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An Interior Point Algorithm for Solving Convex Quadratic Semidefinite Optimization Problems Using a New Kernel Function

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Abstract. In this paper, we consider convex quadratic semidefinite optimization problems and provide a primal-dual Interior Point Method (IPM) based on a new kernel function with a trigonometric barrier term. Iteration complexity of the algorithm is analyzed using some easy to check and mild conditions. Although our proposed kernel function is neither a Self-Regular (SR) function nor logarithmic barrier function, the primal-dual IPMs based on this kernel function enjoy the worst case iteration bound $O\left(\sqrt{n}\log n\log\frac{n}{\epsilon}\right)$ for the large-update methods with the special choice of its parameters. This bound coincides to the so far best known complexity results obtained from SR kernel functions for linear and semidefinite optimization problems. Finally, some numerical issues regarding the practical performance of the new proposed kernel function are reported.

Keywords: Convex quadratic semidefinite optimization problem, Primal-dual interior-point methods, Kernel function, Iteration complexity.

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1. Introduction

In this paper, we deal with the Convex Quadratic Semidefinte Optimization (CQSDO) problem

$$\begin{aligned}(P) \qquad & \min C \bullet X + \frac{1}{2} X \bullet \Omega(X), \\ & s.t.\ A_i \bullet X = b_i, \qquad i = 1, ..., m, \\ & \quad X \succeq \mathbf{0},\end{aligned}$$

along with its dual problem:

$$\begin{aligned}(D) \qquad & \max b^T y - \frac{1}{2} X \bullet \Omega(X), \\ & s.t.\ \sum_{i=1}^{m} y_i A_i - \Omega(X) + S = C, \\ & \quad S \succeq \mathbf{0},\end{aligned}$$

where C and A_i's, for $1 \leq i \leq m$, are symmetric $n \times n$ matrices and $b, y \in \mathbb{R}^m$. Moreover, the self-adjoint positive semidefinite linear operator $\Omega(X) : S^n \mapsto S^n$, defined on the set of all symmetric $n \times n$ matrices S^n, is defined by the properties $\Omega(A) \bullet B = A \bullet \Omega(B)$ and $\Omega(A) \bullet A \succeq \mathbf{0}$, for all $A, B \in S^n$. Note that, the classical Löwner partial ordering $\succeq$ for symmetric matrices is defined by $A \succeq B (A \succ B)$ if and only if $A - B$ is positive semidefinite (positive definite). Moreover, throughout the paper, as LO case [26], we assume that the matrices A_i are linearly independent and the problems (P) and (D) satisfy the Interior Point Condition (IPC), i.e. there exists $\left(X^0, S^0\right) \succ \mathbf{0}$ so that

$$\begin{aligned} A_i \bullet X^0 &= b_i, \qquad i = 1, \ldots, m, \\ \sum_{i=1}^{m} y_i A_i - \Omega(X^0) + S &= C. \end{aligned}$$

The CQSDO problems have some important applications, such as the nearest Euclidian distance and correlation matrix problems, see e.g. [1, 25]. Moreover, it is an extension of the well known Semidefinite Optimization (SDO) problems and can be reformulated as a semidefinite linear complementarity problem [13]. Indeed, the CQSDO problem is reduced to the SDO problem under the operator $\Omega(A) = \mathbf{0}$.

After the seminal paper of Karmarkar [10] in 1984, the polynomial time Interior Point Methods (IPMs) have been revitalized as an active area of research. Since then, many variants of this algorithm have been studied and developed for the many classes of convex optimization problems, including Linear Complementary Problem (LCP), Second Order Cone Optimization (SOCO) problems, Semidefinite Optimization (SDO) problems and most recently CQSDO problems, see e.g. [7, 27, 29, 30, 31, 33]. These methods have shown their power not only in theoretical complexity results but also in practical performance. Due to

these nice behaviors, nowadays, IPMs are of great interest for the researchers in the optimization fields. The so called path following IPMs were first proposed by Kojima et al. [12] and Megiddo [15]. These methods follow the so called central path curve approximately to get close to the optimal point. The existence of the central path for CQSDO was first provided by Nie et al. in [19].

All variants of the interior point methods designed for Linear Optimization (LO) have been successfully extended to SDO. An extension of the primal dual IPMs from LO to SDO was first done by Nesterov and Nemirovski [16] and obtained a polynomial complexity for solving conic problems by introducing the so called self concordant barrier functions which consist of the logarithmic barrier function. Peng et al. [22] proposed a new paradigm in the classical IPM for solving LO and some other extensions of this problem in which the logarithmic barrier function is replaced by the so called Self-Regular (SR) barrier functions. The iteration complexity of LO and its extensions, based on SR barrier functions, led them to obtain the so far best known iteration bound for small and large update IPMs as $O\left(\sqrt{n}L\right)$ and $O\left(\sqrt{n}\log nL\right)$, respectively. Note that, based on the logarithmic barrier functions, these bounds are $O\left(\sqrt{n}L\right)$ and $O\left(nL\right)$, respectively. Moreover, a class of primal-dual interior-point algorithms for linear optimization based on a new family of kernel functions which is fairly general and includes the classical logarithmic function, the prototype self regular function, and some non-self-regular kernel functions as its special case was proposed by Bai et al. in [4].

Nie et al. in [19] proposed an IPM based on potential reduction approach and obtained an iteration bound $O\left(\sqrt{n}\log\frac{n}{\epsilon}\right)$ in the worst case. A predictor-corrector IPM has also been proposed by Nie et al. in [20] with the same order of complexity as the potential reduction approach. An interior point method based on kernel function for solving CQSDO was first proposed by Wang et al. in [31] in which they employed a parametric kernel function, provided in [2] for LO, and obtained $O(\sqrt{n}\log n\log\frac{n}{\epsilon})$ iteration bound for the large update method IPMs.

Recently, Wang et al. in [33] proposed a primal-dual IPM for CQSDO based on a kernel function which was previously introduced for LO in [3]. They obtained the best known iteration complexity for large-update methods, i.e. $O\left(\sqrt{n}\log n\log\frac{n}{\epsilon}\right)$. Later on, Zhang in [30] suggested a new kernel function for solving CQSDO problems for the first time and obtained its iteration complexity as $O\left(\sqrt{n}(\log n)^2\log\frac{n}{\epsilon}\right)$ for the CQSDO problems. Although their complexity was not as good as that of [33], the kernel function and their approach for analyzing the algorithm was rather interesting.

Due to literature, it seems that analyzing of IPMs based on trigonometric kernel function is of interest for the researchers. The first work in this subject for LO problems was done by El Ghami et al. in [6]. They showed

that the primal-dual interior point methods for solving LO meet $O\left(n^{\frac{3}{4}}\log\frac{n}{\epsilon}\right)$ iteration bound in the worst case. Later, Kheirfam in [11] proposed a new kernel function with a trigonometric barrier term and analyzed the complexity of large-update primal-dual IPMs for SDO problems based on this kernel function. He achieved the same complexity of El Ghami et al. in [6]. Moreover, El Ghami in [8] proposed a primal dual IPMs for $P_*(\kappa)$-linear complementarity problem based on a kernel function with trigonometric barrier term which was previously introduced for LO in [6]. He obtained the iteration complexity as $O\left((1+2\kappa)n^{\frac{3}{4}}\log\frac{n}{\epsilon}\right)$ for the large-update methods. In a more recent paper, Peyghami et al. in [24] introduced a new kernel function with trigonometric barrier term for LO problems and achieved the worst case complexity for the large update IPMs as $O\left(n^{\frac{2}{3}}\log\frac{n}{\epsilon}\right)$ which improves the bound obtained in [6] and [8] significantly.

In this paper, we propose a new kernel function with trigonometric barrier term which is not considered in the literature so far. This kernel function is neither the so called self-regular [22] kernel function nor the logarithmic barrier function. The large-update primal-dual IPMs for solving CQSDO problems are analyzed based on this kernel function. Using a simple analysis and under some mild conditions, we show that large-update primal-dual IPMs based on our new kernel function enjoy the so far best known iteration complexity for linear optimization problems, i.e. $O\left(\sqrt{n}\log n\log\frac{n}{\epsilon}\right)$, for specific choices of the function's parameters. To our best knowledge, this is the first work in which the trigonometric kernel function is proposed for the complexity analysis of primal-dual IPMs in solving CQSDO problems. Numerical results on a problem taken from the literature show that the new proposed function is well promising and perform well enough in practice in comparison with some other existing kernel functions in the literature.

The paper is organized as follows: In Section 2, we name some basic concepts of linear algebra and the properties of the matrices. The new kernel function and its properties are given in Section 3. Section 4 is devoted to recall the central path concept and the interior point methods for CQSDO. A default value for the step size together with the proximity reduction are introduced in Section 5. The worst case iteration bound for the primal-dual IPMs based on the new kernel function is provided in Section 6. We illustrate the practical performance of the new proposed kernel function in Section 7. Finally, some concluding remarks are given in Section 8.

The following notational conventions are used throughout this paper: $\mathbb{R}^n_+$ and $\mathbb{R}^n_{++}$ stand for the subsets of nonnegative and positive vectors in $\mathbb{R}^n$, respectively. $\|\,.\,\|$ denotes the Frobenius norm for the matrices, and the 2-norm for the vectors. $\mathbb{R}^{m\times n}$ is the space of all $m\times n$ matrices. S^n, S^n_+ and S^n_{++} are the cone of symmetric, symmetric positive semidefinite and symmetric positive

definite $n\times n$ matrices, respectively. The matrix E indicates the identity matrix of order n. The inner product of the same size matrices A and B is defined by $A \bullet B = Tr(A^T B)$. For given $Q \in S^n_{++}$, the expression $Q^{\frac{1}{2}}$ stands for the symmetric square root of Q. For the vector $\lambda \in \mathbb{R}^n$, the matrix Λ is a diagonal matrix whose diagonal entries are λ_i's. For $V \in S^n_{++}$, the vector $\lambda(V)$ denotes the vector of its eigenvalues arranged in non-increasing order, i.e. $\lambda_1(V) \geq \lambda_2(V) \geq \ldots \geq \lambda_n(V)$.

2. Preliminaries

In this section, we provide an introduction to the CQSDO problem and some related elementary results from linear algebra. The new kernel function along with its properties are also given in this section.

Let us consider the primal and dual CQSDO problem as given by (P) and (D). As in [20] and [31], we restrict the self adjoint positive semidefinite linear operator Ω to the following special case:

$$\Omega(X) = \sum_{i=1}^{l} H_i^T X H_i,$$

where l is a positive integer not greater than n^2 and $H_i \in \mathbb{R}^{n\times n}$. It can be easily seen that

$$\Omega(X) = \Omega(X)^T, \qquad \Omega(X) \bullet X \geq 0, \qquad \forall X \in \mathbb{R}^{n\times n}.$$

In what follows, we present some properties of the symmetric matrices along with the matrix function and its derivatives.

Theorem 2.1. *(Spectral theorem for symmetric matrices* [34] *and* [9]*) The real $n\times n$ matrix A is symmetric if and only if there exists an orthogonal basis with respect to which A is real and diagonal, i.e. if and only if there exists a matrix $U \in \mathbb{R}^{n\times n}$ such that $U^T U = E$ and $U^T A U = \Lambda$.*

Due to Theorem 2.1, let $U \in \mathbb{R}^{n\times n}$ be an orthogonal matrix that diagonalizes $V \in S^n_{++}$, i.e. $V = U^T diag(\lambda(V))U$ and $U^T U = E$. Then, for the real function $\psi(t)$, $t \in \mathbb{R}_+$, the matrix function $\psi : S^n_{++} \to S^n_{++}$ and the real valued matrix function $\Psi : S^n_{++} \to \mathbb{R}_+$ are defined by

$$\psi(V) = U^T diag\left(\psi(\lambda_1(V)), \psi(\lambda_2(V)), \ldots, \psi(\lambda_n(V))\right) U, \tag{2.1}$$

$$\Psi(V) := Tr(\psi(V)) = \sum_{i=1}^{n} \psi(\lambda_i(V)), \tag{2.2}$$

where $Tr(.)$ stands for the trace operator. For the matrix function $\psi(V)$, its first and second derivatives are defined by replacing $\psi(\lambda_i(V))$'s in (2.1) with $\psi'(\lambda_i(V))$'s and $\psi''(\lambda_i(V))$'s, respectively.

Definition 2.1. *A matric $M(t)$ is said to be a matrix of functions if each entry of $M(t)$ is a function of t, i.e., $M(t) = [M_{ij}(t)]$.*

The following properties hold for the matrix of functions. One can find their proof in [9, 14, 22].
For $M, N \in S_+^n$, one has

$$|Tr(MN)| \leq |\lambda_1(M)| \sum_{i=1}^{n} |\lambda_i(N)|, \quad Tr(MN) \leq (Tr(M^2))^{\frac{1}{2}} (Tr(N^2))^{\frac{1}{2}}. \quad (2.3)$$

Moreover, if $M_1 \preceq M_2$ and $N \succeq \mathbf{0}$, then

$$Tr(M_1 N) \leq Tr(M_2 N). \quad (2.4)$$

An extension of the usual concepts of analysis such as continuity, differentiability, and integrability is straight for the matrix of functions by interpreting them as entry-wise. Furthermore, for the matrix functions $M(t)$ and $N(t)$, one can easily see that

$$\begin{aligned} \frac{d}{dt} Tr(M(t)) &= Tr\left(\frac{d}{dt} M(t)\right) = Tr(M'(t)), & (2.5)\\ \frac{d}{dt} Tr(\psi(M(t))) &= Tr(\psi'(M(t))M'(t), & (2.6)\\ \frac{d}{dt}(M(t)N(t)) &= \left(\frac{d}{dt} M(t)\right) N(t) + M(t)\left(\frac{d}{dt} N(t)\right) \\ &= M'(t)N(t) + M(t)N'(t). & (2.7) \end{aligned}$$

3. The New Kernel Function and Its Properties

In this section, a new kernel function with trigonometric barrier term along with its properties are provided. Let us define the new univariate function:

$$\psi(t) = \frac{t^2 - 1}{2} - \int_1^t \frac{4}{(1+x)^2} \tan^{2p}(h(x))dx, \qquad p \geq 1, \quad (3.1)$$

where

$$h(x) = \frac{\pi}{2 + 2x}. \quad (3.2)$$

It can be easily seen that as $t \to 0$ or $t \to \infty$, then $\psi(t) \to \infty$. Therefore, $\psi(t)$ is indeed a kernel function. As we need the first three derivatives of $\psi(t)$, we list them here:

$$\begin{aligned} \psi'(t) &= t - \frac{4}{(1+t)^2} \tan^{2p}(h(t)) & (3.3)\\ \psi''(t) &= 1 + \frac{8}{(1+t)^3} \tan^{2p}(h(t)) + \frac{4p\pi}{(1+t)^4} \tan^{2p-1}(h(t))r(t) & (3.4)\\ \psi'''(t) &= -\frac{4}{(1+t)^4} \tan^{2p-2}(h(t))k(t), & (3.5) \end{aligned}$$

where

$$\begin{aligned} r(t) &= 1+\tan^2(h(t)) \\ k(t) &= 6\tan^2(h(t)) + \frac{6p\pi}{1+t}\tan(h(t))r(t) + \frac{\pi^2 p(2p-1)}{2(1+t)^2}r^2(t) \\ &\quad + \frac{p\pi^2}{(1+t)^2}\tan^2(h(t))r(t). \end{aligned}$$

The following results provide some essential properties of the kernel function $\psi(t)$ which are used in our analysis in the next sections.

Lemma 3.1. *For the function $h(t)$, defined by (3.2), the following inequality holds for all $t \in (0,1]$:*

$$\tan(h(t)) \geq \frac{1}{\pi t}.$$

Proof. The proof is similar to the proof of Lemma 2.1 in [6], however we restate it here. Let $g(t)$ be defined as

$$g(t) := \tan(h(t)) - \frac{1}{\pi t}.$$

For this function, one has:

$$g'(t) = \frac{h'(t)}{\cos^2(h(t))} + \frac{1}{\pi t^2} = \frac{1}{\pi t^2\cos^2(h(t))}\left(h'(t)\pi t^2 + \cos^2(h(t))\right),$$

where $h'(t) = \frac{-\pi}{2(1+t)^2}$. As for all $t \in (0,1]$, the inequality $\frac{\pi}{4} \leq h(t) < \frac{\pi}{2}$ holds, therefore, we have:

$$\sin\left(\frac{\pi}{2} - h(t)\right) = \cos(h(t)) \leq \frac{\pi}{2} - h(t),$$

which implies that

$$\begin{aligned} g'(t) &= \frac{1}{\pi t^2\cos^2(h(t))}\left(h'(t)\pi t^2 + \sin^2\left(\frac{\pi}{2} - h(t)\right)\right) \\ &\leq \frac{1}{\pi t^2\cos^2(h(t))}\left(h'(t)\pi t^2 + \left(\frac{\pi}{2} - h(t)\right)^2\right) \\ &\leq \frac{1}{\pi t^2\cos^2(h(t))}\left(\frac{-\pi^2 t^2}{(2+2t)^2}\right) < 0. \end{aligned}$$

This shows that $g(t)$ is a decreasing function on $(0,1]$ which completes the proof by considering the fact that $g(1) > 0$. □

Lemma 3.2. *For the function $\psi(t)$, defined by (3.1), we have:*

i): $\psi''(t) > 1, \qquad \forall t > 0,$
ii): $t\psi''(t) - \psi'(t) > 0, \qquad \forall t > 1,$
iii): $t\psi''(t) + \psi'(t) > 0, \qquad \forall t > 0$
iv): $\psi'''(t) < 0, \qquad \forall t > 0.$

Proof. First of all, we note that

$$\tan(h(t)) \geq 1, \qquad \text{for all } 0 < t \leq 1, \tag{3.6}$$

$$0 \leq \tan(h(t)) < 1, \qquad \text{for all } t > 1. \tag{3.7}$$

Using (3.6) and (3.7), we obtain:

$$\psi''(t) = 1 + \frac{8}{(1+t)^3}\tan^{2p}(h(t)) + \frac{4p\pi}{(1+t)^4}\tan^{2p-1}(h(t))r(t) > 1,$$

which shows that (i) holds. To prove (ii), for all $t \geq 1$, we have:

$$t\psi''(t) - \psi'(t) = \frac{12t+4}{(1+t)^3}\tan^{2p}(h(t)) + \frac{4p\pi t}{(1+t)^4}\tan^{2p-1}(h(t))r(t) > 0.$$

Now, we prove that (iii) holds. For $t \geq 1$, the statement is trivial as $\psi'(1) = 0$ and $\psi'(t)$ is a strictly increasing function. For $t \in (0,1)$, using Lemma 3.1, we have:

$$\begin{aligned} t\psi''(t) + \psi'(t) &= 2t + \frac{4t-4}{(1+t)^3}\tan^{2p}(h(t)) + \frac{4p\pi t}{(1+t)^4}\tan^{2p-1}(h(t))r(t) \\ &\geq \left(\frac{4t-4}{(1+t)^3} + \frac{4p}{(1+t)^4}\right)\tan^{2p}(h(t)) + \frac{4p\pi t}{(1+t)^4}\tan^{2p-1}(h(t)) \\ &= \frac{4(t^2+(p-1))}{(1+t)^4}\tan^{2p}(h(t)) + \frac{4p\pi t}{(1+t)^4}\tan^{2p-1}(h(t)) > 0. \end{aligned}$$

The statement (iv) can be easily followed by considering (3.5). □

The first and the third parts of Lemma 3.2 are respectively known as super convexity and exponential convexity (e-convexity) of the kernel function $\psi(t)$ in the literature and play important roles in the analysis of the primal-dual IPMs based on kernel functions. It is shown that the e-convexity property is equivalent to the convexity of the function $\psi(e^{\xi})$ and to the following inequality [22]:

$$\psi(\sqrt{t_1 t_2}) \leq \frac{1}{2}(\psi(t_1) + \psi(t_2)), \qquad \text{for all } t_1, t_2 > 0. \tag{3.8}$$

Using the e-convexity property of the kernel function, the following result for the real valued matrix function $\Psi(V)$ can be concluded.

Theorem 3.1. *(Proposition 5.2.6 in* [21]*) Suppose that matrices V_1 and V_2 are symmetric and positive definite. Then*

$$\Psi\left(\left[V_1^{\frac{1}{2}} V_2 V_1^{\frac{1}{2}}\right]^{\frac{1}{2}}\right) \leq \frac{1}{2}\left(\Psi(V_1) + \Psi(V_2)\right).$$

In what follows, we provide some other results related to the new kernel function which are crucial in deriving the iteration complexity of IPMs based

on kernel functions. Let the norm-based proximity measure $\delta(V)$ be defined by:

$$\delta := \delta(V) = \frac{1}{2}\|\psi'(V)\| = \frac{1}{2}\sqrt{\sum_{i=1}^{n}(\psi'(\lambda_i(V)))^2}. \tag{3.9}$$

Moreover, using $\psi(1) = \psi'(1) = 0$, the function $\psi(t)$ can be totally described by its second derivative according to:

$$\psi(t) = \int_1^t \int_1^\xi \psi''(\zeta)d\zeta d\xi. \tag{3.10}$$

From the super convexity property of the kernel function and (3.10), one can easily obtain the following properties for the kernel function and the related real valued matrix function.

Lemma 3.3. *Let the kernel function $\psi(t)$ be defined as in (3.1). Then, we have:*

i): $\frac{1}{2}(t-1)^2 \leq \psi(t) \leq \frac{1}{2}\psi'(t)^2$, *for all* $t > 0$.
ii): $\Psi(V) \leq 2\delta(V)^2, \qquad \forall\ V \succ \mathbf{0}$.
iii): $\|\lambda(V)\| \leq \sqrt{n} + \sqrt{2\Psi(V)}, \qquad \forall\ V \succ \mathbf{0}$.

Proof. The proof is similar to the proof of Lemma 3.4 in [23] and therefore is omitted here. □

Now, we discuss about the growth behavior of the new kernel function and its related real valued matrix function. For this purpose, we have the following results:

Lemma 3.4. *Let $\beta \geq 1$. Then, we have*

$$\psi(\beta t) \leq \psi(t) + \frac{1}{2}(\beta^2 - 1)t^2.$$

Proof. Let the function $\psi(t)$ be defined by $\psi(t) = \frac{t^2-1}{2} + p(t)$, where $p(t)$ is given by

$$p(t) = -\int_1^t \frac{4}{(1+x)^2}\tan^{2p}(h(x))dx.$$

In order to prove the statement, it is sufficient to show that the function $p(t)$ is a decreasing function. To do so, one has

$$p'(t) = -\frac{4}{(1+t)^2}\tan^{2p}(h(t)) < 0.$$

This completes the proof by considering the fact that $\beta t \geq t$, for $\beta \geq 1$. □

An immediate consequence of Lemma 3.4 for the real valued matrix function $\Psi(V)$ is given by the following lemma.

Lemma 3.5. *Let $\beta \geq 1$ and V be a symmetric positive definite matrix. Then, for $V_+ := \beta V$, we have*

$$\Psi(V_+) \leq \Psi(V) + \frac{\beta^2 - 1}{2}\left(2\Psi(V) + 2\sqrt{2n\Psi(V)} + n\right).$$

Proof. Using Lemma 3.4, we have

$$\begin{aligned}
\Psi(\beta V) &= \sum_{i=1}^{n} \psi(\beta\lambda_i(V)) \\
&\leq \sum_{i=1}^{n} \left(\psi(\lambda_i(V)) + \frac{1}{2}(\beta^2 - 1)\lambda_i(V)^2\right) \\
&= \Psi(V) + \frac{1}{2}(\beta^2 - 1)\sum_{i=1}^{n} \lambda_i(V)^2 \\
&= \Psi(V) + \frac{1}{2}(\beta^2 - 1)\|\lambda(V)\|^2.
\end{aligned}$$

The proof is completed by using the third part of Lemma 3.3. □

4. A Primal-dual IPM for CQSDO

The optimality conditions for the problems (P) and (D) are given by:

$$\begin{aligned}
A_i \bullet X &= b_i, \qquad X \succ \mathbf{0}, \qquad i = 1, \ldots, m, \\
\sum_{i=1}^{m} y_i A_i - \Omega(X) + S &= C, \qquad S \succ \mathbf{0}, \\
XS &= \mathbf{0}.
\end{aligned} \tag{4.1}$$

A key idea in the primal-dual IPMs for solving CQSDO problem is to replace the third equality in (4.1), the so called complementarity condition, with the parametric equation $XS = \mu E$, for $\mu > 0$. This leads us to the following system:

$$\begin{aligned}
A_i \bullet X &= b_i, \qquad X \succ \mathbf{0}, \qquad i = 1, \ldots, m, \\
\sum_{i=1}^{m} y_i A_i - \Omega(X) + S &= C, \qquad S \succ \mathbf{0}, \\
XS &= \mu E.
\end{aligned} \tag{4.2}$$

The IPC implies that system (4.2) has a unique solution $(X(\mu), y(\mu), S(\mu))$, for each $\mu > 0$. The set of all solutions of system (4.2) for every $\mu > 0$ is called the central path for the problems (P) and (D). $X(\mu)$ and $(y(\mu), S(\mu))$ are called the μ-centers of (P) and (D), respectively. As $\mu \to 0$, it has been shown that the limit of the central path exists and goes to the so called analytic center of the optimal set of problems (P) and (D) [12]. Most of IPMs follow the central path approximately to get close enough to the optimal solution, see e.g. [5, 15, 28, 34].

An application of Newton method to the system (4.2) leads us to the following linear system for the search direction $(\Delta X, \Delta y, \Delta S)$:

$$\begin{aligned} A_i \bullet \Delta X &= 0, \qquad i = 1, \ldots, m, \\ \sum_{i=1}^{m} \Delta y_i A_i - \Omega(\Delta X) + \Delta S &= \mathbf{0}, \\ X\Delta S + S\Delta X &= \mu E - XS. \end{aligned} \tag{4.3}$$

System (4.3) has a unique solution [34], in which ΔX is not necessarily symmetric. Some symmetrization techniques exist in the literature that are used to obtain a symmetric solution for the ΔX, see e.g [17, 18]. In this paper, we use the Nesterov-Todd symmetrization scheme [17, 18] which leads us to the so called NT direction. Let

$$P := X^{\frac{1}{2}}(X^{\frac{1}{2}}SX^{\frac{1}{2}})^{-\frac{1}{2}}X^{\frac{1}{2}} = S^{-\frac{1}{2}}(S^{\frac{1}{2}}XS^{\frac{1}{2}})^{\frac{1}{2}}S^{-\frac{1}{2}},$$

and $D = P^{\frac{1}{2}}$. Using D, one can define the symmetric and positive definite matrix V as follows:

$$V := \frac{1}{\sqrt{\mu}} D^{-1} X D^{-1} = \frac{1}{\sqrt{\mu}} DSD. \tag{4.4}$$

It can be easily seen that

$$V^2 := \frac{1}{\mu} D^{-1} XSD. \tag{4.5}$$

Let us further define

$$\begin{aligned} \bar{A}_i &:= DA_iD, \qquad 1 \le i \le m, \\ D_X &:= \frac{1}{\sqrt{\mu}} D^{-1}\Delta X D^{-1}, \\ D_S &:= \frac{1}{\sqrt{\mu}} D\Delta S D, \\ \bar{\Omega}(D_X) &:= \sum_{i=1}^{l} DH_i^T D D_X D H_i D. \end{aligned} \tag{4.6}$$

Now, using (4.6), system (4.3) can be scaled as follows to generate the (scaled) NT search direction $(\Delta X, \Delta y, \Delta S)$:

$$\begin{aligned} \bar{A}_i \bullet D_X &= 0, \qquad 1 \le i \le m, \\ \sum_{i=1}^{m} \Delta y_i \bar{A}_i - \bar{\Omega}(D_X) + D_S &= \mathbf{0}, \\ D_X + D_S &= V^{-1} - V. \end{aligned} \tag{4.7}$$

A crucial observation in this system is that the right hand side of the third equation is the negative gradient of the matrix function induced from the so called logarithmic barrier kernel function. This function is a strictly convex function with minimum value zero at $t = 1$. Now, let $\psi(t)$ be any strictly

convex function on $\mathbb{R}_{++}$ with minimizer at $t = 1$, and $\psi(1) = 0$. A new idea in the IPMs was proposed by Peng et al. [22], in which the right hand side of the centering equation is replaced by $-\nabla\Psi(V)$. Now, given the kernel function $\psi(t)$ and its associated matrix function $\psi(V)$, the right-hand side of the third equation in (4.7) is replaced by $-\psi'(V)$. Thus, the new search direction D_X and D_S can be obtained by solving the following system:

$$\begin{aligned} \bar{A}_i \bullet D_X &= 0, \qquad 1 \le i \le m, \\ \sum_{i=1}^{m} \Delta y_i \bar{A}_i - \bar{\Omega}(D_X) + D_S &= \mathbf{0}, \\ D_X + D_S &= -\psi'(V). \end{aligned} \tag{4.8}$$

Again, this system has a unique solution D_X, D_S and Δy [31], which can be used to compute ΔX and ΔS according to (4.6). Note that, due to (4.6), we have $D_X \bullet D_S \ge 0$. Moreover,

$$D_X = D_S = \mathbf{0} \Leftrightarrow \psi'(V) = \mathbf{0} \Leftrightarrow V = E \Leftrightarrow \Psi(V) = 0 \Leftrightarrow XS = \mu E,$$

which implies that $X = X(\mu)$ and $S = S(\mu)$.
By taking an appropriate step along the search direction obtained from (4.8), one can construct a new triple (X_+, y_+, S_+) according to

$$X_+ = X + \alpha\Delta X, \qquad y_+ = y + \alpha\Delta y, \qquad S_+ = S + \alpha\Delta S. \tag{4.9}$$

Summarizing the above arguments, we can describe one step of the IPMs based on the kernel function as follows: Starting with (X^0, y^0, S^0), $\mu_0 > 0$, an accuracy parameter $\epsilon > 0$ and the real valued matrix function $\Psi(V) = \sum_{i=1}^{n} \psi(\lambda_i(V))$, let an approximation of the μ-center $(X(\mu), y(\mu), S(\mu))$ be known for $\mu > 0$. Then, the parameter μ is decreased by a factor $1 - \theta$, for $\theta \in (0, 1)$, and set $\mu := (1 - \theta)\mu$. In this case, an approximate solution of the μ-center is obtained by frequently using Newton method. Indeed, we first solve system (4.8) for D_X and D_S and then find Newton directions ΔX, Δy and ΔS by using (4.6). This procedure is repeated until we get to the point in which $n\mu < \epsilon$. In this case, we say that the current X and (y, S) are ϵ-approximate solutions of the primal and dual problems (P) and (D), respectively.

Now, we can outline and customize the primal-dual interior point scheme based on kernel functions to the CQSDO problems as follows [22]:

Algorithm 1: Generic Primal-dual IPM for CQSDO

Input

A proximity function $\Psi(V) = \Psi(X, S, \mu)$
a threshold parameter $\tau > 0$
an accuracy parameter $\varepsilon > 0$
a fixed barrier update parameter $0 < \theta < 1$

a strictly feasible pair (X^0, S^0) and μ^0 so that $\Psi(X^0, S^0, \mu^0) \leq \tau$;
begin
$X := X^0; S := S^0; \mu := \mu^0;$
while $n\mu > \varepsilon$ **do**
$\mu := (1-\theta)\mu;$
while $\Psi(X, S, \mu) > \tau$ **do**
solve system (4.8) and use (4.6) to obtain $(\Delta X, \Delta y, \Delta S)$;
determine a step size α;
$X_+ := X + \alpha\Delta X$
$y_+ := y + \alpha\Delta y$
$S_+ := S + \alpha\Delta S$
$V := \frac{1}{\sqrt{\mu}}(D^{-1}XSD)^{\frac{1}{2}}$
end do
end do
end

Algorithm 1 consists of inner and outer while loops which are called inner and outer iterations, respectively. The total number of iterations is the multiplication of the inner and outer iterations and is described as a function of the dimension n and ϵ. Choosing the barrier parameter θ plays an important role in theory and practice of IPMs. For a constant θ, let say $\theta = \frac{1}{2}$, we obtain the so called large-update IPMs, while for θ be dependent on n, let say $\theta = \dfrac{1}{\sqrt{n}}$, the algorithm is called small-update IPM. It is well known that small-update methods have the best iteration bound in theory while the large update methods are practically efficient [26].

5. An Estimation for the Step Size

In this section, we will discuss about a default value for the step size during an inner iteration of Algorithm 1. To do so, we first note that, after an inner iteration, the new point is then given by

$$\begin{aligned} X_+ &:= X + \alpha\Delta X = X + \alpha\sqrt{\mu}DD_XD = \sqrt{\mu}D(V + \alpha D_X)D, \\ S_+ &:= S + \alpha\Delta S = S + \alpha\sqrt{\mu}D^{-1}D_SD^{-1} = \sqrt{\mu}D^{-1}(V + \alpha D_S)D^{-1}, \end{aligned}$$

where α is the step size. On the other hand, from (4.5), we obtain

$$V_+ = \frac{1}{\sqrt{\mu}}(D^{-1}X_+S_+D)^{\frac{1}{2}}. \tag{5.1}$$

It is easily seen that the matrix V_+^2 is unitarily similar to the matrices $X_+^{\frac{1}{2}}S_+X_+^{\frac{1}{2}}$ and $\bar{V}_+^2 := (V + \alpha D_X)^{\frac{1}{2}}(V + \alpha D_S)(V + \alpha D_X)^{\frac{1}{2}}$ and therefore have the same

eigenvalues. Thus, from the definition of $\Psi(V)$, we obtain

$$\Psi(V_+) = \Psi\left(\bar{V}_+\right) \leq \frac{1}{2}\left(\Psi(V+\alpha D_X) + \Psi(V+\alpha D_S)\right),$$

where the last inequality is followed from Theorem 3.1. Now, by defining

$$\begin{aligned} f(\alpha) &:= \Psi(V_+) - \Psi(V) = \Psi(\bar{V}_+) - \Psi(V), \\ f_1(\alpha) &:= \frac{1}{2}\left[\Psi(V+\alpha D_X) + \Psi(V+\alpha D_S)\right] - \Psi(V), \end{aligned} \tag{5.2}$$

we have: $f(\alpha) \leq f_1(\alpha)$. Furthermore, the function $f_1(\alpha)$ is a convex function with respect to α due to the fact that Ψ is a convex function and its arguments in the first two terms are linear with respect to α. The first two derivatives of the function $f_1(\alpha)$ with respect to α are as follows:

$$\begin{aligned} f_1'(\alpha) &= \frac{1}{2}Tr\left(\psi'(V+\alpha D_X)D_X + \psi'(V+\alpha D_S)D_S\right), \\ f_1''(\alpha) &= \frac{1}{2}Tr\left(\psi''(V+\alpha D_X)D_X^2 + \psi''(V+\alpha D_S)D_S^2\right). \end{aligned}$$

The third inequality of system (4.8) implies that

$$f_1'(0) = \frac{1}{2}Tr\left(\psi'(V)(D_X+D_S)\right) = \frac{1}{2}Tr(-\psi'(V)^2) = -2\delta^2, \tag{5.3}$$

where the last equality is obtained from (3.9).
In what follows, we are going to introduce conditions on the step size α in which $f_1'(\alpha) < 0$ holds. This allows us to deduce that the function $f(\alpha)$ is a decreasing function using the fact that $f(0) = f_1(0) = 0$ and $f(\alpha) \leq f_1(\alpha)$. It is worth mentioning that properties of the kernel function $\psi(t)$ play major role in providing the conditions in which $f_1'(\alpha) < 0$ rather than the structure of the CQSDO problem. Therefore, the results in this section is very similar to those of existing papers in this subject and we will omit their proof by just providing appropriate references.
The following lemma provides an upper bound for the second derivative of the function $f_1(\alpha)$. One can find its proof in [23].

Lemma 5.1. *Let f_1 be defined as in (5.2). Then, the second derivative of the function f_1 with respect to α satisfies the following inequality:*

$$f_1''(\alpha) \leq 2\delta^2\psi''(\lambda_n(V) - 2\alpha\delta).$$

Our aim in introducing a suitable step size is that it should be chosen so that X_+ and S_+ are feasible and $f(\alpha)$ decreases sufficiently. The procedure for choosing the largest possible step size is almost a "word-by-word" extension of the LO case in [2, 3]. Thus, we omit their proof here.

Lemma 5.2. *$f_1'(\alpha) \leq 0$ holds if α satisfies the following inequality:*

$$-\psi'(\lambda_n(V) - 2\alpha\delta) + \psi'(\lambda_n(V)) \leq 2\delta. \tag{5.4}$$

Lemma 5.3. *Assume that $\rho : [0,\infty) \to (0,1]$ is the inverse of the function $-\frac{1}{2}\psi'(t)$ in the interval $(0,1]$. Then, the largest possible value for the α satisfying (5.4) is given by*

$$\overline{\alpha} = \frac{1}{2\delta}(\rho(\delta) - \rho(2\delta)). \tag{5.5}$$

Lemma 5.4. *Let $\overline{\alpha}$ be defined as in (5.5). Then, we have:*

$$\overline{\alpha} \geq \frac{1}{\psi''(\rho(2\delta))}. \tag{5.6}$$

Due to Lemmas 5.3 and 5.4, we will consider the following value for the step size as the default value in the further analysis:

$$\tilde{\alpha} = \frac{1}{\psi''(\rho(2\delta))}. \tag{5.7}$$

It is easily seen that $\tilde{\alpha} \leq \overline{\alpha}$. Now, an upper bound for the amount of decrease in the real valued matrix function $\Psi(V)$ during an inner iteration can be given by the following lemma.

Lemma 5.5. *(Lemma 5.2 in* [23]*) For the step size α satisfying $\alpha \leq \bar{\alpha}$, we have:*

$$f(\alpha) \leq -\alpha\delta^2. \tag{5.8}$$

The following lemma customizes the result of Lemma 5.5 for the default step size.

Lemma 5.6. *Let $\Psi(V) \geq 1$, and ρ and $\tilde{\alpha}$ be defined as in Lemma 5.3 and (5.7), respectively. Then, we have*

$$f(\tilde{\alpha}) \leq \Theta\left(-\frac{\delta^{\frac{2p-1}{2p}}}{p}\right). \tag{5.9}$$

Proof. Lemma 5.5 and the fact that $\tilde{\alpha} \leq \overline{\alpha}$ imply that $f(\tilde{\alpha}) \leq -\tilde{\alpha}\delta^2 = -\dfrac{\delta^2}{\psi''(\rho(2\delta))}$, the first inequality in (5.9). To prove the second inequality, we need to compute the inverse function of $-\frac{1}{2}\psi'(t)$ in the interval $(0,1]$ for the our proposed kernel function. To do so, the equation $-\frac{1}{2}\psi'(t) = s$ should be solved for t. Equality

$$t - \frac{4}{(1+t)^2}\tan^{2p}(h(t)) = -2s,$$

implies that,

$$\frac{4}{(1+t)^2}\tan^{2p}(h(t)) \leq 2s + t \Rightarrow \tan^{2p}(h(t)) \leq 2s + 1,$$

where the last inequality is obtained by the fact that $t \leq 1$. Now, putting $t = \rho(2\delta)$, we get $4\delta = -\psi'(t)$. Thus, we have

$$\tan^{2p}(h(t)) \leq 4\delta + 1 \Rightarrow \tan(h(t)) \leq (4\delta + 1)^{\frac{1}{2p}}. \tag{5.10}$$

Also, for all $t \in (0,1]$, we have:

$$\tan(h(t)) \geq 1, \tag{5.11}$$

$$\frac{1}{1+t} \leq 1. \tag{5.12}$$

Now, using (5.10)–(5.12), we have:

$$\begin{aligned}
\tilde{\alpha} &= \frac{1}{\psi''(t)} = \frac{1}{1+\frac{8}{(1+t)^3}\tan^{2p}(h(t)) + \frac{4p\pi}{(1+t)^4}\tan^{2p-1}(h(t))r(t)} \\
&\geq \frac{1}{1+8\tan^{2p}(h(t)) + 8p\pi\tan^{2p+1}(h(t))} \\
&\geq \frac{1}{17p\pi\tan^{2p+1}(h(t)} \\
&\geq \Theta\left(\frac{1}{p\delta^{\frac{2p+1}{2p}}}\right),
\end{aligned}$$

which implies that

$$f(\tilde{\alpha}) \leq -\frac{\delta^2}{\psi''(\rho(2\delta))} \leq \Theta\left(-\frac{\delta^2}{p\delta^{\frac{2p+1}{2p}}}\right) = \Theta\left(-\frac{\delta^{\frac{2p-1}{2p}}}{p}\right).$$

This completes the proof. □

A direct application of the second part of Lemma 3.3 in (5.9) provides the following inequality which is crucial in deriving the iteration complexity in the next section:

$$f(\tilde{\alpha}) \leq \Theta\left(-\frac{\delta^{\frac{2p-1}{2p}}}{p}\right) \leq \Theta\left(-\frac{\Psi^{\frac{2p-1}{4p}}}{p}\right). \tag{5.13}$$

6. Iteration Complexity

In this section, we derive the worst case iteration bound for Algorithm 1 based on the real valued matrix function $\Psi(V)$ induced from the kernel function ψ defined by (3.1). In our analysis, we utilize $\tilde{\alpha}$, defined by (5.7), as a default value for the step size during an inner iteration. Since after updating the parameter μ to $(1-\theta)\mu$, for $\theta \in (0,1)$, we have $V_+ = \frac{1}{\sqrt{1-\theta}}V$, then, from Lemma 3.5 with $\beta = \frac{1}{\sqrt{1-\theta}}$, we have

$$\Psi(V_+) \leq \Psi(V) + \frac{\theta}{2(1-\theta)}(2\Psi(V) + 2\sqrt{2n\Psi(V)} + n). \tag{6.1}$$

At the start of an outer iteration, we have $\Psi(V) \leq \tau$ right before updating of the parameter μ, which may cause the function $\Psi(V)$ to be exceeded the threshold τ due to (6.1). Since we are interested to work in large neighborhood of the central path, we assume that $\tau = O(n) \geq 1$. In order to provide the iteration bound, we need to compute the number of inner iterations that are required to

return value of the function $\Psi(V)$ back to the situation where $\Psi(V) \leq \tau$. Let us denote the value of $\Psi(V)$ after μ-update by Ψ_0, and the subsequent values by Ψ_j, for $j = 1, \ldots, L-1$, where L is the total number of inner iterations in an outer iteration.
As we focus on large-update IPMs, we have $\theta = \Theta(1)$. Therefore, from (6.1) and $\Psi(V) \leq \tau = O(n)$, we obtain:

$$\Psi_0 \leq \tau + \frac{\theta}{2(1-\theta)}(2\tau + 2\sqrt{2n\tau} + n) = O(n). \tag{6.2}$$

Using (5.13) and the fact that in the inner iterations, we have $\Psi_j > \tau \geq 1$, the decrease of Ψ in each inner iteration is then given by

$$\Psi_{j+1} \leq \Psi_j - \kappa \Delta\Psi_j, \qquad j = 0, 1, \ldots, L-1, \tag{6.3}$$

where κ is some positive constant and $\Delta\Psi_j$ is defined by

$$\Delta\Psi_j = \frac{\Psi^{\frac{2p-1}{4p}}}{p}. \tag{6.4}$$

The following technical lemma is crucial in deriving the number of inner iterations in an outer iteration. One can find its proof in [22].

Lemma 6.1. *Given $\alpha \in [0,1]$ and $t \geq -1$, one has*

$$(1+t)^\alpha \leq 1 + \alpha t.$$

Now, using Lemma 6.1, we can provide the worst case upper bound for the total number of inner iterations in an outer iteration as follows:

Theorem 6.1. *Let $\tau \geq 1$ and the kernel function ψ be defined by (3.1). Then, considering (6.3), the number of inner iterations that are required to return the iterations back to the situation $\Psi(V) \leq \tau$ is bounded above by*

$$L \leq 1 + \frac{4p^2}{(2p+1)\kappa}\Psi_0^{\frac{2p+1}{4p}}. \tag{6.5}$$

Proof. Using (6.3), for all $j = 0, 1, \ldots, L-1$, we have

$$\begin{aligned}
0 \leq \Psi_{j+1}^{\frac{2p+1}{4p}} &\leq \left(\Psi_j - \frac{\kappa}{p}\Psi_j^{\frac{2p-1}{4p}}\right)^{\frac{2p+1}{4p}} \\
&= \Psi_j^{\frac{2p+1}{4p}}\left(1 - \frac{\kappa}{p}\Psi_j^{-\frac{2p+1}{4p}}\right)^{\frac{2p+1}{4p}} \\
&\leq \Psi_j^{\frac{2p+1}{4p}}\left(1 - \frac{\kappa(2p+1)}{4p^2}\Psi_j^{-\frac{2p+1}{4p}}\right) \\
&= \Psi_j^{\frac{2p+1}{4p}} - \frac{\kappa(2p+1)}{4p^2},
\end{aligned} \tag{6.6}$$

where the last inequality is obtained from Lemma 6.1. By subsequently using (6.6), we obtain

$$\Psi_{j+1}^{\frac{2p+1}{4p}} \leq \Psi_0^{\frac{2p+1}{4p}} - \frac{j\kappa(2p+1)}{4p^2}.$$

Letting $j = L-1$ implies that

$$0 \leq \Psi_L^{\frac{2p+1}{4p}} \leq \Psi_0^{\frac{2p+1}{4p}} - \frac{(L-1)\kappa(2p+1)}{4p^2},$$

which shows that

$$L \leq 1 + \frac{4p^2}{(2p+1)\kappa}\Psi_0^{\frac{2p+1}{4p}}.$$

This completes the proof of the theorem. □

Using (6.2), we have $\Psi_0 = O(n)$. Now, from Theorem 6.1, we obtain the following upper bound for the total number of inner iterations in an outer iteration:

$$L \leq \left\lceil 1 + \frac{4p^2}{(2p+1)\kappa}\Psi_0^{\frac{2p+1}{4p}} \right\rceil = O\left(pn^{\frac{2p+1}{4p}}\right). \tag{6.7}$$

On the other hand, for given accuracy parameter $\epsilon > 0$, the total number of outer iterations for getting $n\mu \leq \epsilon$ in the large update methods is bounded above by $O\left(\frac{1}{\theta}\log\frac{n}{\epsilon}\right)$, see Lemma I.36 in [26]. Therefore, the total number of iterations in Algorithm 1 is obtained by multiplying the total number of inner and outer iterations. Thus, we need the following total number of iterations to get an ϵ solution for the problems (P) and (D), i.e. a solution that satisfies $x^T s = n\mu \leq \epsilon$:

$$L \leq O\left(pn^{\frac{2p+1}{4p}}\log\frac{n}{\epsilon}\right). \tag{6.8}$$

This bound significantly improves the so far iteration bound of large update primal-dual interior point methods based on the trigonometric kernel functions for solving LO problems which was obtained in [6]. By taking $p = O(\log n)$, one can easily see that the obtained upper bound in (6.8) gives the complexity $O\left(\sqrt{n}\log n\log\frac{n}{\varepsilon}\right)$ for the CQSDO problems which coincides to the so far best known complexity result for LO, SDO and LCP cases.

7. Numerical Results

In this section, our main focus is to provide a numerical experiences regarding the practical performance of the new proposed kernel function in comparison with some other kernel functions which have been proposed in the literature. Let us consider the following special case of CQSDO problem, whose

primal-dual pair have the following data [32]:

$$A_1 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -2 & -1 \\ 0 & -1 & 1 & -1 & -2 \end{bmatrix},\ A_2 = \begin{bmatrix} 0 & 0 & -2 & 2 & 0 \\ 0 & 2 & 1 & 0 & 2 \\ -2 & 1 & -2 & 0 & 1 \\ 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 2 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} 2 & 2 & -1 & -1 & 1 \\ 2 & 0 & 2 & 1 & 1 \\ -1 & 2 & 0 & 1 & 0 \\ -1 & 1 & 1 & -2 & 0 \\ 1 & 1 & 0 & 0 & -2 \end{bmatrix},\ C = \begin{bmatrix} 3 & 3 & -3 & 1 & 1 \\ 3 & 5 & 3 & 1 & 2 \\ -3 & -1 & 1 & 2 & \\ 1 & 1 & 1 & -3 & -1 \\ 1 & 2 & 2 & -1 & -1 \end{bmatrix},$$

$$b = [-2, 2, -2]^T, \qquad \Omega(X) = \mathbf{0}.$$

The optimal solution of the problems *(P)* and *(D)* are [32]:

$$X = \begin{bmatrix} 0.0704 & -0.0717 & 0.0133 & 0.0695 & -0.1465 \\ -0.0717 & 0.0753 & -0.0162 & -0.0636 & 0.1631 \\ 0.0133 & -0.0162 & 0.0105 & -0.0124 & -0.0703 \\ 0.0695 & -0.0636 & -0.0124 & 0.1695 & 0.0178 \\ -0.1465 & 0.1631 & -0.0703 & 0.0178 & 0.5829 \end{bmatrix},$$

$$S = \begin{bmatrix} 1.4249 & 0.5669 & -0.0204 & -0.4045 & 0.2125 \\ 0.5669 & 1.0918 & 0.3289 & 0.2125 & -0.1216 \\ -0.0204 & 0.3289 & 1.1921 & 0.2125 & 0.0459 \\ -0.4045 & 0.2125 & 0.2125 & 0.2912 & -0.1420 \\ 0.2125 & -0.1216 & 0.0459 & -0.1420 & 0.0991 \end{bmatrix},\ y = \begin{bmatrix} 0.8580 \\ 1.0960 \\ 0.7875 \end{bmatrix}.$$

Starting by the strictly feasible primal and dual solutions $X = E$, $S = E$ and $y = (1, 1, 1)^T$, we apply Algorithm 1 on the above mentioned CQSDO problem with the new proposed kernel function $\psi(t)$ and the kernel functions listed in Table 1.

Table 1: Considered kernel functions

$\psi_1(t) = \frac{t^2-1}{2} - \log(t)$	[26]
$\psi_2(t) = \frac{t^2-1}{2} - (t-1)e^{\frac{1}{t}-1}$	[30]
$\psi_3(t) = \frac{t^2-1}{2} + \frac{\frac{1}{t}-1}{2} - \frac{t-1}{2}$	[22]
$\psi_4(t) = \frac{t^2-1}{2} + \frac{6}{\pi}\tan(h(t)), \qquad h(t) = \frac{1-t}{2+4t}\pi$	[6]
$\psi_5(t) = \frac{t^2-1}{2} + \frac{4}{\pi}\cot(h(t)), \qquad h(t) = \frac{\pi t}{1+t}$	[11]
$\psi_6(t) = \frac{t^2-1}{2} - \log(t) + \frac{1}{8}\tan^2(h(t)), \qquad h(t) = \frac{1-t}{2+4t}\pi$	[24]

We have also considered the threshold parameter $\tau = 15$ and the accuracy parameter $\varepsilon = 10^{-8}$. We have implemented Algorithm 1 in MATLAB 7.10.0 (R2010a) environment and run the above mentioned problem on a PC with CPU 2.0 GHz and 2G RAM memory and double precision format. The results of applying Algorithm 1 based on the kernel functions given in Table 1 with

different values of θ are given in Table 2. It has to be noted that the value of the stepsize is chosen as an approximation of the default value in the related references. Moreover, for the new proposed kernel function, the results are given in Table 3 with different values of θ and p.

Table 2: The numerical results

θ	0.1	0.2	0.3	0.4	0.5	0.6
$\psi_1(t)$	104	125	128	135	152	163
$\psi_2(t)$	108	130	132	139	150	165
$\psi_3(t)$	112	136	137	143	156	171
$\psi_4(t)$	136	139	137	142	154	175
$\psi_5(t)$	110	132	135	144	153	171
$\psi_6(t)$	101	127	128	136	150	162

Table 3: Numerical results of $\psi(t)$ with different values of p

θ	0.1	0.2	0.3	0.4	0.5	0.6
$p = 1$	91	114	118	130	142	151
$p = 2$	90	113	117	124	139	149
$p = 3$	90	112	117	124	137	149
$p = 4$	90	113	118	124	137	148
$p = 10$	90	114	118	124	137	148

As it is clear from Tables 2 and 3, the performance of the new proposed kernel function is well promising and it provides less number of iterations in solving the considered CQSDO problem in comparison with the other considered kernel functions.

8. Conclusion

In this paper, we present a wide neighborhood large-update primal-dual Interior Point Methods (IPMs) based on a new kernel function with trigonometric barrier term for the Convex Quadratic Semidefinite Optimization (CQSDO) problems. The proposed kernel function is neither the so called self-regular nor the logarithmic barrier functions. Our aim is to investigate the worst case iteration complexity results for the proposed approach. Using a simple analysis and under some suitable conditions, we show that the worst case iteration bound achieves the so far best known complexity result for linear optimization, i.e. $O\left(\sqrt{n}\log n\log\frac{n}{\epsilon}\right)$, with special choices of the kernel function's parameters. To our best knowledge, this is the first work in which the trigonometric kernel function is considered for complexity analysis of IPMs for the CQSDO problems. To illustrate the numerical behavior of the new proposed function against some other kernel functions in the literature, we have implemented Algorithm 1 with these kernel function in MATLAB environment and ran it on a problem in the literature. Numerical results confirm that the new proposed kernel function is well promising.

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