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New Approaches to Duals of Fourier-Like Systems

Elnaz Osgooei

Faculty of Science, Urmia University of Technology, Urmia, Iran.

E-mail: e.osgooei@uut.ac.ir; osgooei@yahoo.com

Abstract. The sequences of the form $\{E_{mb}g_n\}_{m,n\in\mathbb{Z}}$, where E_{mb} is the modulation operator, $b > 0$ and g_n is the window function in $L^2(\mathbb{R})$, construct Fourier-like systems. We try to consider some sufficient conditions on the window functions of Fourier-like systems, to make a frame and find a dual frame with the same structure. We also extend the given two Bessel Fourier-like systems to make a pair of dual frames and prove that the window functions of Fourier-like Bessel sequences share the compactly supported property with their extensions. But for polynomials windows, a result of this type does not happen.

Keywords: Fourier-like systems, Shift-invariant systems, A pair of dual frames, Polynomials.

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1. Introduction

In practice, it is so important to be able to find and control the properties of the associated dual frames and be sure that the structure of the obtained dual frames is the same as primary one.
Using the modulation operator $E_b : L^2(\mathbb{R}) \to L^2(\mathbb{R}), (E_bf)(x) = e^{2\pi ibx}f(x)$, for $b > 0$, and $g_n \in L^2(\mathbb{R})$, we can write the system of functions considered as $\{E_{mb}g_n\}_{m,n\in\mathbb{Z}}$. We call such a system a Fourier-like system, with generators g_n. Via the Fourier transform, Fourier-like systems are equivalent to the well-known shift-invariant systems. We also note that Fourier-like systems

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are closely related to Gabor systems. Denoting the translation operator by $T_a : L^2(\mathbb{R}) \to L^2(\mathbb{R}), (T_a f)(x) = f(x-a)$, the Gabor systems generated by a function $g \in L^2(\mathbb{R})$ and two parameters $a, b > 0$ is $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$. This corresponds to a Fourier-like system with $g_n = T_{na}g$.
Our main goal is to construct a pair of dual Fourier-like systems with generators that are easy to use in practice. In other words, we want to make a pair of dual frames $\{E_{mb}g_n\}_{m,n\in\mathbb{Z}}$ and $\{E_{mb}h_n\}_{m,n\in\mathbb{Z}}$ with compactly supported generators for which the support of h_n is controlled by the given generators g_n. Our paper is organized as follows. In Section 2 by assuming that the window functions $\{g_n\}_{n\in\mathbb{Z}}$ form a partition of unity, we use a method based on a family of polynomials which is closely related to the Daubechies polynomials to construct a pair of dual frames. In Section 3 by extending the given two Fourier-like Bessel sequences with compactly supported generators, we construct a pair of dual frames with the same property of generators.
In the rest of this introduction we collect some definitions and basic results that are needed in this paper.

Definition 1.1. A sequence $\{f_k\}_{k=1}^{\infty}$ of a separable Hilbert space H is said to be a frame for H if there exist constants $A, B > 0$ such that

$$A\|f\|^2 \leq \sum_{k=1}^{\infty} |\langle f, f_k\rangle|^2 \leq B\|f\|^2, \quad f \in H. \tag{1.1}$$

We say that $\{f_k\}_{k=1}^{\infty}$ is a Bessel sequence for H with Bessel bound B, if the second inequality in (1.1) satisfied.

Definition 1.2. Suppose that $\{f_k\}_{k=1}^{\infty}$ and $\{g_k\}_{k=1}^{\infty}$ are two Bessel sequences. We say that they are dual if

$$f = \sum_{k=1}^{\infty} \langle f, f_k\rangle g_k, \quad f \in H.$$

For more information on frame theory we refer to [1, 8, 9, 10, 11]. Given a collection of functions $\{g_n\}_{n\in\mathbb{Z}}$ in $L^2(\mathbb{R})$ and a positive number b, we consider frame properties for the system of functions given by $\{E_{mb}g_n\}_{m,n\in\mathbb{Z}}$. In particular, we find conditions on that system, having dual of the form $\{E_{mb}h_n\}_{m,n\in\mathbb{Z}}$ with the functions h_n given by a clear formula. Via the Fourier transform, the results have immediate consequences for shift-invariant systems, this means that by the following formula:

$$\mathcal{F}E_{mb}f = T_{mb}\mathcal{F}f,$$

where

$$(\mathcal{F}f)(\gamma) = \hat{f}(\gamma) = \int_{-\infty}^{\infty} f(x)e^{-2\pi i x\gamma}dx,$$

the analysis of the system of functions $\{E_{mb}g_n\}_{m,n\in\mathbb{Z}}$ corresponds to analyzing the shift-invariant system $\{T_{mb}\hat{g_n}\}_{m,n\in\mathbb{Z}}$ on the Fourier side. Thus, all of the

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frame properties for the system $\{E_{mb}g_n\}_{m,n\in\mathbb{Z}}$ can easily be derived based on the results of shift-invariant systems. The well-known results for duals of shift-invariant systems given by Ron and Shen [11] and Janssen [10].

Lemma 1.3. *([6]) Let $g_n \in L^2(\mathbb{R})$, $b > 0$ and suppose that*

$$B = \frac{1}{b}\sup_{x\in\mathbb{R}}\sum_{k\in\mathbb{Z}}|\sum_{n\in\mathbb{Z}} g_n(x)\overline{g_n(x-\frac{k}{b})}| < \infty.$$

Then $\{E_{mb}g_n; m,n \in \mathbb{Z}\}$ is a Bessel sequence with upper frame bound B. If also

$$A = \frac{1}{b}inf_{x\in\mathbb{R}}(\sum_{n\in\mathbb{Z}}|g_n(x)|^2 - \sum_{k\neq 0}|\sum_{n\in\mathbb{Z}} g_n(x)\overline{g_n(x-\frac{k}{b})}|) > 0,$$

then $\{E_{mb}g_n; m,n \in \mathbb{Z}\}$ is a frame for $L^2(\mathbb{R})$.

Lemma 1.4. *([6]) Two Bessel sequences $\{E_{mb}g_n\}_{m,n\in\mathbb{Z}}$ and $\{E_{mb}h_n\}_{m,n\in\mathbb{Z}}$ form a pair of dual frames for $L^2(\mathbb{R})$ if and only if*

$$\sum_{n\in\mathbb{Z}}\overline{g_n(x+\frac{k}{b})}h_n(x) = b\delta_{k,0}, \quad k \in \mathbb{Z}. \tag{1.2}$$

Our results in Theorems 2.1 and 2.5 is based on the following theorem.

Theorem 1.5. *([3]) Let $K \in \mathbb{N}_0$ and let $b \in]0, \frac{1}{4K+2}]$. Let g be a real-valued bounded function with suppg $\subseteq [-(2K+1), 2K+1]$, for which*

$$|\sum_{\ell\in\mathbb{Z}} g(x+\ell)| \geq A, \quad x \in [0,1],$$

for a constant $A > 0$. Define $\tilde{G}$ by

$$\tilde{G}(x) = \sum_{\ell\in\mathbb{Z}} g(x+2\ell), \quad x \in [-1,1],$$

Take $\tilde{H} \in L^2(\mathbb{R})$ to satisfy

$$\tilde{G}(x-1)\tilde{H}(x-1) + \tilde{G}(x)\tilde{H}(x) = 1, \ a.e.\ x \in [0,1].$$

Define h by

$$h(x) = b\sum_{\ell=-K}^{K}\tilde{H}(x+2\ell).$$

Then $\{E_{mb}T_ng\}_{m,n\in\mathbb{Z}}$ and $\{E_{mb}T_nh\}_{m,n\in\mathbb{Z}}$ form a pair of dual frames for $L^2(\mathbb{R})$, and h is supported in $[-(2K+1), 2K+1]$.

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2. Duals of the System $\{E_{mb}g_n\}_{m,n\in\mathbb{Z}}$

In the following theorem, we consider some verifiable conditions on functions g_n. Then we try to find functions h_n such that $\{E_{mb}g_n\}_{m,n\in\mathbb{Z}}$ and $\{E_{mb}h_n\}_{m,n\in\mathbb{Z}}$ form a pair of dual frames for $L^2(\mathbb{R})$ with compactly supported generators.

Theorem 2.1. *Let $\{x_n\}_{n\in\mathbb{Z}} \subseteq \mathbb{R}$ be a sequence such that $\lim_{n\to\pm\infty} x_n = \pm\infty$, $x_{n-1} \leq x_n$ and $x_{n+N} - x_n \leq M$, where $M > 0$ and $N \in \mathbb{N}$. Let $g_n \in L^2(\mathbb{R})$ be real-valued and uniformly bounded functions with $suppg_n \subseteq [x_n, x_{n+N}]$, and there exists constant $A > 0$ such that $|g_n(x)| \geq A$, $x \in [x_n, x_{n+N}]$. Then there exist functions h_n such that for $0 < b \leq \frac{1}{M}$, $\{E_{mb}g_n\}_{m,n\in\mathbb{Z}}$ and $\{E_{mb}h_n\}_{m,n\in\mathbb{Z}}$ form a pair of dual frames for $L^2(\mathbb{R})$.*

Proof. Take $h_n(x) = \frac{b}{Ng_n(x)}$ on $[x_n, x_{n+N}]$ and $h_n(x) = 0$, outside $[x_n, x_{n+N}]$. By assumption, the functions h_n are real-valued and uniformly bounded. By Lemma 1.3, $\{E_{mb}g_n\}_{m,n\in\mathbb{Z}}$ and $\{E_{mb}h_n\}_{m,n\in\mathbb{Z}}$ are Bessel sequences. Now, it is enough to show that (1.2) holds. Since the functions g_n and h_n have compact support in $[x_n, x_{n+N}]$ and $0 < b \leq \frac{1}{M}$, for $k \neq 0$, $\sum_{n\in\mathbb{Z}} g_n(x + \frac{k}{b})h_n(x) = 0$. Given $x \in \mathbb{R}$, choose $n \in \mathbb{Z}$ such that $x \in [x_n, x_{n+N}]$. By the definition of h_n, the following N equations are satisfied in each intervals:

$$\begin{cases} g_{n-N+1}(x)h_{n-N+1}(x) + ... + g_n(x)h_n(x) & = b, \quad x \in [x_n, x_{n+1}]; \\ g_{n-N+2}(x)h_{n-N+2}(x) + ... + g_{n+1}(x)h_{n+1}(x) & = b, \quad x \in [x_{n+1}, x_{n+2}]; \\ & . \\ & . \\ & . \\ g_n(x)h_n(x) + ... + g_{n-N+1}(x)h_{n+N-1}(x) & = b, \quad x \in [x_{n+N-1}, x_{n+N}]. \end{cases}$$

Therefore,

$$\sum_{n\in\mathbb{Z}} g_n(x)h_n(x) = b, \quad a.e.\ x \in \mathbb{R}, \quad k \in \mathbb{Z}.$$

So we get the proof. □

Example 2.2. Let $N \in \mathbb{N}$ and $0 < b \leq \frac{1}{N}$. Suppose that $g_n(x) = 2 + \sin(\frac{n\pi x}{3})$ on $[n, n+N]$ and $g_n(x) = 0$ outside $[n, n+N]$. Clearly, $g_n \in L^2(\mathbb{R})$ for each $n \in \mathbb{Z}$. First we need to show that $\{E_{mb}g_n\}_{m,n\in\mathbb{Z}}$ is a Bessel sequence for $L^2(\mathbb{R})$. Since $suppg_n \subseteq [n, n+N]$ and $\frac{1}{b} \geq N$,

$$\sum_{k\neq 0} |\sum_{n\in\mathbb{Z}} g_n(x)g_n(x - \frac{k}{b})| = 0.$$

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Given $x \in \mathbb{R}$, choose $n \in \mathbb{Z}$ such that $x \in [n, n+1]$. Then

$$\sum_{k\in\mathbb{Z}} |\sum_{n\in\mathbb{Z}} g_n(x)g_n(x-\frac{k}{b})| = |\sum_{\ell=0}^{N-1}(g_{n-\ell}(x))^2| \leq 9N.$$

So,

$$B = \frac{1}{b}\sup_{x\in\mathbb{R}}\sum_{k\in\mathbb{Z}} |\sum_{n\in\mathbb{Z}} g_n(x)g_n(x-\frac{k}{b})| < \infty,$$

and by Lemma 1.3, $\{E_{mb}g_n\}_{m,n\in\mathbb{Z}}$ is a Bessel sequence for $L^2(\mathbb{R})$. Now, we have to show that the lower frame condition is satisfied. Similar to the above discussion

$$\sum_{k\neq 0} |\sum_{n\in\mathbb{Z}} g_n(x)g_n(x-\frac{k}{b})| = 0.$$

Given $x \in \mathbb{R}$, choose $n \in \mathbb{Z}$ such that $x \in [n, n+1]$. Then

$$(\sum_{n\in\mathbb{Z}} |g_n(x)|^2 - \sum_{k\neq 0} |\sum_{n\in\mathbb{Z}} g_n(x)g_n(x-\frac{k}{b})|) = \sum_{\ell=0}^{N-1} |g_{n-\ell}(x)|^2 \geq N.$$

So,

$$A = \frac{1}{b}\inf_{x\in\mathbb{R}}(\sum_{n\in\mathbb{Z}} |g_n(x)|^2 - \sum_{k\neq 0} |\sum_{n\in\mathbb{Z}} g_n(x)g_n(x-\frac{k}{b})|) > 0.$$

Hence, by Lemma 1.3, $\{E_{mb}g_n\}_{m,n\in\mathbb{Z}}$ is a frame for $L^2(\mathbb{R})$. Now if we take $h_n(x) = \frac{b}{N(2+\sin\frac{n\pi x}{3})}$ on $[n, n+N]$ and $h_n(x) = 0$ outside $[n, n+N]$, then we can see that $\{E_{mb}g_n\}_{m,n\in\mathbb{Z}}$ and $\{E_{mb}h_n\}_{m,n\in\mathbb{Z}}$ form a pair of dual frames for $L^2(\mathbb{R})$.

Motivated by Theorem 1.5, in the following theorems, we consider functions g_n such that $suppg_n \subseteq [x_n, x_{n+N}]$ and for $x \in [x_n, x_{n+1}]$, $|\sum_{n\in\mathbb{Z}} g_n(x)| \geq A$ for some constant $A > 0$. Then we try to find structure of the functions h_n such that $\{E_{mb}g_n\}_{m,n\in\mathbb{Z}}$ and $\{E_{mb}h_n\}_{m,n\in\mathbb{N}}$ be a pair of dual frames for $L^2(\mathbb{R})$.

Theorem 2.3. *Let $\{x_n\}_{n\in\mathbb{Z}} \subseteq \mathbb{R}$ be the sequence as mentioned in Theorem 2.1. Suppose that g_n are real-valued and uniformly bounded functions with $suppg_n \subseteq [x_n, x_{n+N}]$. Assume that for some constant $A > 0$*

$$|\sum_{\ell=n}^{n+N-1} g_\ell(x)| \geq A, \quad x \in [x_{n+N-1}, x_{n+N}], \quad n \in \mathbb{Z}. \tag{2.1}$$

Then there exist bounded functions h_n with $supph_n \subseteq [x_n, x_{n+N}]$, such that for $0 < b \leq \frac{1}{M}$, $\{E_{mb}g_n\}_{m,n\in\mathbb{Z}}$ and $\{E_{mb}h_n\}_{m,n\in\mathbb{N}}$ are a pair of dual frames for $L^2(\mathbb{R})$.

Proof. Take $h_n(x) = 0$ outside $[x_n, x_{n+N}]$. Since the functions g_n and h_n have compact support in $[x_n, x_{n+N}]$ and $0 < b \leq \frac{1}{M}$, for $k \neq 0$

$$\sum_{n\in\mathbb{Z}} g_n(x+\frac{k}{b})h_n(x) = 0.$$

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Now, it is enough to show that

$$\sum_{n\in\mathbb{Z}} g_n(x)h_n(x) = b, \quad a.e.\ x \in \mathbb{R}. \tag{2.2}$$

Given $x \in \mathbb{R}$, choose $n \in \mathbb{Z}$ such that $x \in [x_n, x_{n+N}]$. Therefore, we have to show that these equations are satisfied on each intervals:

$$g_{n-N+1}(x)h_{n-N+1}(x) + ... + g_n(x)h_n(x) = b, \quad x \in [x_n, x_{n+1}].$$

$$g_{n-N+2}(x)h_{n-N+2}(x) + ... + g_{n+1}(x)h_{n+1}(x) = b, \quad x \in [x_{n+1}, x_{n+2}].$$

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$$g_n(x)h_n(x) + ... + g_{n+N-1}(x)h_{n+N-1}(x) = b, \quad x \in [x_{n+N-1}, x_{n+N}].$$

By (2.1) for each $n \in \mathbb{Z}$, there exists at least one $k \in \{n, ..., n+N-1\}$, such that

$$|g_k(x)| \geq \frac{A}{N}.$$

Suppose that $k_1, k_2, ..., k_L \in \{n, ..., n+N-1\}$, are such that

$$|g_{k_p}(x)| \geq \frac{A}{N}, \quad 1 \leq p \leq L.$$

Then take $h_{k_p}(x) = \frac{b}{Lg_{k_p}(x)}$, $1 \leq p \leq L$ and take $h_t(x) = 0$ for $t \in \{n, ..., n+N-1\} - \{k_1, ...k_L\}$. Therefore (2.2) holds and so $\{E_{mb}g_n\}_{m,n\in\mathbb{Z}}$ and $\{E_{mb}h_n\}_{m,n\in\mathbb{Z}}$ are a pair of dual frames for $L^2(\mathbb{R})$. □

Suppose that g_n are real-valued and uniformly bounded functions such that $supp g_n \subseteq [x_n, x_{n+2N}]$, where $N \in \mathbb{N}$. We define the functions

$$\tilde{G}_n(x) = \sum_{\ell\in\mathbb{Z}} g_{n-2\ell}(x), \quad x \in [x_n, x_{n+2}], \tag{2.3}$$

which due to the compact support of g_n, can also be written as

$$\tilde{G}_n(x) = \sum_{\ell=0}^{N-1} g_{n-2\ell}(x), \quad x \in [x_n, x_{n+2}]. \tag{2.4}$$

Lemma 2.4. *Let $\tilde{G}_n$ be real-valued and uniformly bounded functions. Assume that for some constant $A > 0$*

$$|\tilde{G}_{n-1}(x)| + |\tilde{G}_n(x)| \geq A, \quad x \in [x_n, x_{n+1}]. \tag{2.5}$$

Then there exist real-valued bounded functions $\tilde{H}_n$ with $supp\tilde{H}_n \subseteq supp\tilde{G}_n \cap [x_n, x_{n+2}]$ such that

$$\tilde{G}_{n-1}(x)\tilde{H}_{n-1}(x) + \tilde{G}_n(x)\tilde{H}_n(x) = 1, \quad x \in [x_n, x_{n+1}].$$

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Proof. Let $x \in [x_n, x_{n+1}]$. If $|\tilde{G}_{n-1}(x)| \geq \frac{A}{2}$, we put $\tilde{H}_{n-1}(x) = \frac{1}{\tilde{G}_{n-1}(x)}$ and $\tilde{H}_n(x) = 0$. On the other hand if $|\tilde{G}_{n-1}(x)| < \frac{A}{2}$, then $|\tilde{G}_n(x)| \geq \frac{A}{2}$. In this case, we put $\tilde{H}_{n-1}(x) = 0$ and $\tilde{H}_n(x) = \frac{1}{\tilde{G}_n(x)}$. We can take $\tilde{H}_n = 0$ outside $[x_n, x_{n+2}]$ for each $n \in \mathbb{Z}$. □

Theorem 2.5. *Let $\{x_n\}_{n\in\mathbb{Z}} \subseteq \mathbb{R}$ be a sequence such that*

$$\lim_{n\to\pm\infty} x_n = \pm\infty, \quad x_{n-1} \leq x_n, \quad x_{n+2N} - x_n \leq M, \ n \in \mathbb{Z},$$

where $M > 0$ and $N \in \mathbb{N}$. Let $g_n \in L^2(\mathbb{R})$ be real-valued and uniformly bounded functions with $supp g_n \subseteq [x_n, x_{n+2N}]$ such that

$$|\sum_{\ell=0}^{2N-1} g_{n-\ell}(x)| \geq A, \quad x \in [x_n, x_{n+1}], \tag{2.6}$$

for a constant $A > 0$. Then the following hold:

(i) *The functions $\tilde{G}_n$ in (2.3) satisfy the conditions in Lemma 2.4.*

(ii) *Take $\tilde{H}_n$ as Lemma 2.4 and let*

$$h_n(x) = b \sum_{\ell=0}^{N-1} \tilde{H}_{n+2\ell}(x). \tag{2.7}$$

Then for $0 < b \leq \frac{1}{M}$, $\{E_{mb} g_n\}_{m,n\in\mathbb{Z}}$ and $\{E_{mb} h_n\}_{m,n\in\mathbb{Z}}$ form a pair of dual frames for $L^2(\mathbb{R})$ and h_n is supported in $[x_n, x_{n+2N}]$.

Proof. (i) By the definition of $\tilde{G}_n$ and (2.6), for $x \in [x_n, x_{n+1}]$, $n \in \mathbb{Z}$, we have

$$\begin{aligned} |\tilde{G}_{n-1}(x)| + |\tilde{G}_n(x)| &= |\sum_{\ell=0}^{N-1} g_{n-1-2\ell}(x)| + |\sum_{\ell=0}^{N-1} g_{n-2\ell}(x)| \\ &\geq |\sum_{\ell=0}^{N-1} g_{n-(1+2\ell)}(x) + \sum_{\ell=0}^{N-1} g_{n-2\ell}(x)| \\ &= |\sum_{\ell=0}^{2N-1} g_{n-\ell}(x)| \geq A. \end{aligned}$$

Thus $\tilde{G}_n$ satisfies the condition (2.5). Therefore, we can choose $\tilde{H}_n$ as in Lemma 2.4 such that

$$\tilde{G}_{n-1}(x)\tilde{H}_{n-1}(x) + \tilde{G}_n(x)\tilde{H}_n(x) = 1, \quad x \in [x_n, x_{n+1}]. \tag{2.8}$$

(ii) Since g_n are real-valued and uniformly bounded functions with compact support, by the construction of h_n, h_n share these properties and $supp h_n \subseteq [x_n, x_{n+2N}]$. By Lemma 1.3 $\{E_{mb} g_n\}_{m,n\in\mathbb{Z}}$ and $\{E_{mb} h_n\}_{m,n\in\mathbb{Z}}$ are Bessel sequences. In order to verify that these sequences form a pair of dual frames, we must show that

$$\sum_{n\in\mathbb{Z}} g_n(x + \frac{k}{b}) h_n(x) = b\delta_{k,0}, \quad a.e \ \ x \in \mathbb{R}. \tag{2.9}$$

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Since g_n and h_n have compact supports in $[x_n, x_{n+2N}]$, an interval of length at most M, and $\frac{1}{b} \geq M$, (2.9) is satisfied for $k \neq 0$. For the case of $k = 0$, we have to show that

$$\sum_{n\in\mathbb{Z}} g_n(x)h_n(x) = b, \quad a.e \quad x \in \mathbb{R}.$$

Given $x \in \mathbb{R}$, choose $n \in \mathbb{Z}$ such that $x \in [x_n, x_{n+1}]$. Then

$$\begin{aligned}\sum_{n\in\mathbb{Z}} g_n(x)h_n(x) = \sum_{\ell=0}^{2N-1} g_{n-\ell}(x)h_{n-\ell}(x) &= \sum_{\ell=0}^{N-1} g_{n-(2\ell+1)}(x)h_{n-(2\ell+1)}(x) \\ &+ \sum_{\ell=0}^{N-1} g_{n-2\ell}(x)h_{n-2\ell}(x). \quad (2.10)\end{aligned}$$

For each $\ell \in \{0, 1, ..., N-1\}$, if $x \in [x_n, x_{n+1}]$, then by the definitions of $\tilde{H}_n$ and h_n,

$$h_{n-2\ell}(x) = b\tilde{H}_n(x), \quad h_{n-(2\ell+1)}(x) = b\tilde{H}_{n-1}(x). \quad (2.11)$$

Therefore by (2.8), (2.10) and (2.11) we have

$$\sum_{n\in\mathbb{Z}} g_n(x)h_n(x) = b(\tilde{G}_n(x)\tilde{H}_n(x) + \tilde{G}_{n-1}(x)\tilde{H}_{n-1}(x)) = b.$$

Therefore, $\{E_{mb}g_n\}_{m,n\in\mathbb{Z}}$ and $\{E_{mb}h_n\}_{m,n\in\mathbb{Z}}$ form a pair of dual frames for $L^2(\mathbb{R})$. □

Example 2.6. Let $g_n(x) = 2 + \sin\frac{n\pi x}{3}$ on $[n, n+2]$ and $g_n(x) = 0$ outside $[n, n = 2]$. Suppose that $0 < b \leq \frac{1}{2}$. By the definition of $\tilde{G}_n$ in (2.4), $\tilde{G}_n(x) = g_n(x)$ on $[n, n+2]$. So for $x \in [n, n+1]$

$$|\tilde{G}_n(x)| + |\tilde{G}_{n-1}(x)| \geq 2,$$

and for $x \in [n+1, n+2]$

$$|\tilde{G}_n(x)| + |\tilde{G}_{n+1}(x)| \geq 2.$$

We define $\tilde{H}_n(x) = \frac{1}{\tilde{G}_n(x)}$ on $[n+1, n+2]$ and $\tilde{H}_n(x) = 0$ outside $[n+1, n+2]$. Therefore, $\{E_{mb}g_n\}_{m,n\in\mathbb{Z}}$ and $\{E_{mb}h_n\}_{m,n\in\mathbb{Z}}$, where h_n is as (2.7), form a dual pairs for $L^2(\mathbb{R})$.

The following lemma is a generalization of Lemma 2.2 in [7] to find other explicit structure of functions h_n such that $\{E_{mb}g_n\}_{m,n\in\mathbb{Z}}$ and $\{E_{mb}h_n\}_{m,n\in\mathbb{Z}}$ be a pair of dual frames for $L^2(\mathbb{R})$. In order to reach our goal, we use Daubechies polynomials.

For any $N \in \mathbb{N}$ the Daubechies polynomial of degree $N-1$ is given by

$$P_{N-1}(x) = \sum_{k=0}^{N-1} \binom{2N-1}{k} x^k(1-x)^{N-1-k}, \quad x \in \mathbb{R}.$$

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Daubechies, in her fundamental paper [8] showed that for each $N \in \mathbb{N}$

$$(1-x)^N P_{N-1}(x) + x^N P_{N-1}(1-x) = 1, \quad x \in \mathbb{R}.$$

Lemma 2.7. *Let $\{x_n\}_{n\in\mathbb{Z}} \subseteq \mathbb{R}$ be a sequence mentioned in Theorem 2.5. Suppose that g_n are real-valued and uniformly bounded functions with $suppg_n \subseteq [x_n, x_{n+2N}]$, where $N \in \mathbb{N}$ and*

$$\sum_{n\in\mathbb{Z}} g_n(x) = 1. \tag{2.12}$$

Define $\tilde{G}_n$ by (2.3). Then

$$\tilde{G}_n(x) + \tilde{G}_{n-1}(x) = 1, \quad x \in [x_n, x_{n+1}],$$

and we can find functions $\tilde{H}_n$ such that

$$\tilde{G}_n(x)\tilde{H}_n(x) + \tilde{G}_{n-1}(x)\tilde{H}_{n-1}(x) = 1, \quad x \in [x_n, x_{n+1}].$$

Proof. Since $suppg_n \subseteq [x_n, x_{n+2N}]$, by (2.12) we have

$$\sum_{\ell=0}^{2N-1} g_{n-\ell}(x) = 1, \quad x \in [x_n, x_{n+1}].$$

So, for $x \in [x_n, x_{n+1}]$, we have

$$\begin{aligned} \tilde{G}_n(x) + \tilde{G}_{n-1}(x) &= \sum_{\ell=0}^{N-1} g_{n-2\ell}(x) + \sum_{\ell=0}^{N-1} g_{n-(2\ell+1)}(x) \\ &= \sum_{\ell=0}^{2N-1} g_{n-\ell}(x) = 1. \end{aligned} \tag{2.13}$$

For any $M \in \mathbb{N}$, we define

$$p_{2M-2}(x) = \sum_{k=0}^{M-1} \binom{2M-1}{k} (1-x)^{2(M-1)-k} x^k.$$

By (2.13) for $x \in [x_n, x_{n+1}]$, we have

$$\begin{aligned} 1 = (\tilde{G}_{n-1}(x) + \tilde{G}_n(x))^{2M-1} &= \sum_{k=0}^{2M-1} \binom{2M-1}{k} (1-\tilde{G}_n(x))^k (\tilde{G}_n(x))^{2M-1-k} \\ &= \sum_{k=0}^{M-1} \binom{2M-1}{k} (1-\tilde{G}_n(x))^k (\tilde{G}_n(x))^{2M-1-k} \\ &+ \sum_{k=M}^{2M-1} \binom{2M-1}{k} (1-\tilde{G}_n(x))^k (\tilde{G}_n(x))^{2M-1-k}, \end{aligned}$$

by changing of the index $m = 2M-1-k$ in the second sum and note that

$$\binom{2M-1}{k} = \binom{2M-1}{2M-1-k},$$

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we have

$$\begin{aligned}
1 &= \sum_{k=0}^{M-1} \binom{2M-1}{k}(1-\tilde{G}_n(x))^k(\tilde{G}_n(x))^{2M-1-k} \\
&+ \sum_{m=0}^{M-1} \binom{2M-1}{m}(1-\tilde{G}_n(x))^{2M-1-m}(\tilde{G}_n(x))^m \\
&= \tilde{G}_n(x)\sum_{k=0}^{M-1} \binom{2M-1}{k}(1-\tilde{G}_n(x))^k(\tilde{G}_n(x))^{2(M-1)-k} \\
&+ (1-\tilde{G}_n(x))\sum_{m=0}^{M-1} \binom{2M-1}{k}(1-\tilde{G}_n(x))^{2(M-1)-m}(\tilde{G}_n(x))^m \\
&= \tilde{G}_n(x)p_{2M-2}(\tilde{G}_{n-1}(x)) + \tilde{G}_{n-1}(x)p_{2M-2}(\tilde{G}_n(x)).
\end{aligned}$$

Now if for $x \in [x_n, x_{n+1}]$, we define $\tilde{H}_n(x) = p_{2M-2}(\tilde{G}_{n-1}(x))$ and for $x \in [x_{n+1}, x_{n+2}]$, we define $\tilde{H}_n(x) = p_{2M-2}(\tilde{G}_{n+1}(x))$ and $\tilde{H}_n(x) = 0$, outside $[x_n, x_{n+2}]$, the proof is complete. □

Theorem 2.8. *Let $\{x_n\}_{n\in\mathbb{Z}} \subseteq \mathbb{R}$ be a sequence mentioned in Theorem 2.5. Suppose that g_n are real-valued and uniformly bounded functions with $suppg_n \subseteq [x_n, x_{n+2N}]$, where $N \in \mathbb{N}$ and*

$$\sum_{n\in\mathbb{Z}} g_n(x) = 1.$$

Define

$$h_n(x) = \sum_{\ell=0}^{2N-1} \tilde{H}_{n+2\ell}(x),$$

where we can choose $\tilde{H}_n$ as Lemma 2.7. Then $\{E_{mb}g_n\}_{m,n\in\mathbb{Z}}$ and $\{E_{mb}h_n\}_{m,n\in\mathbb{Z}}$ form a pair of dual frames for $L^2(\mathbb{R})$.

Proof. The proof is clear by Lemma 2.7 and Theorem 2.5. □

3. Extension of Fourier-like Bessel Sequences to Dual Pairs

For any Bessel sequences $\{f_i\}_{i\in I}$ and $\{g_i\}_{i\in I}$ in a Hilbert space H, it was shown by Christensen, et al. [2] that there exists Bessel sequences $\{p_j\}_{j\in J}$ and $\{q_j\}_{j\in J}$ in H such that $\{f_i\}_{i\in I} \cup \{p_j\}_{j\in J}$ and $\{g_i\}_{i\in I} \cup \{q_j\}_{j\in J}$ form a pair of dual frames for H. Also they showed that any pair of Gabor Bessel sequences can be extended to a pair of dual frames with Gabor structure.

Proposition 3.1. *([2]) Let $\{f_i\}_{i\in I}$ and $\{g_i\}_{i\in I}$ be Bessel sequences in a Hilbert space H. Then there exist Bessel sequences $\{p_j\}_{j\in J}$ and $\{q_j\}_{j\in J}$ in H such that $\{f_i\}_{i\in I} \cup \{p_j\}_{j\in J}$ and $\{g_i\}_{i\in I} \cup \{q_j\}_{j\in J}$ form a pair of dual frames for H.*

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In this section, motivated by Theorem 3.1 in [2] we follow our main goal. This means that we want to show that we can add the generators h_n and $\tilde{h_n}$ with compactly supported for the given compactly supported generators g_n and $\tilde{g_n}$.

Theorem 3.2. *Let $\{E_{mb}g_n\}_{m,n\in\mathbb{Z}}$ and $\{E_{mb}\tilde{g_n}\}_{m,n\in\mathbb{Z}}$ be Bessel sequences in $L^2(\mathbb{R})$. Then the following statements hold:*

(i) *There exist Fourier-like systems $\{E_{mb}h_n\}_{m,n\in\mathbb{Z}}$ and $\{E_{mb}\tilde{h_n}\}_{m,n\in\mathbb{Z}}$ in $L^2(\mathbb{R})$ such that $\{E_{mb}g_n\}_{m,n\in\mathbb{Z}}\cup\{E_{mb}h_n\}_{m,n\in\mathbb{Z}}$ and $\{E_{mb}\tilde{g_n}\}_{m,n\in\mathbb{Z}}\cup\{E_{mb}\tilde{h_n}\}_{m,n\in\mathbb{Z}}$ form a pair of dual frames for $L^2(\mathbb{R})$.*

(ii) *Let $\{x_n\}_{n\in\mathbb{Z}}\subset\mathbb{R}$ be a sequence such that $\lim_{n\to\pm\infty}x_n=\pm\infty$, $x_{n-1}\leq x_n$ and $x_{n+2N-1}-x_{n-N+1}\leq M$, where $M>0$ and $N\in\mathbb{N}$. If the functions g_n and $\tilde{g_n}$ are real-valued and uniformly bounded with compact support such that $suppg_n\subseteq[x_n,x_{n+N}]$ and $supp\tilde{g_n}\subseteq[x_n,x_{n+N}]$, then the functions h_n and $\tilde{h_n}$ can be chosen to be compactly supported as well.*

Proof. Let T and U denote the synthesis operators for $\{E_{mb}g_n\}_{m,n\in\mathbb{Z}}$ and $\{E_{mb}\tilde{g_n}\}_{m,n\in\mathbb{Z}}$, respectively. Then

$$UT^*f=\sum_{m,n\in\mathbb{Z}}\langle f,E_{mb}g_n\rangle E_{mb}\tilde{g_n}.$$

Consider the operator $\phi=I-UT^*$. Let $\{E_{mb}f_n\}_{m,n\in\mathbb{Z}}$ and $\{E_{mb}\tilde{f_n}\}_{m,n\in\mathbb{Z}}$ denote any pair of dual frames for $L^2(\mathbb{R})$ (see Appendix A in [2]). By the proof of Proposition 3.1, $\{E_{mb}g_n\}_{m,n\in\mathbb{Z}}\cup\{\phi^*E_{mb}f_n\}_{m,n\in\mathbb{Z}}$ and $\{E_{mb}\tilde{g_n}\}_{m,n\in\mathbb{Z}}\cup\{E_{mb}\tilde{f_n}\}_{m,n\in\mathbb{Z}}$ form a pair of dual frames for $L^2(\mathbb{R})$. Since $\phi^*=I-TU^*$, we have

$$\phi^*f=f-\sum_{m,n\in\mathbb{Z}}\langle f,E_{mb}\tilde{g_n}\rangle E_{mb}g_n. \tag{3.1}$$

Now it is enough to show that ϕ^* commutes with all the modulation operators E_{mb}. By (3.1) for each $m,n\in\mathbb{Z}$ we have

$$\begin{aligned}
\phi^*E_{mb}f_n(.) &= E_{mb}f_n(.)-\sum_{m',n'\in\mathbb{Z}}\langle E_{mb}f_n,E_{m'b}\tilde{g_{n'}}\rangle E_{m'b}g_{n'}(.)\\
&= E_{mb}f_n(.)-\sum_{m',n'\in\mathbb{Z}}\langle f_n,E_{(m'-m)b}\tilde{g_{n'}}\rangle E_{m'b}g_{n'}(.)\\
&= E_{mb}f_n(.)-e^{-2\pi i(m'-m)b(.)}\langle f_n,\tilde{g_{n'}}\rangle e^{2\pi im'b(.)}g_{n'}(.)\\
&= E_{mb}f_n(.)-e^{2\pi imb(.)}\langle f_n,\tilde{g_{n'}}\rangle g_{n'}(.)\\
&= E_{mb}\phi^*f_n(.).
\end{aligned}$$

Thus we conclude that $\{E_{mb}g_n\}_{m,n\in\mathbb{Z}}\cup\{E_{mb}\phi^*f_n\}_{m,n\in\mathbb{Z}}$ and $\{E_{mb}\tilde{g_n}\}_{m,n\in\mathbb{Z}}\cup\{E_{mb}\tilde{f_n}\}_{m,n\in\mathbb{Z}}$ form a pair of dual frames for $L^2(\mathbb{R})$.

We now prove (ii). By Theorem 2.2 in [5] we can choose functions f_n and $\tilde{f_n}$ to be compactly supported with $suppf_n\subseteq[x_{n-N+1},x_{n+2N-1}]$ and $supp\tilde{f_n}\subseteq$

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$[x_n, x_{n+N}]$ such that for $0 < b \leq \frac{1}{M}$, $\{E_{mb}f_n\}_{m,n\in\mathbb{Z}}$ and $\{E_{mb}\tilde{f}_n\}_{m,n\in\mathbb{Z}}$ form a pair of dual frames for $L^2(\mathbb{R})$. By part (i) we just need to show that $\phi^* f_n$ is compactly supported. Due to the compact support of the functions f_n, g_n and $\tilde{g_n}$, by (3.1) for each $n \in \mathbb{Z}$ we have

$$\begin{aligned}\phi^* f_n &= f_n - \sum_{m',n'\in\mathbb{Z}} \langle f_n, E_{m'b}\tilde{g_{n'}}\rangle E_{m'b}g_{n'} \\ &= f_n - \sum_{n'=n-N+1}^{n+N-1} \sum_{m'\in\mathbb{Z}} \langle f_n, E_{m'b}\tilde{g_{n'}}\rangle E_{m'b}g_{n'},\end{aligned}$$

which is compactly supported. □

Theorem 3.2 shows that if $\{E_{mb}g_n\}_{m,n\in\mathbb{Z}}$ and $\{E_{mb}\tilde{g_n}\}_{m,n\in\mathbb{Z}}$ are Bessel sequences with compactly supported generators. Then there exist sequences $\{E_{mb}h_n\}_{m,n\in\mathbb{Z}}$ and $\{E_{mb}\tilde{h_n}\}_{m,n\in\mathbb{Z}}$ such that $\{E_{mb}g_n\}_{m,n\in\mathbb{Z}} \cup \{E_{mb}h_n\}_{m,n\in\mathbb{Z}}$ and $\{E_{mb}\tilde{g_n}\}_{m,n\in\mathbb{Z}} \cup \{E_{mb}\tilde{h_n}\}_{m,n\in\mathbb{Z}}$ form a pair of dual frames with compactly supported generators. But in the following example we show that if the generators g_n and $\tilde{g_n}$ are polynomials, then h_n and $\tilde{h_n}$ may not be polynomials in general.

Example 3.3. Suppose that $f(x) = (25x^2 - 10x^3 + x^4)\chi_{[0,5]}(x)$ and $\tilde{f}(x) = \frac{b}{k}B_5(x)$, where $k = 103.3$, $0 \leq b \leq \frac{1}{5}$ and B_5 is a B-spline. By Example 2.9 in [4] $\{E_{mb}f_n\}_{m,n\in\mathbb{Z}}$ and $\{E_{mb}\tilde{f}_n\}_{m,n\in\mathbb{Z}}$, where $f_n = T_n f$ and $\tilde{f}_n = T_n\tilde{f}$ form a pair of dual frames. Now let $g_n(x) = (x-n)(x-n-2)(x-n-1)\chi_{[0,5]}$ and $\tilde{g_n}(x) = (x-n)(x-n+1)(x-n+2)\chi_{[0,5]}$. Then by Theorem 3.2 $\{E_{mb}g_n\}_{m,n\in\mathbb{Z}} \cup \{E_{mb}\phi^* f_n\}_{m,n\in\mathbb{Z}}$ and $\{E_{mb}\tilde{g_n}\}_{m,n\in\mathbb{Z}} \cup \{E_{mb}\tilde{f}_n\}_{m,n\in\mathbb{Z}}$ form a pair of dual frames but calculations shows that $\phi^* f_n$ is not a polynomial.

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