New Jensen and Ostrowski Type Inequalities for General Lebesgue Integral with Applications

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Abstract. Some new inequalities related to Jensen and Ostrowski inequalities for general Lebesgue integral are obtained. Applications for $f$-divergence measure are provided as well.

Keywords: Ostrowski’s inequality, Jensen’s inequality. $f$-Divergence measures.


1. Introduction

Let $(\Omega, \mathcal{A}, \mu)$ be a measurable space consisting of a set $\Omega$, a $\sigma$-algebra $\mathcal{A}$ of parts of $\Omega$ and a countably additive and positive measure $\mu$ on $\mathcal{A}$ with values in $\mathbb{R} \cup \{\infty\}$. Assume, for simplicity, that $\int_{\Omega} d\mu = 1$. Consider the Lebesgue space

$$L(\Omega, \mu) := \{ f : \Omega \rightarrow \mathbb{R}, \ f \text{ is } \mu\text{-measurable and } \int_{\Omega} |f(t)| \, d\mu(t) < \infty \}.$$ 

For simplicity of notation we write everywhere in the sequel $\int_{\Omega} w(t) \, d\mu(t)$ instead of $\int_{\Omega} w(t) \, d\mu(t)$.

In order to provide a reverse of the celebrated Jensen’s integral inequality for convex functions, S.S. Dragomir obtained in 2002 [29] the following result:
Theorem 1.1. Let \( \Phi : [m, M] \subset \mathbb{R} \to \mathbb{R} \) be a differentiable convex function on \((m, M)\) and \( f : \Omega \to [m, M] \) so that \( \Phi \circ f, f, \Phi' \circ f, (\Phi' \circ f) : f \in L(\Omega, \mu) \). Then we have the inequality:

\[
0 \leq \int_{\Omega} \Phi \circ f \, d\mu - \Phi \left( \int_{\Omega} f \, d\mu \right) \tag{1.1}
\]

\[
\leq \int_{\Omega} f : (\Phi' \circ f) \, d\mu - \int_{\Omega} \Phi' \circ f \, d\mu \int f \, d\mu
\]

\[
\leq \frac{1}{2} \left[ \Phi' (M) - \Phi' (m) \right] \int_{\Omega} \left| f - \int_{\Omega} f \, d\mu \right| \, d\mu.
\]

In the case of discrete measure, we have:

Corollary 1.2. Let \( \Phi : [m, M] \to \mathbb{R} \) be a differentiable convex function on \((m, M)\). If \( x_i \in [m, M] \) and \( w_i \geq 0 \) \((i = 1, \ldots, n)\) with \( W_n := \sum_{i=1}^{n} w_i = 1 \), then one has the counterpart of Jensen’s weighted discrete inequality:

\[
0 \leq \sum_{i=1}^{n} w_i \Phi (x_i) - \Phi \left( \sum_{i=1}^{n} w_i x_i \right) \tag{1.2}
\]

\[
\leq \sum_{i=1}^{n} w_i \Phi' (x_i) x_i - \sum_{i=1}^{n} w_i \Phi' (x_i) \sum_{i=1}^{n} w_i x_i
\]

\[
\leq \frac{1}{2} \left[ \Phi' (M) - \Phi' (m) \right] \sum_{i=1}^{n} w_i \left| x_i - \sum_{j=1}^{n} w_j x_j \right|.
\]

Remark 1.3. We notice that the inequality between the first and the second term in (1.2) was proved in 1994 by Dragomir & Ionescu, see [36].

If \( f, g : \Omega \to \mathbb{R} \) are \( \mu \)-measurable functions and \( f, g, fg \in L(\Omega, \mu) \), then we may consider the Čebyšev functional

\[
T (f, g) := \int_{\Omega} f g \, d\mu - \int_{\Omega} f \, d\mu \int_{\Omega} g \, d\mu. \tag{1.3}
\]

The following result is known in the literature as the Grüss inequality

\[
|T (f, g)| \leq \frac{1}{4} (\Gamma - \gamma) (\Delta - \delta), \tag{1.4}
\]

provided

\[-\infty < \gamma \leq f (t) \leq \Gamma < \infty, \quad -\infty < \delta \leq g (t) \leq \Delta < \infty \tag{1.5}\]

for \( \mu \)-a.e. \( t \in \Omega \).

The constant \( \frac{1}{4} \) is sharp in the sense that it cannot be replaced by a smaller quantity.

If we assume that \(-\infty < \gamma \leq f (t) \leq \Gamma < \infty \) for \( \mu \)-a.e. \( t \in \Omega \), then by the Grüss inequality for \( g = f \) and by the Schwarz’s integral inequality, we have

\[
\int_{\Omega} \left| f - \int_{\Omega} f \, d\mu \right| \, d\mu \leq \left[ \int_{\Omega} f^2 \, d\mu - \left( \int_{\Omega} f \, d\mu \right)^2 \right]^{\frac{1}{2}} \leq \frac{1}{2} (\Gamma - \gamma). \tag{1.6}
\]
On making use of the results (1.1) and (1.6), we can state the following string of reverse inequalities

\[
0 \leq \int_{\Omega} \Phi \circ f d\mu - \Phi \left( \int_{\Omega} f d\mu \right) \\
\leq \int_{\Omega} f \cdot (\Phi' \circ f) d\mu - \int_{\Omega} \Phi' \circ f \int_{\Omega} f d\mu \\
\leq \frac{1}{2} \left[ \Phi'(M) - \Phi'(m) \right] \left[ \int_{\Omega} f^2 d\mu - \left( \int_{\Omega} f d\mu \right)^2 \right]^{\frac{1}{2}} \\
\leq \frac{1}{4} \left[ \Phi'(M) - \Phi'(m) \right] (M - m),
\]

provided that \( \Phi : [m, M] \subset \mathbb{R} \to \mathbb{R} \) is a differentiable convex function on \((m, M)\) and \( f : \Omega \to [m, M] \) so that \( \Phi \circ f, f, \Phi' \circ f, f \cdot (\Phi' \circ f) \in L(\Omega, \mu) \), with \( \int_{\Omega} d\mu = 1 \).

The following reverse of the Jensen’s inequality also holds [33]:

**Theorem 1.4.** Let \( \Phi : I \to \mathbb{R} \) be a continuous convex function on the interval of real numbers \( I \) and \( m, M \in \mathbb{R} \), \( m < M \) with \([m, M] \subset \mathbb{I} \), where \( \mathbb{I} \) is the interior of \( I \). If \( f : \Omega \to \mathbb{R} \) is \( \mu \)-measurable, satisfies the bounds

\[-\infty < m \leq f(t) \leq M < \infty \text{ for } \mu\text{-a.e. } t \in \Omega\]

and such that \( f, \Phi \circ f \in L(\Omega, \mu) \), then

\[
0 \leq \int_{\Omega} \Phi \circ f d\mu - \Phi \left( \int_{\Omega} f d\mu \right) \\
\leq \left( M - \int_{\Omega} f d\mu \right) \left( \int_{\Omega} f d\mu - m \right) \frac{\Phi'(M) - \Phi'(m)}{M - m} \\
\leq \frac{1}{4} \left( M - m \right) \left[ \Phi'_{-}(M) - \Phi'_{+}(m) \right],
\]

where \( \Phi'_{-} \) is the left and \( \Phi'_{+} \) is the right derivative of the convex function \( \Phi \).

For other reverse of Jensen inequality and applications to divergence measures see [33].

In 1938, A. Ostrowski [55], proved the following inequality concerning the distance between the integral mean \( \frac{1}{b-a} \int_{a}^{b} \Phi(t) dt \) and the value \( \Phi(x) \), \( x \in [a, b] \).

For various results related to Ostrowski’s inequality see [6]-[9], [15]-[41], [43] and the references therein.

**Theorem 1.5.** Let \( \Phi : [a, b] \to \mathbb{R} \) be continuous on \([a, b]\) and differentiable on \((a, b)\) such that \( \Phi' : (a, b) \to \mathbb{R} \) is bounded on \((a, b)\), i.e., \( \| \Phi' \|_{\infty} := \sup_{t \in (a, b)} |\Phi'(t)| < \)
∞. Then
\[ \left| \Phi(x) - \frac{1}{b-a} \int_a^b f(t) \, dt \right| \leq \left[ \frac{1}{4} + \left( \frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right] \| \Phi' \|_\infty (b-a), \quad (1.9) \]

for all \( x \in [a, b] \) and the constant \( \frac{1}{4} \) is the best possible.

Now, for \( \gamma, \Gamma \in \mathbb{C} \) and \([a, b]\) an interval of real numbers, define the sets of complex-valued functions [34]
\begin{equation}
\tilde{U}_{[a,b]}(\gamma, \Gamma) := \{ f : [a, b] \to \mathbb{C} | \Re \left( (\Gamma - f(t)) \left( \overline{f(t)} - \gamma \right) \right) \geq 0 \text{ for almost every } t \in [a, b] \}.
\end{equation}

and
\begin{equation}
\tilde{\Delta}_{[a,b]}(\gamma, \Gamma) := \left\{ f : [a, b] \to \mathbb{C} \left| \left| f(t) - \frac{\gamma + \Gamma}{2} \right| \leq \frac{1}{2} |\Gamma - \gamma| \right. \text{ for a.e. } t \in [a, b] \right\}.
\end{equation}

The following representation result may be stated [34].

**Proposition 1.6.** For any \( \gamma, \Gamma \in \mathbb{C}, \gamma \neq \Gamma \), we have that \( \tilde{U}_{[a,b]}(\gamma, \Gamma) \) and \( \tilde{\Delta}_{[a,b]}(\gamma, \Gamma) \) are nonempty, convex and closed sets and
\begin{equation}
\tilde{U}_{[a,b]}(\gamma, \Gamma) = \tilde{\Delta}_{[a,b]}(\gamma, \Gamma). \quad (1.10)
\end{equation}

On making use of the complex numbers field properties we can also state that:

**Corollary 1.7.** For any \( \gamma, \Gamma \in \mathbb{C}, \gamma \neq \Gamma \), we have that
\begin{equation}
\tilde{U}_{[a,b]}(\gamma, \Gamma) = \{ f : [a, b] \to \mathbb{C} | (\Re \Gamma - \Re f(t)) (\Re f(t) - \Re \gamma) + (\Im \Gamma - \Im f(t)) (\Im f(t) - \Im \gamma) \geq 0 \text{ for a.e. } t \in [a, b] \}. \quad (1.11)
\end{equation}

Now, if we assume that \( \Re(\Gamma) \geq \Re(\gamma) \) and \( \Im(\Gamma) \geq \Im(\gamma) \), then we can define the following set of functions as well:
\begin{equation}
\tilde{S}_{[a,b]}(\gamma, \Gamma) := \{ f : [a, b] \to \mathbb{C} | \Re(\Gamma) \geq \Re f(t) \geq \Re(\gamma) \text{ and } \Im(\Gamma) \geq \Im f(t) \geq \Im(\gamma) \text{ for a.e. } t \in [a, b] \}. \quad (1.12)
\end{equation}

One can easily observe that \( \tilde{S}_{[a,b]}(\gamma, \Gamma) \) is closed, convex and
\begin{equation}
\emptyset \neq \tilde{S}_{[a,b]}(\gamma, \Gamma) \subseteq \tilde{U}_{[a,b]}(\gamma, \Gamma). \quad (1.13)
\end{equation}

The following result holds [34]:

**Theorem 1.8.** Let \( \Phi : I \to \mathbb{C} \) be an absolutely continuous functions on \([a, b] \subseteq I\), the interior of \( I \). For some \( \gamma, \Gamma \in \mathbb{C}, \gamma \neq \Gamma \), assume that \( \Phi' \in \tilde{U}_{[a,b]}(\gamma, \Gamma) (= \tilde{\Delta}_{[a,b]}(\gamma, \Gamma)) \). If \( g : \Omega \to [a, b] \) is Lebesgue \( \mu \)-measurable on \( \Omega \) and such that \( \Phi \circ g, g \in L(\Omega, \mu) \), then we have the inequality
\begin{equation}
\left| \int_\Omega \Phi \circ g d\mu - \Phi(x) - \frac{\gamma + \Gamma}{2} \left( \int_\Omega g d\mu - x \right) \right| \leq \frac{1}{2} |\Gamma - \gamma| \int_\Omega |g - x| d\mu \quad (1.14)
\end{equation}
for any $x \in [a, b]$. In particular, we have

$$\left| \int_{\Omega} \Phi \circ g d\mu - \Phi \left( \frac{a+b}{2} \right) - \frac{\gamma + \Gamma}{2} \left( \int_{\Omega} g d\mu - \frac{a+b}{2} \right) \right| \leq \frac{1}{2} |\Gamma - \gamma| \int_{\Omega} \left| g - \frac{a+b}{2} \right| d\mu \leq \frac{1}{4} (b-a) |\Gamma - \gamma| \tag{1.15}$$

and

$$\left| \int_{\Omega} \Phi \circ g d\mu - \Phi \left( \int_{\Omega} g d\mu \right) \right| \leq \frac{1}{2} |\Gamma - \gamma| \int_{\Omega} \left| g - \int_{\Omega} g d\mu \right| d\mu \leq \frac{1}{4} (b-a) |\Gamma - \gamma| \tag{1.16}$$

Motivated by the above results, in this paper we provide more upper bounds for the quantity

$$\left| \int_{\Omega} \Phi \circ g d\mu - \Phi (x) - \lambda \left( \int_{\Omega} g d\mu - x \right) \right|, \ x \in [a, b],$$

under various assumptions on the absolutely continuous function $\Phi$, which in the particular case of $x = \int_{\Omega} g d\mu$ provides some results connected with Jensen’s inequality while in the case $\lambda = 0$ provides some generalizations of Ostrowski’s inequality. Applications for divergence measures are provided as well.

### 2. SOME IDENTITIES

The following result holds [34]:

**Lemma 2.1.** Let $\Phi : I \to \mathbb{C}$ be an absolutely continuous functions on $[a, b] \subset \bar{I}$, the interior of $I$. If $g : \Omega \to [a, b]$ is Lebesgue $\mu$-measurable on $\Omega$ and such that $\Phi \circ g, g \in L(\Omega, \mu)$, then we have the equality

$$\int_{\Omega} \Phi \circ g d\mu - \Phi (x) = \int_{\Omega} \left[ (g-x) \int_{0}^{1} (\Phi'((1-s)x+sg) - \lambda) \ ds \right] d\mu \tag{2.1}$$

for any $\lambda \in \mathbb{C}$ and $x \in [a, b]$.

In particular, we have

$$\int_{\Omega} \Phi \circ g d\mu - \Phi (x) = \int_{\Omega} \left[ (g-x) \int_{0}^{1} \Phi'((1-s)x+sg) \ ds \right] d\mu, \tag{2.2}$$

for any $x \in [a, b]$. 

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Remark 2.2. With the assumptions of Lemma 2.1 we have
\[
\int_{\Omega} \Phi \circ g d\mu - \Phi \left( \frac{a + b}{2} \right) = \int_{\Omega} \left[ \left( g - \frac{a + b}{2} \right) \int_{0}^{1} \Phi' \left( (1 - s) \frac{a + b}{2} + sg \right) ds \right] d\mu.
\] (2.3)

Corollary 2.3. With the assumptions of Lemma 2.1 we have
\[
\int_{\Omega} \Phi \circ g d\mu - \Phi \left( \int_{\Omega} g d\mu \right) = \int_{\Omega} \left[ \left( g - \int_{\Omega} g d\mu \right) \int_{0}^{1} \Phi' \left( (1 - s) \int_{\Omega} g d\mu + sg \right) ds \right] d\mu.
\] (2.4)

Proof. We observe that since \( g : \Omega \to [a, b] \) and \( \int_{\Omega} d\mu = 1 \) then \( \int_{\Omega} g d\mu \in [a, b] \) and by taking \( x = \int_{\Omega} g d\mu \) in (2.2) we get (2.4). \( \square \)

Corollary 2.4. With the assumptions of Lemma 2.1 we have
\[
\int_{\Omega} \Phi \circ g d\mu - \left( \frac{b - a}{b - a} \right) \int_{a}^{b} \Phi (x) dx - \lambda \left( \int_{\Omega} g d\mu - \int_{\Omega} h d\mu \right) = \int_{\Omega} \int_{\Omega} \left[ (g(t) - h(\tau)) \int_{0}^{1} \Phi' \left( (1 - s) h(\tau) + sg(t) \right) ds \right] d\mu(t) d\mu(\tau),
\] (2.5)

for any \( \lambda \in \mathbb{C} \) and \( x \in [a, b] \).

Proof. Follows by integrating the identity (2.1) over \( x \in [a, b] \), dividing by \( b - a > 0 \) and using Fubini’s theorem. \( \square \)

Corollary 2.5. Let \( \Phi : I \to \mathbb{C} \) be an absolutely continuous functions on \([a, b] \subset \hat{I}, \) the interior of \( I \). If \( g, h : \Omega \to [a, b] \) are Lebesgue \( \mu \)-measurable on \( \Omega \) and such that \( \Phi \circ g, \Phi \circ h, g, h \in L(\Omega, \mu) \), then we have the equality
\[
\int_{\Omega} \Phi \circ g d\mu - \int_{\Omega} \Phi \circ h d\mu - \lambda \left( \int_{\Omega} g d\mu - \int_{\Omega} h d\mu \right) = \int_{\Omega} \int_{\Omega} \int_{0}^{1} \Phi' \left( (1 - s) h(\tau) + sg(t) \right) ds \times d\mu(t) d\mu(\tau) - \lambda \left( \int_{\Omega} g d\mu - \int_{\Omega} h d\mu \right),
\] (2.6)

for any \( \lambda \in \mathbb{C} \) and \( x \in [a, b] \).

In particular, we have
\[
\int_{\Omega} \Phi \circ g d\mu - \int_{\Omega} \Phi \circ h d\mu = \int_{\Omega} \int_{\Omega} \int_{0}^{1} \Phi' \left( (1 - s) h(\tau) + sg(t) \right) ds \times d\mu(t) d\mu(\tau),
\] (2.7)

for any \( x \in [a, b] \).
Remark 2.6. The above inequality (2.6) can be extended for two measures as follows
\[
\int_{\Omega_1} \Phi \circ g d\mu_1 - \int_{\Omega_2} \Phi \circ h d\mu_2 - \lambda \left( \int_{\Omega_1} g d\mu_1 - \int_{\Omega_2} h d\mu_2 \right)
\]
\[
= \int_{\Omega_1} \int_{\Omega_2} \left[ (g(t) - h(\tau)) \int_0^1 (\Phi'((1-s)h(\tau) + sg(t)) - \lambda) ds \right]
\times d\mu_1(t) d\mu_2(\tau),
\]
for any \( \lambda \in \mathbb{C} \) and \( x \in [a, b] \) and provided that \( \Phi \circ g, g \in L(\Omega_1, \mu_1) \) while \( \Phi \circ h, h \in L(\Omega_2, \mu_2) \).

Remark 2.7. If \( w \geq 0 \) \( \mu \)-almost everywhere (\( \mu \)-a.e.) on \( \Omega \) with \( \int_{\Omega} wd\mu > 0 \), then by replacing \( d\mu \) with \( \int_{\Omega} w d\mu \) in (2.1) we have the weighted equality
\[
\frac{1}{\int_{\Omega} w d\mu} \int_{\Omega} w (\Phi \circ g) d\mu - \Phi(x) - \Phi'(a) + \Phi'(b)
\]
\[
\leq \frac{1}{2} \int_a^b (\Phi') \int_{\Omega} |g - x| d\mu
\]
for any \( \lambda \in \mathbb{C} \) and \( x \in [a, b] \), provided \( \Phi \circ g, g \in L_w(\Omega, \mu) \) where
\[
L_w(\Omega, \mu) := \{ g | \int_{\Omega} w|g| d\mu < \infty \}.
\]
The other equalities have similar weighted versions. However the details are omitted.

3. Inequalities for Derivatives of Bounded Variation

The following result holds:

Theorem 3.1. Let \( \Phi : I \to \mathbb{C} \) be an absolutely continuous functions on \( [a, b] \subset \tilde{I} \), the interior of \( I \) and with the property that the derivative \( \Phi' \) is of bounded variation on \( [a, b] \). If \( g : \Omega \to [a, b] \) is Lebesgue \( \mu \)-measurable on \( \Omega \) and such that \( \Phi \circ g, g \in L(\Omega, \mu) \), then we have
\[
\left| \int_{\Omega} \Phi \circ g d\mu - \Phi(x) - \frac{\Phi'(a) + \Phi'(b)}{2} \left( \int_{\Omega} g d\mu - x \right) \right|
\]
\[
\leq \frac{1}{2} \sqrt{\Phi'} \int_{\Omega} |g - x| d\mu
\]
for any \( x \in [a, b] \).

In particular, we have
\[
\left| \int_{\Omega} \Phi \circ g d\mu - \Phi \left( \frac{a+b}{2} \right) - \frac{\Phi'(a) + \Phi'(b)}{2} \left( \int_{\Omega} g d\mu - \frac{a+b}{2} \right) \right|
\]
\[
\leq \frac{1}{2} \sqrt{\Phi'} \int_{\Omega} |g - \frac{a+b}{2}| d\mu \leq \frac{1}{2} (b-a) \sqrt{\Phi'}
\]
and
\[
\left| \int_{\Omega} \Phi \circ g d\mu - \Phi \left( \int_{\Omega} g d\mu \right) \right| \leq \frac{1}{2} \left( \Phi' \right) \left( \int_{\Omega} g^2 d\mu \right)^{1/2} \left( \int_{\Omega} g d\mu \right) \leq \frac{1}{4} (b - a) \left( \Phi' \right).
\]

Proof. From the identity (2.1) we have
\[
\int_{\Omega} \Phi \circ g d\mu - \Phi (x) - \Phi' (a) + \Phi' (b) = \int_{\Omega} \left[ (g - x) \int_{0}^{1} \left( \Phi' ((1 - s) x + sg) - \frac{\Phi'(a) + \Phi'(b)}{2} \right) ds \right] d\mu
\]
for any \( x \in [a, b] \).

Taking the modulus in (3.4) we get
\[
\left| \int_{\Omega} \Phi \circ g d\mu - \Phi (x) - \Phi' (a) + \Phi' (b) \left( \int_{\Omega} g d\mu - x \right) \right| \leq \int_{\Omega} \left| (g - x) \int_{0}^{1} \left( \Phi' ((1 - s) x + sg) - \frac{\Phi'(a) + \Phi'(b)}{2} \right) ds \right| d\mu
\]
for any \( x \in [a, b] \).

Since \( \Phi' \) is of bounded variation on \([a, b]\), then for any \( s \in [0, 1] \), \( x \in [a, b] \) and \( t \in \Omega \) we have
\[
\left| \Phi' ((1 - s) x + sg (t)) - \frac{\Phi'(a) + \Phi'(b)}{2} \right|
= \frac{1}{2} \left| \Phi' ((1 - s) x + sg (t)) - \Phi' (a) + \Phi' ((1 - s) x + sg (t)) - \Phi' (b) \right|
\leq \frac{1}{2} \left| \Phi' ((1 - s) x + sg (t)) - \Phi' (a) \right| + \left| \Phi' (b) - \Phi' ((1 - s) x + sg (t)) \right|
\leq \frac{1}{2} \left( \Phi' \right).
\]

Then we have
\[
\int_{\Omega} |g - x| \int_{0}^{1} \left| \Phi' ((1 - s) x + sg) - \frac{\Phi'(a) + \Phi'(b)}{2} \right| ds d\mu \leq \frac{1}{2} \left( \Phi' \right) \int_{\Omega} |g - x| d\mu
\]
for any $x \in [a, b]$.

Making use of (3.5) and (3.6) we deduce the desired result (3.1). □

**Remark 3.2.** Let $\Phi : I \to \mathbb{C}$ be an absolutely continuous functions on $[a, b] \subset \hat{I}$, the interior of $I$ and with the property that the derivative $\Phi'$ is of bounded variation on $[a, b]$. If $x_i \in [m, M]$ and $w_i \geq 0 \ (i = 1, \ldots, n)$ with $W_n := \sum_{i=1}^{n} w_i = 1$, then one has the weighted discrete inequality:

$$\left| \sum_{i=1}^{n} w_i \Phi (x_i) - \Phi (x) - \frac{\Phi' (a) + \Phi' (b)}{2} \left( \sum_{i=1}^{n} w_i x_i - x \right) \right| (3.7)$$

$$\leq \frac{1}{2} \sqrt{\Phi' \sum_{i=1}^{n} w_i |x_i - x|}$$

for any $x \in [a, b]$.

In particular, we have

$$\sum_{i=1}^{n} w_i \Phi (x_i) - \Phi \left( \frac{a + b}{2} \right) - \frac{\Phi' (a) + \Phi' (b)}{2} \left( \sum_{i=1}^{n} w_i x_i - \frac{a + b}{2} \right) \right| (3.8)$$

$$\leq \frac{1}{2} \sqrt{\Phi' \sum_{i=1}^{n} w_i |x_i - \frac{a + b}{2}|} \leq \frac{1}{4} (b - a) \sqrt{\Phi'}$$

and

$$\left| \sum_{i=1}^{n} w_i \Phi (x_i) - \Phi \left( \sum_{i=1}^{n} w_i x_i \right) \right| \leq \frac{1}{2} \sqrt{\Phi' \sum_{i=1}^{n} w_i |x_i - \sum_{i=1}^{n} w_i x_i|} \right| (3.9)$$

$$\leq \frac{1}{2} \sqrt{\Phi' \left( \sum_{j=1}^{n} w_j x_j^2 - \left( \sum_{k=1}^{n} w_k x_k \right)^2 \right) \right}^{1/2}$$

$$\leq \frac{1}{4} (b - a) \sqrt{\Phi'}.$$

### 4. Inequalities for Lipschitzian Derivatives

The following result holds:

**Theorem 4.1.** Let $\Phi : I \to \mathbb{C}$ be an absolutely continuous functions on $[a, b] \subset \hat{I}$, the interior of $I$ and with the property that the derivative $\Phi'$ is Lipschitzian with the constant $K > 0$ on $[a, b]$. If $g : \Omega \to [a, b]$ is Lebesgue $\mu$-measurable on $\Omega$ and such that $\Phi \circ g, g \in L (\Omega, \mu)$, then we have

$$\left| \int_{\Omega} \Phi \circ g d\mu - \Phi (x) - \Phi' (x) \left( \int_{\Omega} g d\mu - x \right) \right| (4.1)$$

$$\leq \frac{1}{2} K \left[ \sigma^2 \mu (g) + \left( \int_{\Omega} g d\mu - x \right)^2 \right]$$
for any $x \in [a, b]$, where $\sigma_{\mu}(g)$ is the dispersion or the standard variation, namely

$$\sigma_{\mu}(g) := \left( \int_{\Omega} (g - \int_{\Omega} gd\mu)^2 \, d\mu \right)^{1/2} = \left( \int_{\Omega} g^2 d\mu - \left( \int_{\Omega} gd\mu \right)^2 \right)^{1/2}.$$ 

In particular, we have

$$\left| \int_{\Omega} \Phi \circ gd\mu - \Phi \left( \int_{\Omega} gd\mu \right) \right| \leq \frac{1}{2} K \left[ \sigma_{\mu}^2(g) + \left( \int_{\Omega} gd\mu - \frac{a + b}{2} \right)^2 \right]$$

and

$$\left| \int_{\Omega} \Phi \circ gd\mu - \Phi \left( \int_{\Omega} gd\mu \right) \right| \leq \frac{1}{2} K \sigma_{\mu}^2(g) \leq \frac{1}{8} K (b - a)^2. \quad (4.3)$$

**Proof.** From the identity (2.1) we have for $\lambda = \Phi'(x)$ that

$$\int_{\Omega} \Phi \circ gd\mu - \Phi \left( \int_{\Omega} gd\mu \right) - \Phi'(x) \left( \int_{\Omega} gd\mu - x \right)$$

$$= \int_{\Omega} \left[ (g - x) \int_0^1 \Phi'(s(1 - s) x + sg) - \Phi'(x) \right] ds \, d\mu \quad (4.4)$$

for any $x \in [a, b]$.

Taking the modulus in (4.4) we get

$$\left| \int_{\Omega} \Phi \circ gd\mu - \Phi \left( \int_{\Omega} gd\mu \right) - \Phi'(x) \left( \int_{\Omega} gd\mu - x \right) \right| \leq \int_{\Omega} |g - x| \left| \int_0^1 \Phi'(s(1 - s) x + sg) - \Phi'(x) \right| ds \, d\mu$$

$$\leq \int_{\Omega} \left[ |g - x| \int_0^1 |\Phi'(s(1 - s) x + sg) - \Phi'(x)| \right] ds \, d\mu$$

$$\leq K \int_{\Omega} \left[ |g - x| \int_0^1 s |g - x| \right] ds \, d\mu = \frac{1}{2} K \int_{\Omega} (g - x)^2 \, d\mu$$

for any $x \in [a, b]$. 
However, 
\[ \int_{\Omega} (g - x)^2 \, d\mu \]
\[ = \int_{\Omega} \left( g - \int_{\Omega} gd\mu + \int_{\Omega} gd\mu - x \right)^2 \, d\mu \]
\[ = \int_{\Omega} \left( g - \int_{\Omega} gd\mu \right)^2 \, d\mu + 2 \int_{\Omega} \left( g - \int_{\Omega} gd\mu \right) \left( \int_{\Omega} gd\mu - x \right) \, d\mu \]
\[ + \int_{\Omega} \left( \int_{\Omega} gd\mu - x \right)^2 \, d\mu \]
\[ = \int_{\Omega} \left( g - \int_{\Omega} gd\mu \right)^2 \, d\mu + \left( \int_{\Omega} gd\mu - x \right)^2 \]
for any \( x \in [a, b] \), and by (4.5) we get the desired result (4.1).

**Corollary 4.2.** Let \( \Phi : I \to \mathbb{C} \) be a twice differentiable functions on \( [a, b] \subset \bar{I} \) with \( \| \Phi'' \|_{[a, b], \infty} := \text{ess sup}_{t \in [a, b]} | \Phi''(t) | < \infty \). Then the inequalities (4.1)-(4.3) hold for \( K = \| \Phi'' \|_{[a, b], \infty} \).

**Remark 4.3.** Let \( \Phi : I \to \mathbb{C} \) be an absolutely continuous functions on \( [a, b] \subset \bar{I} \) and with the property that the derivative \( \Phi' \) is Lipschitzian with the constant \( K > 0 \) on \( [a, b] \). If \( x_i \in [m, M] \) and \( w_i \geq 0 \ (i = 1, \ldots, n) \) with \( W_n := \sum_{i=1}^{n} w_i = 1 \), then one has the weighted discrete inequality:
\[ \left| \sum_{i=1}^{n} w_i \Phi(x_i) - \Phi(x) \left( \sum_{i=1}^{n} w_i x_i - x \right) \right| \]
\[ \leq \frac{1}{2} K \left[ \sigma_w^2(x) + \left( \sum_{i=1}^{n} w_i x_i - x \right)^2 \right] \]
for any \( x \in [a, b] \), where
\[ \sigma_w(x) := \left( \sum_{i=1}^{n} w_i \left( x_i - \sum_{k=1}^{n} w_k x_k \right)^2 \right)^{1/2} = \left( \sum_{i=1}^{n} w_i x_i^2 - \left( \sum_{k=1}^{n} w_k x_k \right)^2 \right)^{1/2} \]

The following lemma may be stated:

**Lemma 4.4.** Let \( u : [a, b] \to \mathbb{R} \) and \( l, L \in \mathbb{R} \) with \( L > l \). The following statements are equivalent:

(i) The function \( u - \frac{l + L}{2} e \), where \( e(t) = t, t \in [a, b] \) is \( \frac{1}{2} (L - l) \) -Lipschitzian;

(ii) We have the inequalities \[ l \leq \frac{u(t) - u(s)}{t - s} \leq L \quad \text{for each} \ t, s \in [a, b] \quad \text{with} \ t \neq s; \]

(iii) We have the inequalities \[ l(t - s) \leq u(t) - u(s) \leq L(t - s) \quad \text{for each} \ t, s \in [a, b] \quad \text{with} \ t > s. \]
Following [53], we can introduce the definition of \((l, L)\)-Lipschitzian functions:

**Definition 4.5.** The function \(u : [a, b] \to \mathbb{R}\) which satisfies one of the equivalent conditions (i) – (iii) from Lemma 4.4 is said to be \((l, L)\)-Lipschitzian on \([a, b]\).

If \(L > 0\) and \(l = -L\), then \((-L, L)\)-Lipschitzian means \(L\)-Lipschitzian in the classical sense.

Utilising Lagrange’s mean value theorem, we can state the following result that provides examples of \((l, L)\)-Lipschitzian functions.

**Proposition 4.6.** Let \(u : [a, b] \to \mathbb{R}\) be continuous on \([a, b]\) and differentiable on \((a, b)\). If \(-\infty < l = \inf_{t \in [a, b]} u'(t)\) and \(\sup_{t \in [a, b]} u'(t) = L < \infty\), then \(u\) is \((l, L)\)-Lipschitzian on \([a, b]\).

The following result holds.

**Corollary 4.7.** Let \(\Phi : I \to \mathbb{R}\) be an absolutely continuous functions on \([a, b] \subset I\), with the property that the derivative \(\Phi'\) is \((l, L)\)-Lipschitzian on \([a, b]\), where \(l, L \in \mathbb{R}\) with \(L > l\). If \(g : \Omega \to [a, b]\) is Lebesgue \(\mu\)-measurable on \(\Omega\) and such that \(\Phi \circ g, g \in L(\Omega, \mu)\), then we have

\[
\left| \int_\Omega \Phi \circ gd\mu - \Phi \left( \frac{a + b}{2} \right) \right| - \Phi' \left( \frac{a + b}{2} \right) \left( \int_\Omega gd\mu - x \right) \leq \frac{1}{4} \left( L + l \right) \left( \int_\Omega gd\mu - x \right)^2 \tag{4.9}
\]

for any \(x \in [a, b]\).

In particular, we have

\[
\left| \int_\Omega \Phi \circ gd\mu - \Phi \left( \frac{a + b}{2} \right) \right| - \Phi' \left( \frac{a + b}{2} \right) \left( \int_\Omega gd\mu - \frac{a + b}{2} \right) \leq \frac{1}{4} \left( L + l \right) \left( \int_\Omega gd\mu - \frac{a + b}{2} \right)^2 \tag{4.10}
\]

and

\[
\left| \int_\Omega \Phi \circ gd\mu - \Phi \left( \int_\Omega gd\mu \right) - \frac{1}{4} \left( L + l \right) \sigma^2 (g) \right| \leq \frac{1}{4} \left( L - l \right) \sigma^2 (g) \tag{4.11}
\]

\[
\leq \frac{1}{16} \left( L - l \right) \sigma^2 (g) \leq \frac{1}{16} \left( L - l \right) (b - a)^2.
\]
Proof. Consider the auxiliary function \( \Psi : [a, b] \to \mathbb{R} \) given by
\[
\Psi (x) = \Phi (x) - \frac{1}{4} (L + l) x^2.
\]
We observe that \( \Psi \) is differentiable and
\[
\Psi' (x) = \Phi' (x) - \frac{1}{2} (L + l) x.
\]
Since \( \Phi' \) is \((l, L)\)-Lipschitzian on \([a, b]\) it follows that \( \Psi' \) is Lipschitzian with the constant \( \frac{1}{2} (L - l) \), so we can apply Theorem 4.1 for \( \Psi \), i.e. we have the inequality
\[
\left| \int _{\Omega} \Psi \circ g d\mu - \Psi (x) - \Psi' (x) \left( \int _{\Omega} g d\mu - x \right) \right| \leq \frac{1}{4} (L - l) \left[ \sigma _\mu ^2 (g) + \left( \int _{\Omega} g d\mu - x \right)^2 \right].
\] (4.12)

However
\[
\int _{\Omega} \Psi \circ g d\mu - \Psi (x) - \Psi' (x) \left( \int _{\Omega} g d\mu - x \right)
= \int _{\Omega} \Phi \circ g d\mu - \Phi (x) - \Phi' (x) \left( \int _{\Omega} g d\mu - x \right)
- \frac{1}{4} (L + l) \left[ \int _{\Omega} g^2 d\mu - x^2 - 2x \left( \int _{\Omega} g d\mu - x \right) \right]
= \int _{\Omega} \Phi \circ g d\mu - \Phi (x) - \Phi' (x) \left( \int _{\Omega} g d\mu - x \right)
- \frac{1}{4} (L + l) \left[ \sigma _\mu ^2 (g) + \left( \int _{\Omega} g d\mu - x \right)^2 \right]
\]
and by (4.12) we get the desired result (4.9). \( \square \)

Remark 4.8. We observe that if the function \( \Phi \) is twice differentiable on \( \bar{I} \) and for \([a, b] \subseteq \bar{I}\) we have
\[-\infty < l \leq \Phi'' (x) \leq L < \infty \text{ for any } x \in [a, b],
\]
then \( \Phi' \) is \((l, L)\)-Lipschitzian on \([a, b]\) and the inequalities (4.9)-(4.11) hold true.

The following result also holds:

**Theorem 4.9.** Let \( \Phi : I \to \mathbb{C} \) be an absolutely continuous functions on \([a, b] \subseteq \bar{I}\), the interior of \( I \) and with the property that the derivative \( \Phi' \) is Lipschitzian with the constant \( K > 0 \) on \([a, b]\). If \( g : \Omega \to [a, b] \) is Lebesgue \( \mu \)-measurable on
and such that $\Phi \circ g, g \in L(\Omega, \mu)$, then we have

\[
\left| \int_{\Omega} \Phi \circ gd\mu - \Phi (x) - \Phi' \left( \int_{\Omega} gd\mu \right) \left( \int_{\Omega} gd\mu - x \right) \right| \leq \frac{1}{2} K \left[ \left| x - \int_{\Omega} gd\mu \right| \left| g - x \right| d\mu + \left| \int_{\Omega} gd\mu \right| d\mu \right]
\]

\[
\leq \frac{1}{2} K \left[ \left| x - \int_{\Omega} gd\mu \right| + \left| g - \int_{\Omega} gd\mu \right| \right] \int_{\Omega} \left| g - x \right| d\mu
\]

for any $x \in [a, b]$, where

\[
\left\| g - \int_{\Omega} gd\mu \right\|_{\Omega, \infty} := \text{ess sup}_{t \in \Omega} \left| g(t) - \int_{\Omega} gd\mu \right| < \infty.
\]

In particular, we have

\[
\left| \int_{\Omega} \Phi \circ gd\mu - \Phi \left( \frac{a+b}{2} \right) - \Phi' \left( \int_{\Omega} gd\mu \right) \left( \int_{\Omega} gd\mu - \frac{a+b}{2} \right) \right| \leq \frac{1}{2} K \left[ \left| \frac{a+b}{2} - \int_{\Omega} gd\mu \right| \left| g - \frac{a+b}{2} \right| d\mu + \left| g - \int_{\Omega} gd\mu \right| \right] \int_{\Omega} \left| g - \frac{a+b}{2} \right| d\mu.
\]

Proof. From the identity (2.1) we have for $\lambda = \Phi' \left( \int_{\Omega} gd\mu \right)$ that

\[
\int_{\Omega} \Phi \circ gd\mu - \Phi (x) - \Phi' \left( \int_{\Omega} gd\mu \right) \left( \int_{\Omega} gd\mu - x \right) = \int_{\Omega} \left[ (g - x) \int_{0}^{1} \Phi' \left( (1-s)x + sg \right) - \Phi' \left( \int_{\Omega} gd\mu \right) \right] ds d\mu
\]

for any $x \in [a, b]$. 
Taking the modulus in (4.15) we get
\[
\left| \int_{\Omega} \Phi \circ g d\mu - \Phi (x) - \Phi' \left( \int_{\Omega} g d\mu \right) \left( \int_{\Omega} g d\mu - x \right) \right| \tag{4.16}
\]
\[
\leq \int_{\Omega} |g - x| \left| \int_{0}^{1} \left( \Phi' \left( (1-s) x + sg \right) - \Phi' \left( \int_{\Omega} g d\mu \right) \right) ds \right| d\mu
\]
\[
\leq \int_{\Omega} \left| g - x \right| \left| \int_{0}^{1} \left( \Phi' \left( (1-s) x + sg \right) - \Phi' \left( \int_{\Omega} g d\mu \right) \right) ds \right| d\mu
\]
\[
\leq K \int_{\Omega} \left| g - x \right| \int_{0}^{1} (1-s) x + sg - \int_{\Omega} g d\mu \right| ds d\mu
\]
\[
= K \int_{\Omega} \left| g - x \right| \int_{0}^{1} (1-s) x + sg - (1-s) \int_{\Omega} g d\mu - s \int_{\Omega} g d\mu \right| ds d\mu
\]
\[
:= B.
\]
Using the triangle inequality we have for any \( t \in \Omega \)
\[
\int_{0}^{1} (1-s) x + sg \left( t \right) - (1-s) \int_{\Omega} g d\mu - s \int_{\Omega} g d\mu \right| ds
\]
\[
\leq \int_{0}^{1} (1-s) \left| x - \int_{\Omega} g d\mu \right| ds + \int_{0}^{1} s \left| g \left( t \right) - \int_{\Omega} g d\mu \right| ds
\]
\[
= \frac{1}{2} \left[ \left| x - \int_{\Omega} g d\mu \right| + \left| g \left( t \right) - \int_{\Omega} g d\mu \right| \right]
\]
and then
\[
B \leq \frac{1}{2} K \int_{\Omega} \left| g - x \right| \left[ \left| x - \int_{\Omega} g d\mu \right| + \left| g \left( t \right) - \int_{\Omega} g d\mu \right| \right] d\mu \tag{4.17}
\]
\[
= \frac{1}{2} K \left[ \left| x - \int_{\Omega} g d\mu \right| \int_{\Omega} \left| g - x \right| d\mu + \int_{\Omega} \left| g - x \right| \left| g - \int_{\Omega} g d\mu \right| d\mu \right].
\]
Making use of (4.16) and (4.17) we deduce the desired result (4.13). \( \square \)

**Corollary 4.10.** Let \( \Phi : I \to \mathbb{R} \) be an absolutely continuous functions on \([a, b] \subset I\), with the property that the derivative \( \Phi' \) is \((l, L)\)-Lipschitzian on \([a, b]\), where \( l, L \in \mathbb{R} \) with \( L > l \). If \( g : \Omega \to [a, b] \) is Lebesgue \( \mu \)-measurable on \( \Omega \) and such that \( \Phi \circ g, g \in L(\Omega, \mu) \), then we have
\[
\left| \int_{\Omega} \Phi \circ g d\mu - \Phi (x) - \Phi' \left( \int_{\Omega} g d\mu \right) \left( \int_{\Omega} g d\mu - x \right) \right| \tag{4.18}
\]
\[
\leq \frac{1}{4} (L + l) \left[ \sigma_{\mu}^{2} (g) - \left( x - \int_{\Omega} g d\mu \right)^{2} \right]
\]
\[
\leq \frac{1}{4} (L - l) \left[ \left| x - \int_{\Omega} g d\mu \right| \int_{\Omega} \left| g - x \right| d\mu + \int_{\Omega} \left| g - x \right| \left| g - \int_{\Omega} g d\mu \right| d\mu \right]
\]
\[
\leq \frac{1}{4} (L - l) \left[ \left| x - \int_{\Omega} g d\mu \right| + \left| g - \int_{\Omega} g d\mu \right| \right] \int_{\Omega} \left| g - x \right| d\mu
\]
for any $x \in [a, b]$.

In particular, we have

\[
\left| \int_{\Omega} \Phi \circ g d\mu - \Phi \left( \frac{a + b}{2} \right) - \Phi' \left( \int_{\Omega} g d\mu \right) \left( \int_{\Omega} g d\mu - \frac{a + b}{2} \right) \right| \leq \frac{1}{4} (L + l) \left[ \sigma^2_{\mu} (g) - \left( \frac{a + b}{2} - \int_{\Omega} g d\mu \right)^2 \right].
\]

5. APPLICATIONS FOR $f$-DIVERGENCE

One of the important issues in many applications of Probability Theory is finding an appropriate measure of distance (or difference or discrimination) between two probability distributions. A number of divergence measures for this purpose have been proposed and extensively studied by Jeffreys [47], Kullback and Leibler [52], Rényi [58], Havrda and Charvat [44], Kapur [50], Sharma and Mittal [62], Burbea and Rao [5], Rao [57], Lin [53], Csiszár [12], Ali and Silvey [1], Vajda [68], Shioya and Da-te [63] and others (see for example [54] and the references therein).

These measures have been applied in a variety of fields such as: anthropology [57], genetics [54], finance, economics, and political science [60], [66], [67], biology [56], the analysis of contingency tables [42], approximation of probability distributions [11], [51], signal processing [48], [49] and pattern recognition [4], [10]. A number of these measures of distance are specific cases of Csiszár $f$-divergence and so further exploration of this concept will have a flow on effect to other measures of distance and to areas in which they are applied.

Assume that a set $\Omega$ and the $\sigma$-finite measure $\mu$ are given. Consider the set of all probability densities on $\mu$ to be $\mathcal{P} := \{ p | p : \Omega \to \mathbb{R}, p(t) \geq 0, \int_{\Omega} p(t) d\mu(t) = 1 \}$.

The Kullback-Leibler divergence [52] is well known among the information divergences. It is defined as:

\[
D_{KL}(p, q) := \int_{\Omega} p(t) \ln \left( \frac{p(t)}{q(t)} \right) d\mu(t), \quad p, q \in \mathcal{P},
\]  

where $\ln$ is to base $e$.

In Information Theory and Statistics, various divergences are applied in addition to the Kullback-Leibler divergence. These are the: variation distance $D_v$, Hellinger distance $D_H$ [45], $\chi^2$-divergence $D_{\chi^2}$, $\alpha$-divergence $D_\alpha$, Bhattacharyya distance $D_B$ [3], Harmonic distance $D_{H\alpha}$, Jeffrey’s distance $D_J$ [47],...
triangular discrimination $D_\Delta$ [65], etc... They are defined as follows:

\[ D_v (p, q) := \int_\Omega |p(t) - q(t)| \, d\mu(t), \quad p, q \in \mathcal{P}; \]  

\[ D_H (p, q) := \int_\Omega \left| \sqrt{p(t)} - \sqrt{q(t)} \right| \, d\mu(t), \quad p, q \in \mathcal{P}; \]  

\[ D_{\chi^2} (p, q) := \int_\Omega \left[ \left( \frac{q(t)}{p(t)} \right)^2 - 1 \right] \, d\mu(t), \quad p, q \in \mathcal{P}; \]  

\[ D_\alpha (p, q) := \frac{4}{1 - \alpha^2} \left[ 1 - \int_\Omega \left[ p(t)^{1-\alpha} \frac{q(t)}{p(t)} \right]^{\frac{1+\alpha}{2}} \, d\mu(t) \right], \quad p, q \in \mathcal{P}; \]  

\[ D_B (p, q) := \int_\Omega \sqrt{p(t)q(t)} \, d\mu(t), \quad p, q \in \mathcal{P}; \]  

\[ D_{Ha} (p, q) := \int_\Omega \left( \frac{2p(t)q(t)}{p(t) + q(t)} \right) \, d\mu(t), \quad p, q \in \mathcal{P}; \]  

\[ D_J (p, q) := \int_\Omega [p(t) - q(t)] \ln \left( \frac{p(t)}{q(t)} \right) \, d\mu(t), \quad p, q \in \mathcal{P}; \]  

\[ D_\Delta (p, q) := \int_\Omega \left[ \frac{p(t) - q(t)}{p(t) + q(t)} \right]^2 \, d\mu(t), \quad p, q \in \mathcal{P}. \]  

For other divergence measures, see the paper [50] by Kapur or the book on line [64] by Taneja.

Csiszár $f$-divergence is defined as follows [13]

\[ I_f (p, q) := \int_\Omega p(t) f \left( \frac{q(t)}{p(t)} \right) \, d\mu(t), \quad p, q \in \mathcal{P}, \]  

where $f$ is convex on $(0, \infty)$. It is assumed that $f(u)$ is zero and strictly convex at $u = 1$. By appropriately defining this convex function, various divergences are derived. Most of the above distances (5.1)-(5.9), are particular instances of Csiszár $f$-divergence. There are also many others which are not in this class (see for example [64]). For the basic properties of Csiszár $f$-divergence see [13], [14] and [68].

The following result holds:

**Proposition 5.1.** Let $f : (0, \infty) \to \mathbb{R}$ be a twice differentiable convex function with the property that $f(1) = 0$ and there exists the constants $\gamma, \Gamma$ so that

\[ -\infty < \gamma < f(t) \leq \Gamma < \infty. \]

Assume that $p, q \in \mathcal{P}$ and there exists the constants $0 < r < 1 < R < \infty$ such that

\[ r \leq \frac{q(t)}{p(t)} \leq R \text{ for } \mu\text{-a.e. } t \in \Omega. \]
If $x \in [r, R]$, then we have the inequality
\[
\left| I_f (p, q) - f(x) - f'(x)(1 - x) - \frac{1}{4} (L + l) \left[ D_{\chi^2} (p, q) + (1 - x)^2 \right] \right| \leq \frac{1}{4} (L - l) \left[ D_{\chi^2} (p, q) + (1 - x)^2 \right].
\] (5.12)

In particular, we have
\[
\left| I_f (p, q) - f \left( \frac{r + R}{2} \right) - f' \left( \frac{r + R}{2} \right) \left( 1 - \frac{r + R}{2} \right) \right| \leq \frac{1}{4} (L - l) \left[ D_{\chi^2} (p, q) + \left( 1 - \frac{r + R}{2} \right)^2 \right].
\] (5.13)

and
\[
\left| I_f (p, q) - \frac{1}{4} (L + l) D_{\chi^2} (p, q) \right| \leq \frac{1}{4} (L - l) D_{\chi^2} (p, q).
\] (5.14)

Proof. From (4.9) we have
\[
\left| \int_{\Omega} p(t) f \left( \frac{q(t)}{p(t)} \right) d\mu(t) - f(x) - f'(x)(1 - x) \right|
\]
\[
- \frac{1}{4} (L + l) \left[ \int_{\Omega} p(t) \left( \frac{q(t)}{p(t)} \right)^2 d\mu(t) - 1 + (1 - x)^2 \right]
\]
\[
\leq \frac{1}{4} (L - l) \left[ \int_{\Omega} p(t) \left( \frac{q(t)}{p(t)} \right)^2 d\mu(t) - 1 + (1 - x)^2 \right]
\]
for any $x \in [r, R]$, which is equivalent to (5.12). \qed

Utilising Corollary 4.10 we can state the following result as well:

Proposition 5.2. With the assumptions in Proposition 5.1, we have
\[
\left| I_f (p, q) - f(x) - f'(1)(1 - x) - \frac{1}{4} (L + l) \left[ D_{\chi^2} (p, q) - (1 - x)^2 \right] \right| \leq \frac{1}{4} (L - l) \left[ x - 1 \right] \int_{\Omega} |q - xp| d\mu + \int_{\Omega} |q - xp| \left| \frac{q}{p} - 1 \right| d\mu
\]
\[
\leq \frac{1}{4} (L - l) \left[ x - 1 \right] + \left\| \frac{q}{p} - 1 \right\|_{\Omega, \infty} \int_{\Omega} |q - xp| d\mu
\]
for any $x \in [r, R]$.

If we consider the convex function $f : (0, \infty) \rightarrow \mathbb{R}$, $f(t) = t \ln t$ then
\[
I_f (p, q) := \int_{\Omega} p(t) \frac{q(t)}{p(t)} \ln \left( \frac{q(t)}{p(t)} \right) d\mu(t) = \int_{\Omega} q(t) \ln \left( \frac{q(t)}{p(t)} \right) d\mu(t)
\]
\[
= D_{KL} (q, p).
\]
We have \( f'(t) = \ln t + 1 \) and \( f''(t) = \frac{1}{t} \) and then we can choose \( l = \frac{1}{R} \) and \( L = \frac{1}{r} \). Applying the inequality (5.14) we get

\[
\left| D_{KL}(q,p) - \left( \frac{R + r}{4rR} \right) D_{\chi^2}(p,q) \right| \leq \frac{R - r}{4rR} D_{\chi^2}(p,q). \tag{5.16}
\]

If we consider the convex function \( f : (0, \infty) \to \mathbb{R} \), \( f(t) = -\ln t \) then

\[
I_f(p,q) := -\int_{\Omega} p(t) \ln \left( \frac{q(t)}{p(t)} \right) d\mu(t) = \int_{\Omega} p(t) \ln \left( \frac{p(t)}{q(t)} \right) d\mu(t)
= D_{KL}(p,q).
\]

We have \( f'(t) = -\frac{1}{t} \) and \( f''(t) = \frac{1}{t^2} \) and then we can choose \( l = \frac{1}{R^2} \) and \( L = \frac{1}{r^2} \). Applying the inequality (5.14) we get

\[
\left| D_{KL}(p,q) - \frac{R^2 + r^2}{4R^2r^2} D_{\chi^2}(p,q) \right| \leq \frac{R^2 - r^2}{4R^2r^2} D_{\chi^2}(p,q). \tag{5.17}
\]

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