graphs G_1 and G_2 is defined to be the graph $G = G_1 \circ G_2$ formed from one copy of G_1 and $|V(G_1)|$ copies of G_2 where the i^{th} vertex of G_1 is adjacent to every vertex in the i^{th} copy of G_2. The *distance* $d(u, v)$ between two vertices u and v of a connected graph G is defined to be the length of a shortest path joining u and v. The *diameter* of a connected graph G, denoted by $diam(G)$, is the maximum distance among all pairs of vertices in G. A $u - v$ path in G with $d(u, v) = diam(G)$ is called a *diametrical path*.

A set S of vertices of G where no two vertices are adjacent in G is called an *independent set* of G. The *independence number* $\alpha(G)$ of G is defined to be the maximum cardinality of an independent set of G. A set S of vertices of G is an *open packing set* of G if the open neighbourhoods of the vertices of S are pairwise disjoint in G. The *open packing number* of G, denoted by $\rho^o(G)$, is the maximum cardinality of an open packing set of G. An open packing set of cardinality $\rho^o(G)$ is called an ρ^o-*set* of G. In fact, a ρ^o-set of a graph G is an α-set of a common neighbourhood graph $con(G)$ of G (see [1]). More details about the open packing number can be seen in [5], [6], [7], [8] and [9]. This paper extends the study of the open packing number. We state the following results that are needed in the subsequent sections.

Theorem 1.1. [8] *Let G be a graph of order at least 3. Then $\rho^o(G) = 1$ if and only if $diam(G) \leq 2$ and every edge of G lies on a triangle.*

Theorem 1.2. [8] *If G is a connected graph on n vertices with $\Delta(G) = n - 1$, then $\rho^o(G) \leq 2$. Further, $\rho^o(G) = 2$ if and only if $\delta(G) = 1$.*

Theorem 1.3. [8] *For any connected graph G of order $n \geq 3$, $\rho^o(G) \leq n - \omega(G) + 1$ with equality if and only if G is either K_n or a graph obtained from K_{n-1} by adding a vertex and joining it to exactly one vertex of K_{n-1}.*

2. Open Packing, Clique Number and Maximum Degree

In this section, we obtain some bounds for the open packing number in terms of order, clique number and the maximum degree. It is obvious that for a connected graph G of order n, $\rho^o(G) = n$ if and only if G is either K_1 or K_2; and $\rho^o(G) = n - 1$ if and only if G is P_3. The class of connected graphs G of order $n \geq 3$ for which $\rho^o(G) = n - 2$ is determined below.

Theorem 2.1. *Let G be a connected graph of order $n \geq 3$. Then $\rho^o(G) = n-2$ if and only if G is one of the graphs P_4, P_5, P_6, C_3, C_4, $K_{1,3}$ and the graph H of Figure 1.*

Proof. Let $\rho^o(G) = n - 2$. If $\rho^o(G) = 1$, then $n = 3$ and consequently by Theorem 1.1, we have $G \cong C_3$. Assume that $\rho^o(G) \geq 2$. Let S be a ρ^o- set of G and let $V - S = \{x, y\}$. Then each of x and y has at most one neighbour in S. Consider the case that x and y have neighbours in S, say u and v

respectively. Suppose $u = v$. Since G is connected and S is open packing it follows that $|S| = 2$ and $\langle S \rangle = K_2$. Hence G is isomorphic either to $K_{1,3}$ or to the graph H of Figure 1 according to $xy \notin E(G)$ or $xy \in E(G)$. Suppose $u \neq v$. If $uv \in E(G)$, then $S = \{u, v\}$ so that G is isomorphic either to P_4 or to C_4 according to $xy \notin E(G)$ or $xy \in E(G)$. If $uv \notin E(G)$, then $xy \in E(G)$. Further, each of u and v has at most one neighbour in S and that are distinct. Therefore, G is isomorphic either to P_4 or P_5 or P_6.

On the other hand, suppose exactly one of x and y, say x, has a neighbour u in S. Then x and y must be adjacent. Certainly, u will have a neighbour in S; for otherwise $G = P_3$ for which $\rho^o(G) = n - 1$. Therefore, $G \cong P_4$. The converse is just a simple verification. □

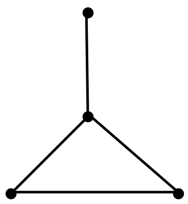

Figure 1: The graph H said in Theorem 2.1.

It has been proved in [8] that for a connected graph G of order n, $\rho^o(G) \leq n - \omega(G) + 1$. The characterization of graphs attaining this bound is also obtained in [8]. Here, we characterize those graphs for which $\rho^o(G) = n - \omega(G)$. For this purpose, we describe the following families of graphs.

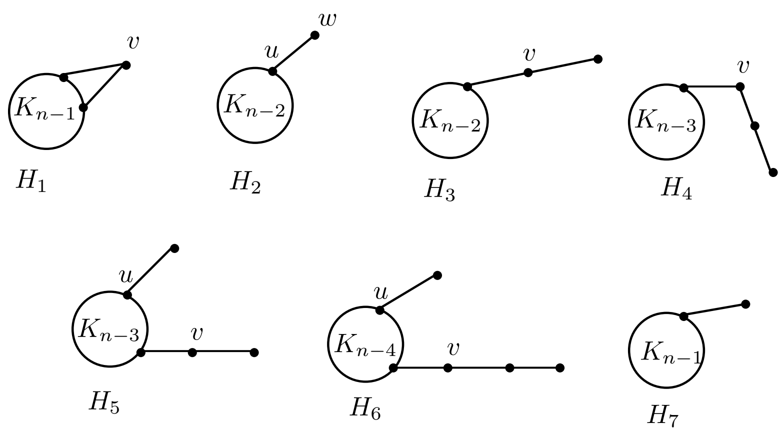

Figure 2: The graphs H_1 to H_7

(i) Let $\mathcal{F}_1$ be the family of non-complete connected graphs of order $n \geq 4$ obtained from the graph H_1 of Figure 2 by joining the vertex v with any number of vertices of the clique K_{n-1}.

(ii) Let $\mathcal{F}_2$ be the family of connected graphs G of order $n \geq 3$ that are constructed as follows. Consider the graph H_2 of Figure 2. Add a new vertex v. Join the vertex v with any number of vertices of H_2 satisfying the conditions that (i) the vertex v has at most $n-3$ neighbors in the clique K_{n-2}. (ii) the vertex v cannot be adjacent with both u and w simultaneously. (In particular, P_4, C_4 and $K_{1,3}$ belong to this family).

(iii) Let $\mathcal{F}_3$ be the family of connected graphs G of order $n \geq 4$ obtained from the graph H_3 of Figure 2 by joining the vertex v with at most $n-3$ vertices of the clique K_{n-2}. (In particular, P_4 belongs to this family).

(iv) Let $\mathcal{F}_4$ be the family of connected graphs G of order $n \geq 5$ obtained from the graph H_4 of Figure 2 by joining the vertex v with at most $n-4$ vertices of the clique K_{n-3}. (In particular, P_5 belongs to this family).

(v) Let $\mathcal{F}_5$ be the family of connected graphs G of order $n \geq 5$ obtained from the graph H_5 of Figure 2 by joining the vertex v with any of the vertices of the clique K_{n-3} except the vertex u. (In particular, P_5 belongs to this family).

(vi) Let $\mathcal{F}_6$ be the family of connected graphs G of order $n \geq 6$ obtained from the graph H_6 of Figure 2 by joining the vertex v with any of the vertices of the clique K_{n-4} except the vertex u. (In particular, P_6 belongs to this family).

(vii) Let $\mathcal{F}_7$ be the family consisting of the connected graphs G of order $n \geq 4$, other than the graph H_7 of Figure 2, that are constructed as follows. Consider the complete graph $H = K_l$ on l vertices, where $2 \leq l \leq n-1$. Attach either a path P_2 or a path P_3, but not both, at the vertices of H(not necessarily at all the vertices of H). The vertices outside H can be made adjacent with the vertices of H; and a pendant vertex of a P_2 attached can also be made adjacent with a pendant vertex of another P_2 attached, satisfying the following conditions (i) $\omega(G) = l$ (ii) degree of each vertex lying on H is either l or $l-1$ and (iii) each component of $G - H$ is either K_1 or K_2. (In particular, P_4, P_5 and P_6 belong to this family).

Theorem 2.2. *Let G be a connected graph of order $n \geq 3$. Then $\rho^o(G) = n - \omega(G)$ if and only if $G \in \bigcup_{i=1}^{7} \mathcal{F}_i$.*

Proof. Let us first verify that for a graph G belonging to one of the families $\mathcal{F}_1$ to $\mathcal{F}_7$, $\rho^o(G) = n - \omega(G)$. If $G \in \mathcal{F}_1$, then $\omega(G) = n-1$ and $\Delta(G) = n-1$.

As $\delta(G) \geq 2$, by Theorem 1.2, $\rho^o(G) = 1 = n - (n-1) = n - \omega(G)$. Suppose that $G \in \mathcal{F}_2$. Then $\omega(G) = n - 2$ and the set $\{u, w\}$ forms an open packing set of G so that $\rho^o(G) \geq 2$. Further, any open packing set S of G can have at most one vertex from the clique and in case S has a vertex from the clique, then it has at most one vertex from outside the clique. Thus $\rho^o(G) \leq 2$ and hence $\rho^o(G) = 2 = n - (n-2) = n - \omega(G)$.

Suppose that $G \in \mathcal{F}_3$. Then $\omega(G) = n-2$. Obviously, the pendant neighbour of the vertex v together with v form a maximum open packing set of G so that $\rho^o(G) = 2 = n - (n-2) = n - \omega(G)$. Now, suppose that $G \in \mathcal{F}_4$. Then $\omega(G) = n - 3$. Let w be the pendant vertex of G and let u be the support vertex of w that is adjacent with the vertex v in G. Let $v' \neq w$ be a non-neighbour of v. Obviously, the set $\{u, w, v'\}$ is an open packing set of G so that $\rho^o(G) \geq 3$. Also, any open packing set of G can have at most one vertex from the clique and at most two vertices from outside the clique and so $\rho^o(G) \leq 3$. Therefore $\rho^o(G) = 3 = n - (n-3) = n - \omega(G)$.

Suppose that $G \in \mathcal{F}_5$. Let u' and v' be the pendant neighbours of u and v respectively in G. Since v is not adjacent with u, the set $\{u', v, v'\}$ forms an open packing set of G so that $\rho^o(G) \geq 3$. Also, it is clear that any open packing set of G can have at most three vertices (one vertex from the clique and two vertices from outside of the clique) so that $\rho^o(G) \leq 3$. Hence $\rho^o(G) = 3 = n - (n-3) = n - \omega(G)$. Suppose that $G \in \mathcal{F}_6$. Then $\omega(G) = n - 4$ and it is not difficult to see that any open packing set of G has at most four vertices so that $\rho^o(G) \leq 4$. Further, the two pendant vertices of G and their respective supports form an open packing set of G so that $\rho^o(G) \geq 4$. Therefore $\rho^o(G) = 4 = n - (n-4) = n - \omega(G)$.

Suppose that $G \in \mathcal{F}_7$. Then the vertices outside the clique together form an open packing set of G so that $\rho^o(G) \geq n - \omega(G)$. Now, in view of Theorem 1.3, $\rho^o(G) \leq n - \omega(G)$ and hence $\rho^o(G) = n - \omega(G)$.

Conversely, assume that $\rho^o(G) = n - \omega(G)$. If $\omega(G) = 2$, then it follows from Theorem 2.1 that G is one of the graphs P_4, P_5, P_6, C_4 and $K_{1,3}$. Certainly, the graphs P_4, C_4 and $K_{1,3}$ belong to $\mathcal{F}_2$; $P_5 \in \mathcal{F}_5$ and $P_6 \in \mathcal{F}_6$. Assume that $\omega(G) \geq 3$. Let H be a clique in G with order $\omega(G)$ and let S be a ρ^o- set of G.

Case 1. $S \cap V(H) \neq \emptyset$.

Then $S \cap V(H)$ consists of exactly one vertex, say u. Let v be the vertex of G that lies neither in H nor in S. The following facts are easy to observe.

Fact 1: No vertex of $S - \{u\}$ has a neighbour in $H - \{u\}$.

Fact 2: Each component of $\langle S \rangle$ is either K_1 or K_2.

Fact 3: The vertex v can have at most one neighbour in S.

Fact 4: The vertex v must have at least one non-neighbour in H.

These facts along with the connectedness of G imply that $|S| \leq 4$. Now, suppose that $|S| = 1$. Then $\omega(G) = n - 1$ and so by Fact 4, the vertex v is adjacent to at most $n-2$ vertices of H. At the same time, the vertex v must have at least two neighbours in H; for otherwise $\rho^o(G)$ would become 2. Therefore, $G \in \mathcal{F}_1$. Suppose that $|S| = 2$; say $S = \{u, w\}$. If u and w are adjacent, then $G \in \mathcal{F}_2$; otherwise $G \in \mathcal{F}_3$. Suppose that $|S| = 3$; say $S = \{u, w_1, w_2\}$. By Fact 1, Fact 3 and by the connectedness of G, we have $\langle S \rangle = K_1 \cup K_2$. If u is the isolate vertex of $\langle S \rangle$, again by Fact 3, exactly one of w_1 and w_2, say w_1, is adjacent to v; and also v must have at least one neighbour in $H - \{u\}$, so that $G \in \mathcal{F}_5$. If u is not an isolate in $\langle S \rangle$, then u is adjacent either to w_1 or to w_2(but not both), say w_1. Then by Fact 1, w_2 is adjacent with v and so, by Fact 3, v is adjacent neither to u nor to w_1 which in turn implies that v has a neighbour in $H - \{u\}$. Therefore, by Fact 4, $G \in \mathcal{F}_4$. When $|S| = 4$, we can prove by the similar argument that $G \in \mathcal{F}_6$.

Case 2. $S \cap V(H) = \emptyset$.

As we observed, each component of $\langle S \rangle$ is either K_1 or K_2. Being G connected, each isolate of $\langle S \rangle$ has at least one neighbour in H and if uv is an edge of $\langle S \rangle$, then either u or v has at least one neighbour in H. Further, no two vertices of S have a common neighbour in H so that each vertex lying on H has degree either $\omega(G)$ or $\omega(G) - 1$ and each component of $G - H$ is either K_1 or K_2. Therefore, $G \in \mathcal{F}_7$. □

In the following we obtain a bound for ρ^o in terms of the maximum degree.

Theorem 2.3. *If G is a connected graph of order $n \geq 3$, then $\rho^o(G) \leq n - \Delta(G)+1$. Further, the equality holds if and only if $\Delta(G) = n-1$ and $\delta(G) = 1$.*

Proof. If S is an open packing set of G, then for any vertex u of G, we have $|N(u) \cap S| \leq 1$; this means that $|V - S| \geq deg u - 1$ and consequently $|S| \leq n + 1 - deg u$. Hence the inequality follows.

Now, suppose that $\rho^o(G) = n - \Delta(G) + 1$. Let v be a vertex of G with deg $v = \Delta(G)$. Consider a ρ^o- set D of G. Since $|V - D| = \Delta(G)-1$ and any vertex in $V - D$ has at most one neighbour in D, it follows that $v \in D$. Also, exactly one neighbour of v, say w, lies in D. This means that $V - D = N(v) - \{w\}$. Since G is connected and D is an open packing set, we have $D = \{v, w\}$ and w is a pendant vertex. Therefore, $\Delta(G) = n - 1$ and $\delta(G) = 1$. Conversely, if $\Delta(G) = n - 1$ and $\delta(G) = 1$, then it follows from Theorem 1.2 that $\rho^o(G) = 2 = n - \Delta(G) + 1$. □

In the following theorem, we characterize the family of trees T of order $n \geq 4$ for which $\rho^o(T) = n - \Delta(T)$. In this connection, we define $\Im$ to be the family of trees that are obtained from a star on at least 3 vertices by subdividing each edge of the star at most two times such that (i) at least one edge of the star is subdivided (ii) not all the edges of the star are subdivided twice.

Theorem 2.4. *Let T be a tree of order $n \geq 4$. Then $\rho^o(T) = n - \Delta(T)$ if and only if $T \in \Im$ or T is one of the graphs T_1, T_2 and T_3 given in Figure 3.*

Proof. Suppose that $\rho^o(T) = n - \Delta(T)$. Certainly, $\rho^o(T) \geq 2$. Let u be a vertex of T such that $deg\ u = \Delta(T)$ and let S be a ρ^o-set of T. We prove the theorem by considering the following two cases.

Case 1. $u \in S$

Since S contains at least two vertices and $|V - S| = \Delta(T)$, exactly one neighbour of u, say v, lies in S. Let $x \notin N(u)$ be the vertex of T lying in $V - S$. Now, the definition of open packing set implies that $|S| \leq 4$. Suppose that $|S| = 2$. Then $S = \{u, v\}$. Certainly, the vertex x is adjacent either to v or to exactly one vertex lying in $N(u) \cap (V - S)$. In either case $T \cong T_1$. Suppose that $|S| = 3$. Let w be the vertex of S other than u and v. Then x must be adjacent with w and it has exactly one neighbour in $N(u) \cap (V - S)$. In this case $T \cong T_2$. If $|S| = 4$, let w_1 and w_2 be the vertices of S distinct from u and v. Then $w_1 w_2 \in E(T)$. Also, x is adjacent to exactly one of w_1 and w_2; and it has exactly one neighbour in $N(u) \cap (V - S)$. This implies that $T \cong T_3$.

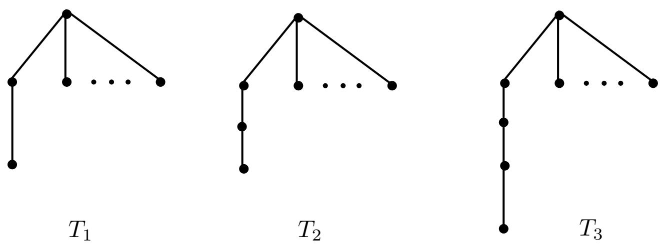

Figure 3: The trees T_1, T_2 and T_3 described in Theorem 2.4

Case 2. $u \notin S$

Then S contains exactly one neighbour of the vertex u. Since no two vertices of S have a common neighbour in T and T is a tree, it follows that exactly one vertex of each component of $\langle S \rangle$ has exactly one neighbour in $V - S$ and those neighbours are distinct. Therefore $T \in \Im$.

Conversely, suppose $T \in \Im$ or T is one of the graphs T_1, T_2 and T_3 given in Figure 3. We need to verify that $\rho^o(T) = n - \Delta(T)$. The result is easily follows for the case that T is isomorphic to one of the graphs T_1, T_2 and T_3. So, assume that $T \in \Im$. Then $\Delta(T) \neq n - 1$ and thus $\rho^o(T) \leq n - \Delta(T)$ by Theorem 2.3. Let v be the center vertex of the star. Obviously, all the vertices in $V(T) - N[v]$ are together with exactly one vertex from $N(v)$ form an open packing set of T so that $\rho^o(T) \geq |V(T) - N[v]| + 1 = n - (\Delta(T) + 1) - 1 = n - \Delta(T)$ and hence $\rho^o(T) = n - \Delta(T)$. □

3. More Results

Here we discuss how the open packing number is related with some existing graph theoretic parameters such as diameter, independence number and the equivalence number.

Theorem 3.1. *Let G be a connected graph with $diam(G)$=k. Then $\rho^o(G) \geq \lceil \frac{k+1}{3} \rceil$ and the bound is sharp.*

Proof. Let P=$(v_1, v_2, ..., v_{k+1})$ be a diametrical path of length k in G. Consider the set $S = \{v_{3i+1} : 0 \leq i \leq \lfloor \frac{k}{3} \rfloor\}$. Since distance between any two vertices belonging to S is at least 3, we have S is an open packing set of G with cardinality $\lceil \frac{k+1}{3} \rceil$ and so the result follows. A graph attaining this bound is given in Figure 4. □

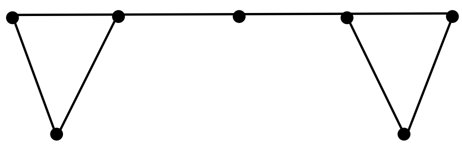

Figure 4: A graph G with $diam(G) = 4$ and $\rho^o(G) = 2$

Theorem 3.2. *Let G be a graph of order at least 2. Then $\rho^o(G) \leq 2\alpha(G)$. Further, equality holds if and only if each component of G is K_2.*

Proof. As each component of the subgraph induced by a ρ^o- set S of G is either K_1 or K_2, the set consisting of exactly one vertex from each component of $\langle S \rangle$ is an independent set of G and so the inequality follows.

Now, suppose that $\rho^o(G) = 2\alpha(G)$. Let S be any ρ^o- set of G. Then $\langle S \rangle$ has no isolates. So, it is enough to prove that $V - S = \emptyset$. On the contrary, suppose that there is a vertex w in $V - S$. Then w has exactly one neighbour in S, say u. Let v be the neighbour of u in S. Now, an independent set of cardinality more than $\frac{n}{2}$ is possible by choosing one vertex from each component of $\langle S - \{u, v\} \rangle$ along with the vertices v and w, which is a contradiction. Thus $G = \langle S \rangle = \cup K_2$. The converse is just a simple verification. □

The concept of equivalence set was introduced in [2]. A subset S of V is called an *equivalence set* if every component of the induced subgraph $\langle S \rangle$ is complete. The *equivalence number* $\beta_e(G)$ is the maximum cardinality of an equivalence set of G. The following theorem connects ρ^o and the equivalence number β_e.

Theorem 3.3. *For any graph G, we have $\rho^o(G) \leq \beta_e(G)$. Further,*

(i) *If G is a connected graph of order at least two with $\rho^o(G) = \beta_e(G)$, then $\rho^o(G)$ is even. Further, given even integer $k \geq 2$, there exists a connected graph G for which $\rho^o(G) = \beta_e(G) = k$.*

(ii) *Given integers a and b with $a \leq b-1$ and $b \geq 3$, there exists a connected graph G for which $\rho^o(G) = a$ and $\beta_e(G) = b$.*

Proof. Every open packing set of G is also an equivalence set of G because each component of the subgraph induced by an open packing set is either K_1 or K_2. This establishes the inequality.

(i) Let G be a connected graph of order at least 2 with $\rho^o(G) = \beta_e(G)$. Suppose that $\rho^o(G)$ is an odd integer. Then $\langle S \rangle$ has an isolated vertex, say u. Being G connected, the vertex u has a neighbour in $V - S$, say v. As no two vertices of S have a common neighbour, v has no neighbour in S other than u. This implies that $S \cup \{v\}$ is an equivalence set of G so that $\beta_e(G) > |S|$, a contradiction.

Now, let $k \geq 2$ be an even integer. We construct a graph G with $\rho^o(G) = \beta_e(G) = k$ as follows. Consider a path $P = (v_1, v_2, ..., v_k)$ on k vertices. Add k vertices, say v'_1, v'_2, ... , v'_k, and join by an edge the vertex v'_i with v_i for each $i = 1, 2, ..., k$. Also join by an edge the vertex v_{2i-1} with the vertex v_{2i}, for each i with $1 \leq i \leq \frac{k}{2}$. Finally, add the edges $v'_1v'_2$, $v'_3v'_4$..., $v'_{k-1}v'_k$. Let G be the resultant graph. For $k = 6$, the graph G is illustrated in Figure 5.

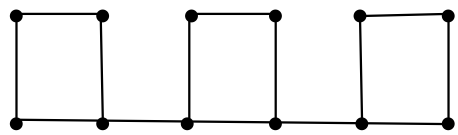

Figure 5: A graph G with $\rho^o(G) = \beta_e(G) = 6$

Obviously, the set $S = \{v'_i : 1 \leq i \leq k\}$ forms a maximal open packing set as well as a maximal equivalence set of G so that $\rho^o(G) \geq k$ and $\beta_e(G) \geq k$. On the other hand, any maximal equivalence set D of G can have at most two vertices from each cycle(on 4 vertices) of G and so $|D| \leq k$. Therefore $\beta_e(G) \leq k$. The inequality $\rho^o(G) \leq k$ follows from the fact that $\rho^o(G) \leq \beta_e(G)$.

(ii) Let a and b be positive integers with $a \leq b - 1$ and $b \geq 3$. When $a = 1$, take $G = K_b$ and when $a \geq 2$, let G be the graph obtained from the complete graph K_b by attaching exactly one pendant edge at a vertices of K_b. Now, it is an easy verification that $\rho^o(G) = a$ and $\beta_e(G) = b$. □

4. Split Graphs

We present here an upper bound for the open packing number for the family of split graphs in terms of order and characterize those split graphs attaining the bound. Recall that, a graph G is said to be a *split graph* if the vertex set

$V(G)$ can be partitioned into two non-empty sets V_1 and V_2 such that $\langle V_1 \rangle$ is complete and $\langle V_2 \rangle$ is totally disconnected. Here (V_1, V_2) is called a *split partition* of G.

Proposition 4.1. *If G is a connected split graph of order $n \geq 4$, then $\rho^o(G) \leq \frac{n}{2}$.*

Proof. Let (V_1, V_2) be a split partition of G, where V_1 is a clique and V_2 is independent. If $|V_1| = 1$, then $G \cong K_{1,n-1}$ and so $\rho^o(G) = 2 \leq \frac{n}{2}$ as $n \geq 4$. Now, assume that $|V_1| \geq 2$. If $|V_1| = 2$ and G contains no triangle, then $\rho^o(G) = 2 \leq \frac{n}{2}$. So, assume the case that G is a split graph with the property that either $|V_1| > 2$ or G has a triangle when $|V_1| = 2$. Let S be a ρ^o-set of G. Being G connected, S contains at most one vertex of V_1. If S contains a vertex of V_1, then it can have at most one vertex of V_2 and so $\rho^o(G) \leq 2 \leq \frac{n}{2}$ as $n \geq 4$. Suppose $S \cap V_1 = \phi$. Certainly, every vertex of V_2 is adjacent to a vertex of V_1 and therefore S may contain maximum $|V_1|$ vertices of V_2; for otherwise two of its vertices will have a neighbour in common. Hence $\rho^o(G) \leq min\{|V_1|, |V_2|\}$. Now, if $|V_1| \leq |V_2|$, then $2|V_1| \leq |V_1| + |V_2| = n$ and so $|V_1| \leq \frac{n}{2}$. If $|V_1| \geq |V_2|$, then $2|V_2| \leq n$ so that $|V_2| \leq \frac{n}{2}$. Thus $\rho^o(G) \leq \frac{n}{2}$. □

In the following theorem, we characterize the split graphs of even order attaining the bound given in Proposition 4.1.

Theorem 4.2. *Let G be a connected split graph of even order $n \geq 4$. Then $\rho^o(G) = \frac{n}{2}$ if and only if G is one of the graphs $K_{1,3}$, $K_{\frac{n}{2}} \circ K_1$ and the graph H of Figure 1.*

Proof. The graphs $K_{1,3}$, K_4, $K_4 - e$, $P_4 \cong K_2 \circ K_1$ and the graph H of Figure 1 are the only connected split graphs on 4 vertices. For all these graphs but K_4 and $K_4 - e$, the value of ρ^o is 2. That is, P_4, $K_{1,3}$ and the graph H are the only connected split graphs of order $n = 4$ with $\rho^o(G) = 2 = \frac{n}{2}$. So, consider a connected split graph G of even order $n \geq 6$. Let (V_1, V_2) be a split partition of G and let S be a ρ^o- set of G. Being G a connected split graph, every vertex of V_2 has a neighbour in V_1. Therefore, if we prove that $S \subseteq V_2$, then every vertex of V_2 lying in S has exactly one neighbour in V_1 and that are distinct so that $|V_1| = |V_2| = \frac{n}{2}$ as $|S| = \frac{n}{2}$; which will imply that $G = K_{\frac{n}{2}} \circ K_1$. So, let us prove that $S \subseteq V_2$. Certainly, S can have at most two vertices of V_1. If $|S \cap V_1| = 2$, then $|V_1| = 2$. As $|S| \geq 3$, S must contain at least one vertex of V_2 and since this vertex is adjacent to a vertex of V_1, the set S is no longer an open packing set of G, a contradiction. Now, if $|S \cap V_1| = 1$, then S has at least two vertices of V_2. If x is the vertex lying in $V_1 \cap S$; and u and v are vertices lying in $V_2 \cap S$, then one of u and v, say v, has a neighbour y in V_1 other than x. Now, y is a common neighbour to both x and v, a contradiction. Thus $S \subseteq V_2$ as desired. □

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