Iranian Journal of Mathematical Sciences and Informatics
Vol. 9, No. 1 (2014), pp 81-98
DOI: 10.7508/ijmsi.2014.01.008

On Hyperideal Structure of Ternary Semihypergroups

Kostaq Hila[a], Bijan Davvaz[b*] , Krisanthi Naka[a]

[a]Department of Mathematics and Computer Sciences,
Faculty of Natural Sciences, University of Gjirokastra, Albani
[b]Department of Mathematics, Yazd University, Yazd, Iran

E-mail: kostaq_hila@yahoo.com, khila@uogj.edu.al
E-mail: davvaz@yazd.ac.ir, bdavvaz@yahoo.com

Abstract. In this paper, we introduce and study the concepts of prime left, semiprime left and irreducible left hyperideals in ternary semihypergroups and investigate some basic properties of them. We introduce the concepts of hyperfilter and hypersemilattice congruence of ternary semihypergroups. We give some characterizations of hyperfilters in ternary semihypergroups. Some relationships between hyperfilters, prime hyperideals and hypersemilattice congruences in ternary semihypergroups are considered. We also introduce the notion of hyperideals extensions in ternary semihypergroups and some properties of them are investigated.

Keywords: Semihypergroup, Ternary semihypergroup, Hyperideal, Prime left hyperideal, Semiprime left hyperideal, Irreducible left hyperideal, Hyperfilter, Left m-system, Left i-system, Left p-system.

2000 Mathematics subject classification: 20N20, 16Y99.

1. Introduction and preliminaries

In 1965, Sioson [13] studied ideal theory in ternary semigroups. He also introduced the notion of regular ternary semigroups and characterized them by using the notion of quasi-ideals. In 1995, Dixit and Dewan [5] introduced

*Corresponding Author

Received 12 November 2012; Accepted 12 January 2013

and studied the properties of ideals in ternary semigroups. Also, see [6, 7, 8, 10, 12, 13].

Hyperstructure theory was born in 1934 at the 8th congress of Scandinavian Mathematicions, where Marty [11] introduced the hypergroup notion as a generalization of groups and after, he proved its utility in solving some problems of groups, algebraic functions and rational fractions. Surveys of the theory can be found in the books of Corsini [1] and Davvaz [3]. Also, see [2, 9]. Recently, Davvaz and Leoreanu-Fotea [4] studied binary relations on ternary semihypergroups and studied some basic properties of compatible relations on them. The main purpose of this paper is to introduce and study prime left, semiprime left and irreducible left hyperideals in ternary semihypergroups and investigate some basic properties of them. We introduce the concepts of hyperfilters and hypersemilattice congruence of ternary semihypergroups. We give some characterizations of hyperfilters in ternary semihypergroups. Some relationships between the hyperfilters and the prime hyperideals and hypersemilattice congruences in ternary semihypergroups are considered. We also introduce the notion of hyperideal extensions in ternary semihypergroups and some properties of them are investigated.

First, we recall the definition of a semihypergroup.

Definition 1.1. Let H be a non-empty set and let $\wp^*(H)$ be the set of all non-empty subsets of H. A *hyperoperation* on H is a map $\circ : H \times H \longrightarrow \wp^*(H)$ and the couple $(H, \circ)$ is called a *hypergroupoid*. If A and B are non-empty subsets of H, then we denote

$$A \circ B = \bigcup_{a\in A,\, b\in B} a \circ b, \quad x \circ A = \{x\} \circ A \ \text{ and } \ A \circ x = A \circ \{x\}.$$

A hypergroupoid $(H, \circ)$ is called a *semihypergroup* if for all x, y, z of H we have $(x \circ y) \circ z = x \circ (y \circ z)$, which means that

$$\bigcup_{u\in x\circ y} u \circ z = \bigcup_{v\in y\circ z} x \circ v.$$

Definition 1.2. [4] A map $f : H \times H \times H \to \mathcal{P}^*(H)$ is called *ternary hyperoperation* on the set H, where H is a nonempty set and $\mathcal{P}^*(H) = \mathcal{P}(H)\backslash\{\emptyset\}$ denotes the set of all nonempty subsets of H.

Definition 1.3. [4] A *ternary hypergroupoid* is called the pair (H, f) where f is a ternary hyperoperation on the set H.

If A, B, C are nonempty subsets of H, then we define

$$f(A, B, C) = \bigcup_{a\in A, b\in B, c\in C} f(a, b, c).$$

Definition 1.4. [4] A ternary hypergroupoid (H, f) is called a *ternary semihypergroup* if for all $a_1, \ldots, a_5 \in H$, we have

$$f(f(a_1, a_2, a_3), a_4, a_5) = f(a_1, f(a_2, a_3, a_4), a_5) = f(a_1, a_2, f(a_3, a_4, a_5)). \quad (*)$$

When we have a semihyperring, we show $(*)$ by $f(a_1, a_2, a_3, a_4, a_5)$, and so on.

Since the set $\{x\}$ can be identified with the element x, any ternary semigroup is a ternary semihypergroup. A ternary semigroup does not necessarily reduce to an ordinary semigroup. This has been shown by the following example.

EXAMPLE 1.5. [5] Let $S = \{-i, 0, i\}$ be a ternary semigroup under the multiplication over complex numbers while S is not a binary semigroup under the multiplication over complex numbers.

Los [10] showed that a ternary semigroup however may be embedded in an ordinary semigroup in such a way that the operation in the ternary semigroup is an (ternary) extension of the (binary) operation of the containing semigroup.

Definition 1.6. [4] Let (H, f) be a ternary semihypergroup. Then, H is called a *ternary hypergroup* if for all $a, b, c \in H$, there exist $x, y, z \in H$ such that:

$$c \in f(x, a, b) \cap f(a, y, b) \cap f(a, b, z).$$

Definition 1.7. [4] Let (H, f) be a ternary hypergroupoid. Then,

(1) (H, f) is $(1,3)$-*commutative* if for all $a_1, a_2, a_3 \in H$, $f(a_1, a_2, a_3) = f(a_3, a_2, a_1)$;
(2) (H, f) is $(2,3)$-*commutative* if for all $a_1, a_2, a_3 \in H$, $f(a_1, a_2, a_3) = f(a_1, a_3, a_2)$;
(3) (H, f) is $(1,2)$-*commutative* if for all $a_1, a_2, a_3 \in H$, $f(a_1, a_2, a_3) = f(a_2, a_1, a_3)$;
(4) (H, f) is *commutative* if for all $a_1, a_2, a_3 \in H$ and for all $\sigma \in \mathbb{S}_3$, $f(a_1, a_2, a_3) = f(a_{\sigma(1)}, a_{\sigma(2)}, a_{\sigma(3)})$.

Different examples of ternary semihypergroups can be found in [4].

Definition 1.8. Let (H, f) be a ternary semihypergroup and T a nonempty subset of H. Then, T is called a ternary subsemihypergroup of H if and only if $f(T, T, T) \subseteq T$.

Definition 1.9. A ternary semihypergroup (H, f) is said to have a *zero element* if there exists an element $0 \in H$ such that for all $a, b \in H, f(0, a, b) = f(a, 0, b) = f(a, b, 0) = \{0\}$.

Definition 1.10. Let (H, f) be a ternary semihypergroup. An element $e \in H$ is called a *left identity* element of H if for all $a \in H$, $f(e, a, a) = \{a\}$. An element $e \in H$ is called an *identity* element of H if for all $a \in H$, $f(a, a, e) = f(e, a, a) = f(a, e, a) = \{a\}$. It is clear that $f(e, e, a) = f(e, a, e) = f(a, e, e) = \{a\}$ [4].

Definition 1.11. A nonempty subset I of a ternary semihypergroup H is called a *left* (*right, lateral*) *hyperideal* of H if

$$f(H,H,I) \subseteq I(f(I,H,H) \subseteq I, f(H,I,H) \subseteq I).$$

A nonemtpy subset I of a ternary semihypergroup H is called a *hyperideal* of H if it is a left, right and lateral hyperideal of H. A nonemtpy subset I of a ternary semihypergroup H is called *two-sided hyperideal* of H if it is a left and right hyperideal of H.

Lemma 1.12. *The union of hyperideals is a hyperideal.*

Proof. Suppose that I and J are hyperideals of a semihypergroup (H,f). We then have

$$\begin{aligned} f(H,H,I\cup J) &= \bigcup_{a\in I\cup J} f(H,H,a) \\ &= \left(\bigcup_{a\in I} f(H,H,a)\right) \cup \left(\bigcup_{a\in J} f(H,H,a)\right) \\ &\subseteq I \cup J. \end{aligned}$$

□

Definition 1.13. Let (H,f) be a ternary semihypergroup. For every element $a \in H$, the left, right, lateral, two-sided and hyperideal generated by a are respectively given by

$$\begin{aligned} \langle a\rangle_l &= \{a\} \cup f(H,H,a) \\ \langle a\rangle_r &= \{a\} \cup f(a,H,H) \\ \langle a\rangle_m &= \{a\} \cup f(H,a,H) \\ \langle a\rangle_t &= \{a\} \cup f(H,H,a) \cup f(a,H,H) \cup f(H,H,a,H,H) \\ \langle a\rangle &= \{a\} \cup f(H,H,a) \cup f(a,H,H) \cup f(H,a,H) \cup f(H,H,a,H,H) \end{aligned}$$

Definition 1.14. Let H be a ternary semihypergroup and M a left hyperideal of H. Then, M is called *maximal left hyperideal* of H if $M \neq H$ and there does not exist any proper left hyperideal I of H such that $M \subset I$.

Definition 1.15. A left hyperideal I of a ternary semihypergroup H is called *idempotent* if $f(I,I,I) = I$.

Definition 1.16. A ternary semihypergroup H is called *semisimple* if all its left hyperideals are idempotent.

It is obvious that the union of any family of left hyperideals of a ternary semihypergroup H is again a left hyperideal of H, and the intersection of any family of left hyperideals of a ternary semihypergroup H is a left hyperideal of H.

The following lemma is obviously.

Lemma 1.17. *If A,B,C are any left three hyperideals of a ternary semihypergroup (H,f), then $f(A,B,C)$ is a left hyperideal of H.*

Proof. We have $f(H,H,f(A,B,C)) = f(f(H,H,A),B,C) \subseteq f(A,B,C)$. □

2. On prime left hyperideals in ternary semihypergroups

In this section we introduce and characterize the prime hyperideals and left hyperideals, the semiprime hyperideals and left hyperideals in ternary semihypergroups. Some properties of them are investigated.

Definition 2.1. Let (H, f) be a ternary semihypergroup. A proper hyperideal (left hyperideal) P of H is called *prime hyperideal (left hyperideal)* of H if $f(A, B, C) \subseteq P$ implies $A \subseteq P$ or $B \subseteq P$ or $C \subseteq P$ for any three hyperideals (left hyperideals) A, B, C of H.

Definition 2.2. Let (H, f) be a ternary semihypergroup. A proper hyperideal (left hyperideal) P of H is called *semiprime hyperideal (left hyperideal)* of H if $f(I, I, I) \subseteq P$ implies $I \subseteq P$ for any hyperideal (left hyperideal) I of H.

It is clear that every prime left hyperideal of a ternary semihypergroup H is also a semiprime left hyperideal of H.

Definition 2.3. Let (H, f) be a ternary semihypergroup. A proper hyperideal P of H is said to be *irreducible*, if for hyperideals H and K of H, $H \cap K = P$ implies that $P = H$ or $P = K$ or equivalently, $H \cap K = P$ implies that $H \subseteq P$ or $K \subseteq P$.

Example 2.4. Let $H = \{a, b, c, d, e, f\}$ and $f(x, y, z) = (x * y) * z$ for all $x, y, z \in H$, where $*$ is defined by the table:

$*$	a	b	c	d	e	f
a	a	$\{a, b\}$	c	$\{c, d\}$	e	$\{e, f\}$
b	b	b	d	d	f	f
c	c	$\{c, d\}$	c	$\{c, d\}$	c	$\{c, d\}$
d	d	d	d	d	d	d
e	e	$\{e, f\}$	c	$\{c, d\}$	e	$\{e, f\}$
f	f	f	d	d	f	f

Then, (H, f) is a ternary semihypergroup. Clearly, $I_1 = \{c, d\}$, $I_2 = \{c, d, e, f\}$ and H are left hyperideals of H. It can be easily verified that I_2 and H are prime left hyperideals. I_1 is irreducible left hyperideal.

Theorem 2.5. *Let H be a ternary semihypergroup, P a semiprime left hyperideal of H and $a \in H$. Then, $a \in P$ if and only if $f(H, H, a) \subseteq P$.*

Proof. Assume that P is a semiprime left hyperideal of H. If $a \in P$, then $f(H, H, a) \subseteq f(H, H, P) \subseteq P$.

Conversely, assume that $f(H, H, a) \subseteq P$ where P is a semiprime left hyperideal of H. Then,

$$\begin{aligned} f(\langle a\rangle_l, \langle a\rangle_l, \langle a\rangle_l) &= f(a \cup f(H, H, a), \{a\} \cup f(H, H, a), \{a\} \cup f(H, H, a)) \subseteq \\ &\subseteq f(H, H, a) \subseteq P. \end{aligned}$$

Since P is a semiprime left hyperideal, we have $\langle a\rangle_l \subseteq P$ and hence $a \in P$. $\square$

Theorem 2.6. *Let H be a ternary semihypergroup and P a left hyperideal of H. The following statements are equivalent:*

(1) *P is a prime left hyperideal of H.*
(2) *$f(a, H, H, b, H, H, c) \subseteq P$ implies $a \in P$ or $b \in P$ or $c \in P$.*
(3) *$f(\langle a\rangle_l, \langle b\rangle_l, \langle c\rangle_l) \subseteq P$ implies $a \in P$ or $b \in P$ or $c \in P$.*

Proof. (1) $\Rightarrow$ (2). Assume that P is a prime left hyperideal of H and $f(a, H, H, b, H, H, c) \subseteq P$. Then, $f(H, H, f(a, H, H, b, H, H, c)) \subseteq f(H, H, P) \subseteq P$. This implies $f(f(H, H, a), f(H, H, b), f(H, H, c)) \subseteq P$. Since P is a prime left hyperideal of H, so $f(H, H, a) \subseteq P$ or $f(H, H, b) \subseteq P$ or $f(H, H, c) \subseteq P$. Since every prime left hyperideal is semiprime, so P is semiprime left hyperideal. Hence, by Theorem 2.5, $a \in P$ or $b \in P$ or $c \in P$.

(2) $\Rightarrow$ (3). Let $f(\langle a\rangle_l, \langle b\rangle_l, \langle c\rangle_l) \subseteq P$ for some $a, b, c \in P$. Then,

$$f(a, H, H, b, H, H, c) = f(a, f(H, H, b), f(H, H, c)) \subseteq f(\langle a\rangle_l, \langle b\rangle_l, \langle c\rangle_l) \subseteq P.$$

Thus, by (2) we have $a \in P$ or $b \in P$ or $c \in P$.

(3) $\Rightarrow$ (1). Let A, B, C be three left hyperideals of H such that $f(A, B, C) \subseteq P$. Let $B \nsubseteq P$ and $C \nsubseteq P$. Let $b \in B$ and $c \in C$ such that $b, c \notin P$. Then, for every $a \in A$ we have $f(\langle a\rangle_l, \langle b\rangle_l, \langle c\rangle_l) \subseteq f(A, B, C) \subseteq P$. By (3) we get $a \in P$ or $b \in P$ or $c \in P$, but $b \notin P$ and $c \notin P$, so $a \in P$ and hence $A \subseteq P$. Therefore, P is a prime left hyperideal of H. $\square$

Corollary 2.7. Let H be a commutative ternary semihypergroup and P a left hyperideal of H. Then, P is a prime left hyperideal if and only if $f(a, b, c) \subseteq P$ implies $a \in P$ or $b \in P$ or $c \in P$ for all $a, b, c \in H$.

It can be easily seen that the above result is also valid for $(1, 3)$-commutative ternary semihypergroup.

Definition 2.8. Let H be a ternary semihypergroup. A nonempty subset A of H is called a *left m-system* if for every $x, y, z \in A$ there exist the elements $a_1, a_2, a_3, a_4 \in H$ such that $f(x, a_1, a_2, y, a_3, a_4, z) \subseteq A$.

Theorem 2.9. *Let H be a ternary semihypergroup and P a proper left hyperideal of H. P is a prime left hyperideal if and only if its complement P^c is a left m-system.*

Proof. Let P be a prime left hyperideal of H. Assume that $x, y, z \notin P$. Then, $x, y, z \in P^c$. Let assume that P^c is not a left m-system. Then, for all $a_1, a_2, a_3, a_4 \in H$, $f(x, a_1, a_2, y, a_3, a_4, z) \nsubseteq P^c$. Thus, $f(x, a_1, a_2, y, a_3, a_4, z) \subseteq P$. Since P is prime left hyperideal of H, so by Theorem 2.6 we get $a \in P$ or $b \in P$ or $c \in P$. It is impossible. Hence, P^c is a left m-system.

Conversely, suppose that P^c is a left m-system. Let $x, y, z \in P^c$. Then, there exist $a_1, a_2, a_3, a_4 \in H$ such that $f(x, a_1, a_2, y, a_3, a_4, z) \subseteq P^c$. Thus,

$f(x, a_1, a_2, y, a_3, a_4, z) \nsubseteq P$. Hence, if $x, y, z \notin P$, then $f(x, H, H, y, H, H, z) \nsubseteq P$. Thus, by Theorem 2.6, P is a prime left hyperideal of H. □

Definition 2.10. Let H be a ternary semihypergroup and P a left hyperideal of H. Then, P is called a *completely prime* left hyperideal of H if $f(a, b, c) \subseteq P$ implies $a \in P$ or $b \in P$ or $c \in P$ for every element $a, b, c \in H$.

It is clear that every completely prime left hyperideal of H is a prime left hyperideal of H, but the converse in general may not be true. For commutative ternary semihypergroup the concepts of completely prime and prime left hyperideal coincide.

Lemma 2.11. *Let H be a ternary semihypergroup. A left hyperideal P of a ternary semihypergroup H is completely prime if and only if P^c is ternary subsemihypergroup of H.*

Proof. It is straightforward. □

Theorem 2.12. *Let H be a ternary semihypergroup. A left hyperideal P of H is completely prime if and only if for every pair $(m, n) \in Z^+$ with even sum, $f(f(\underbrace{H, \ldots, H}_{m\ times}), f(\underbrace{H, \ldots, H}_{n\ times}), f(a, b, c)) \subseteq P$ implies $a \in P$ or $b \in P$ or $c \in P$.*

Proof. Assume that P is completely prime left hyperideal of H and

$$f(f(\underbrace{H, \ldots, H}_{m\ times}), f(\underbrace{H, \ldots, H}_{n\ times}), f(a, b, c)) \subseteq P.$$

Then,

$$f(\underbrace{f(a, b, c), f(a, b, c), \ldots, f(a, b, c)}_{m+n+1\ times}) \subseteq P.$$

Since P is completely prime, by mathematical induction we get $f(a, b, c) \subseteq P$ and hence $a \in P$ or $b \in P$ or $c \in P$.

Conversely, assume that $f(a, b, c) \subseteq P$. Then, for every pair $(m, n) \in Z^+$ with even sum,

$$\begin{aligned} f(f(\underbrace{H, \ldots, H}_{m\ times}), f(\underbrace{H, \ldots, H}_{n\ times}), f(a, b, c)) &\subseteq f(f(\underbrace{H, \ldots, H}_{m\ times}), f(\underbrace{H, \ldots, H}_{n\ times}), P) \\ &\subseteq f(H, H, P) \subseteq P. \end{aligned}$$

Consequently, $a \in P$ or $b \in P$ or $c \in P$. Therefore, P is completely prime. □

Proposition 2.13. Let H be a ternary semihypergroup with left identity. Then, every maximal left hyperideal of H is prime hyperideal of H.

Proof. Let P be a maximal left hyperideal of H. Let A, B, C be three left hyperideals of H such that $f(A, B, C) \subseteq P$. Suppose that $A, B \nsubseteq P$. Then, $A \cup P = H$ and $B \cup P = H$. Since $e \in H$, we have $e \in A \cup P$ and $e \in B \cup P$. Thus, $e \in A$ or $e \in P$ and $e \in B$ or $e \in P$. Since $e \notin P$, we have

$e \in A$ and $e \in B$ implies $A = H$ and $B = H$. Now since $e \in H$, we have $C = f(H, H, C) = f(A, B, C) \subseteq P$ implies $C \subseteq P$. Therefore, P is a prime left hyperideal of H. □

Theorem 2.14. *Let H be a ternary semihypergroup, P be a left m-system and Q be a left hyperideal of H such that $P \cap Q = \emptyset$. Then, there exists a maximal left hyperideal M of H containing Q and $M \cap Q = \emptyset$. Further, M is also a prime left hyperideal of H.*

Proof. Let $\tau_P = \{B : B$ is a prime left hyperideal of $H, Q \subseteq B, B \cap P = \emptyset\}$. Since $Q \in \tau_P$, $\tau_P \neq \emptyset$. τ_P is partially ordered set by set inclusion. Let $\{M_i\}$ be an arbitrary chain in τ_P. Since union of left hyperideals is a left hyperideal, $\bigcup_{i \in I} M_i$ is a left hyperideal of H. Since $Q \subseteq M_i$ for all $i \in I$, we have $Q \subseteq \bigcup_{i \in I} M_i$. Assume that $(\bigcup_{i \in I} M_i) \cap P \neq \emptyset$. Then, there exist some $a \in H$ such that $a \in (\bigcup_{i \in I} M_i) \cap P$. This implies $a \in \bigcup_{i \in I} M_i$ and $a \in P$. Thus, $a \in M_i$ for some $i \in I$ and $a \in P$. Thus, $M_i \cap P \neq \emptyset$. It is impossible. Hence, $(\bigcup_{i \in I} M_i) \cap P = \emptyset$. Thus, $\bigcup_{i \in I} M_i$ is an upper bound of $\{M_i\}$. Since $\{M_i\}$ is an arbitrary chain, we have that every chain in τ_P has an upper bound in τ_P. Hence, by Zorn's Lemma the family τ_P contains a maximal element M. We will show that M is a prime left hyperideal of H. Let A, B, C be three left hyperideals of H such that $f(A, B, C) \subseteq M$. Assume that $A \nsubseteq M, B \nsubseteq M$ and $C \nsubseteq M$. Then, there exist $a \in A, b \in B$ and $c \in C$ such that $a, b, c \notin M$. Now $\langle a \rangle_l \cup M, \langle b \rangle_l \cup M$ and $\langle c \rangle_l \cup M$ are left hyperideals of H properly containing M, so $(\langle a \rangle_l \cup M) \cap P \neq \emptyset, (\langle b \rangle_l \cup M) \cap P \neq \emptyset$ and $(\langle c \rangle_l \cup M) \cap P \neq \emptyset$. Let $x \in (\langle a \rangle_l \cup M) \cap P, y \in (\langle b \rangle_l \cup M) \cap P$ and $z \in (\langle c \rangle_l \cup M) \cap P$. Since $x, y, z \in P$ and P is left m-system, we have $f(x, x_1, x_2, y, x_3, x_4, z) \subseteq P$ for some $x_1, x_2, x_3, x_4 \in H$. Also $f(x, x_1, x_2, y, x_3, x_4, z) \subseteq f((\langle a \rangle_l \cup M), H, H, (\langle b \rangle_l \cup M), H, H, (\langle c \rangle_l \cup M))$. Thus, we have

Case I: If $x \in \langle a \rangle_l \subseteq A, y \in \langle b \rangle_l \subseteq B, z \in \langle c \rangle_l \subseteq C$, then

$$f(x, x_1, x_2, y, x_3, x_4, z) \subseteq f(A, f(H, H, B), f(H, H, C)) \subseteq f(A, B, C) \subseteq M.$$

Case II: If $x \in M$, then

$$f(x, x_1, x_2, y, x_3, x_4, z) \subseteq f(M, f(H, H, M), f(H, H, M)) \subseteq f(M, M, M) \subseteq M.$$

Similarly, if $y \in M$ or $z \in M$, then

$$f(x, x_1, x_2, y, x_3, x_4, z) \subseteq f(H, H, f(H, M, H), H, H) \subseteq f(H, M, H) \subseteq M$$

and

$$f(x, x_1, x_2, y, x_3, x_4, z) \subseteq f(H, H, H, H, f(H, H, M)) \subseteq f(H, H, M) \subseteq M.$$

Hence, $P \cap M \neq \emptyset$, it is impossible. Thus, $A \subseteq M$ or $B \subseteq M$ or $C \subseteq M$. Therefore, M is a prime left hyperideal of H. □

3. On hyperfilters in ternary semihypergroups

In this section we introduce the concepts of hyperfilters and hypersemilattice congruence of ternary semihypergroups. We give some characterizations of hyperfilters in ternary semihypergroups. Some relationships between the hyperfilters and the prime hyperideals and hypersemilattice congruences in ternary semihypergroups are considered.

Let H be a ternary semihypergroup. A ternary subsemihypergroup $F_l(F_m, F_r)$ of ternary semihypergroup H is called a *left (lateral, right) hyperfilter* of H if for any $a, b, c \in H, f(a, b, c) \subseteq F_l(f(a, b, c) \subseteq F_m, f(a, b, c) \subseteq F_r)$ implies $c \in F_l(b \in F_m, c \in F_r)$. If a ternary subsemihypergroup of ternary semihypergroup H is a left, lateral and right hyperfilter of H, then it is called a *hyperfilter* of H. The intersection of all hyperfilters of a ternary semihypergroup H containing a nonempty subset A of H is the hyperfilter of H generated by A. For $A = \{x\}$, let $n(x)$ denote the hyperfilter of H generated by $\{x\}$. An equivalence relation ρ on a ternary semihypergroup H is called a *congruence* if for any $a, b \in H$, $a\rho b$ implies $\forall u \in f(a, x, y), \exists v \in f(b, x, y)$ such that $u\rho v$ and $\forall t \in f(y, x, a), \exists s \in f(y, x, b)$ such that $t\rho s$ for all $x, y \in H$. A congruence ρ on a ternary semihypergroup H is called a *hypersemilattice congruence* if $a\rho x$, $\forall x \in f(a, a, a), \forall a \in H$ and $\forall x \in f(a, b, c), \exists y \in f(a, c, b)$ such that $x\rho y$, $\exists z \in f(c, b, a)$ such that $x\rho z$, $\exists w \in f(b, a, c)$ such that $x\rho w$ for all $a, b, c \in H$. For a nonempty subset A of a ternary semihypergroup H, we define a relation on H as follows:

$$n := \{(x, y) \in H \times H : n(x) = n(y)\}.$$

It can be easily proved that n is an equivalence relation on H.

Theorem 3.1. *Let H be a ternary semihypergroup and F be a nonempty subset of H. Then, the following statements are equivalent:*

(1) *F is a left(lateral, right) hyperfilter of H.*
(2) *$H\backslash F = \emptyset$ or $H\backslash F$ is a prime left (lateral, right) hyperideal of H.*

Proof. Assume that F is a left hyperfilter of H and $H\backslash F \neq \emptyset$. Now let $a, b \in H$ and $c \in H\backslash F$. Then, $f(a, b, c) \subseteq H\backslash F$ because F is a left hyperfilter of H and $c \notin F$. Hence, $H\backslash F$ is a left hyperideal of H. Next let $a, b, c \in H$ be such that $f(a, b, c) \subseteq H\backslash F$. Then, $a \in H\backslash F$ or $b \in H\backslash F$ or $c \in H\backslash F$ because F is a ternary subsemihypergroup of H. Hence, $H\backslash F$ is a prime left hyperideal of H.

Conversely, if $H\backslash F = \emptyset$, then $F = H$. Hence, F is a left hyperfilter of H. Assume that $H\backslash F$ is a prime left hyperideal of H. Now let $a, b, c \in F$. Then, $f(a, b, c) \subseteq F$ because $H\backslash F$ is prime. Hence, F is a ternary subsemihypergroup of H. Next let $a, b, c \in H$ be such that $f(a, b, c) \subseteq F$. Then, $c \in F$ because $H\backslash F$ is a left hyperideal of H. Therefore, F is a left hyperfilter of H. So the proof is completed. $\square$

Corollary 3.2. Let H be a ternary semihypergroup and F be a nonempty subset of H. Then, the following statements are equivalent:

(1) F is a hyperfilter of H.
(2) $H\backslash F=\emptyset$ or $H\backslash F$ is a prime hyperideal of H.

Theorem 3.3. *If H is a ternary semihypergroup, then n is a hypersemilattice congruence on H.*

Proof. Let $a,x,y\in H$ be such that anb. Then, $n(a)=n(b)$. Since $f(b,x,y)\subseteq \bigcup n(t), \forall t\in f(b,x,y)$, we have $b,x,y\in \bigcup n(t), \forall t\in f(b,x,y)$. Thus, $n(a)=n(b)\subseteq \bigcup n(t), \forall t\in f(b,x,y)$, so $a,x,y\in \bigcup n(t), \forall t\in f(b,x,y)$. Hence, $f(a,x,y)\subseteq \bigcup n(t), \forall t\in f(b,x,y)$, so $\bigcup n(s)\subseteq \bigcup n(t), \forall t\in f(b,x,y), \forall s\in f(a,x,y)$. Similarly, $\bigcup n(t)\subseteq \bigcup n(s), \forall t\in f(b,x,y), \forall s\in f(a,x,y)$. Therefore, $\bigcup n(t)=\bigcup n(s), \forall t\in f(b,x,y), \forall s\in f(a,x,y)$. So $\forall s\in f(a,x,y), \exists t\in f(b,x,y)$ such that snt. Similarly we can show that $\forall s\in f(y,x,a), \exists t\in f(y,x,b)$ such that snt. This proves that n is a congruence on H. If $a\in H$, then $a\in n(x), \forall x\in f(a,a,a)$ because $f(a,a,a)\subseteq \bigcup n(x), \forall x\in f(a,a,a)$. Thus, $n(a)\subseteq \bigcup n(x), \forall x\in f(a,a,a)$. Since $a\in n(a)$, we have $f(a,a,a)\subseteq n(a)$. Hence, $\bigcup n(x)\subseteq n(a), \forall x\in f(a,a,a)$. Therefore, $anx, \forall x\in f(a,a,a)$. If $a,b,c\in H$, then $f(a,b,c)\subseteq \bigcup n(x), \forall x\in f(a,b,c)$. Thus, $a,b,c\in \bigcup n(x), \forall x\in f(a,b,c)$, so $f(c,b,a)\subseteq \bigcup n(x), \forall x\in f(a,b,c)$. Thus, $\bigcup n(x)\subseteq \bigcup n(y), \forall x\in f(c,b,a), \forall y\in f(a,b,c)$. Similarly, $\bigcup n(x)\subseteq \bigcup n(y), \forall x\in f(a,b,c), \forall y\in f(c,b,a)$. Hence, $\bigcup n(x)=\bigcup n(y), \forall x\in f(a,b,c), \forall y\in f(c,b,a)$, so $\forall x\in f(a,b,c), \exists y\in f(c,b,a)$, such that xny. Similarly, we can show another cases that $\forall x\in f(a,b,c), \exists y\in f(a,c,b)$, such that xny and $\forall x\in f(a,b,c), \exists y\in f(b,a,c)$, such that xny. Therefore, n is a hypersemilattice congruence on H. □

4. On semiprime left hyperideals in ternary semihypergroups

In this section we characterize the semiprime hyperideals and left hyperideals in ternary semihypergroups. Some properties of them are investigated.

Theorem 4.1. *Let H be a ternary semihypergroup and P a left hyperideal of H. The following statements are equivalent:*

(1) *P is a semiprime left hyperideal of H.*
(2) $f(a,H,H,a,H,H,a)\subseteq P\Rightarrow a\in P$.
(3) $f(\langle a\rangle_l, \langle a\rangle_l, \langle a\rangle_l)\subseteq P\Rightarrow a\in P$.

Proof. (1) $\Rightarrow$ (2). Assume that P is semiprime left hyperideal of H and $f(a,H,H,a,H,H,a)\subseteq P$. Then,

$$f(H,H,f(a,H,H,a,H,H,a))\subseteq f(H,H,P)\subseteq P.$$

Thus, $f(f(H,H,a),f(H,H,a),f(H,H,a))\subseteq P$. Since P is semiprime left hyperideal of H, we have $f(H,H,a)\subseteq P$ and by Theorem 2.5 we get $a\in P$.

(2) $\Rightarrow$ (3). Let $f(\langle a\rangle_l, \langle a\rangle_l, \langle a\rangle_l) \subseteq P$ for some $a \in P$. Then,

$$f(a,H,H,a,H,H,a) = f(a, f(H,H,a), f(H,H,a)) \subseteq f(\langle a\rangle_l, \langle a\rangle_l, \langle a\rangle_l) \subseteq P.$$

By (2) we get $a \in P$.

(3) $\Rightarrow$ (1). Let I be any left hyperideal of H such that $f(I,I,I) \subseteq P$. Let $I \nsubseteq P$. Then, there exists an element $a \in I$ such that $a \notin P$. So, $f(\langle a\rangle_l, \langle a\rangle_l, \langle a\rangle_l) \subseteq f(I,I,I) \subseteq P$. By (3) we have $a \in P$. It is impossible. Therefore, $I \subseteq P$, and so P is a semiprime left hyperideal of H. $\square$

Corollary 4.2. Let H be a commutative ternary semihypergroup and P be a proper left hyperideal of H. Then, P is semiprime if and only if for every $a \in H$, $f(a,a,a) \subseteq P$ implies $a \in P$.

Proof. Let H be a commutative ternary semihypergroup. Assume that P is a semiprime left hyperideal of H and $f(a,a,a) \subseteq P$ for some $a \in H$. Then, we have

$$f(H,H,f(H,H,f(a,a,a))) \subseteq f(H,H,P) \subseteq P.$$

Since H is commutative, we have $f(a,H,H,a,H,H,a) \subseteq P$. By Theorem 3.1, since P is semiprime left hyperideal of H, we have $a \in P$.

Conversely, assume that $f(a,a,a) \subseteq P \Rightarrow a \in P$. Suppose that for any left hyperideal I of H, $f(I,I,I) \subseteq P$. Assume that $I \nsubseteq P$. Then, there exists an element $x \in I$ such that $x \notin P$. Thus, $f(x,x,x) \subseteq f(I,I,I) \subseteq P$ implies $x \in P$. It is impossible. Therefore, $I \subseteq P$ and hence P is a semiprime left hyperideal of H. $\square$

Definition 4.3. Let H be a ternary semihypergroup. A nonempty subset A of H is called a *left p-system* if for every $a \in A$ there exist elements $b_1, b_2, b_3, b_4 \in H$ such that $f(a, b_1, b_2, a, b_3, b_4, a) \subseteq A$.

It is clear that in a ternary semihypergroup H every left m-system is a left p-system and the union of left p-systems is a left p-system too.

Theorem 4.4. *Let H be a ternary semihypergroup and P be a proper left hyperideal of H. Then, P is semiprime if and only if its compliment P^c is a left p-system.*

Proof. Let P be a semiprime left hyperideal of H. Assume that $b \notin P$. Then, $b \in P^c$. Assume that P^c is not a left p-system. Then, for all $a_1, a_2, a_3, a_4 \in H$, $f(b, a_1, a_2, b, a_3, a_4, b) \nsubseteq P^c$. This implies that for all $a_1, a_2, a_3, a_4 \in H$, $f(b, a_1, a_2, b, a_3, a_4, b) \subseteq P$. By Theorem 3.1 it follows that $b \in P$. It is impossible. Therefore, P^c is a left p-system.

Conversely, assume that P^c is a left p-system. Let $b \in P^c$. Then, there exist $a_1, a_2, a_3, a_4 \in H$ such that $f(b, a_1, a_2, b, a_3, a_4, b) \subseteq P^c$. Hence,

$$f(b, a_1, a_2, b, a_3, a_4, b) \nsubseteq P.$$

Thus, $b \notin P$ which implies that $f(b, H, H, b, H, H, b) \nsubseteq P$. By Theorem 3.1 it follows that P is a semiprime left hyperideal of H. $\square$

Definition 4.5. Let H be a ternary semihypergroup. A left hyperideal P of H is called a *completely semiprime* left hyperideal of H if for any $a \in H$, $f(a, a, a) \subseteq P$ implies $a \in P$.

Theorem 4.6. *Let H be a ternary semihypergroup. A proper left hyperideal P of H is completely semiprime if and only if for every pair $(m, n) \in Z^+$ with even sum, $f(f(\underbrace{H, \ldots, H}_{m\ times}), f(\underbrace{H, \ldots, H}_{n\ times}), f(a, a, a)) \subseteq P$ implies $a \in P$.*

Proof. Assume that P is completely semiprime left hyperideal of H and

$$f(f(\underbrace{H, \ldots, H}_{m\ times}), f(\underbrace{H, \ldots, H}_{n\ times}), f(a, a, a)) \subseteq P.$$

Then,

$$f(\underbrace{f(a, a, a), f(a, a, a), \ldots, f(a, a, a)}_{m+n+1\ times}) \subseteq P.$$

Since P is completely left semiprime, by mathematical induction we have $f(a, a, a) \subseteq P$ and hence $a \in P$.

Conversely, suppose that $f(a, a, a) \subseteq P$. Then, for every pair $(m, n) \in Z^+$ with even sum,

$$f(f(\underbrace{H, \ldots, H}_{m\ times}), f(\underbrace{H, \ldots, H}_{n\ times}), f(a, a, a)) \subseteq f(H, H, P) \subseteq P.$$

This implies that $a \in P$. Hence, P is completely semiprime. $\square$

Theorem 4.7. *Let H be a ternary semihypergroup, A be a p-system and I be a left hyperideal of H such that $A \cap I = \emptyset$. Then, there exists a maximal left hyperideal M of H containing I such that $A \cap M = \emptyset$. Further, M is also a semiprime left hyperideal of H.*

Proof. Let $\tau_A = \{M : M \text{ is proper left hyperideal of } H, I \subseteq M, M \cap A = \emptyset\}$. Since $I \in \tau_A$, we have $\tau_A \neq \emptyset$. τ_A is partially ordered set by set inclusion. Let $\{M_i\}$ be an arbitrary chain in τ_A. We show that $\bigcup\limits_{i \in J} M_i$ is an element of τ_A. Since the union of left hyperideals is a left hyperideals, we have $\bigcup\limits_{i \in J} M_i$ is a left hyperideal of H. Also, $I \subseteq M_i$ for all $i \in J$. Therefore, $I \subseteq \bigcup\limits_{i \in J} M_i$. Now we show that $(\bigcup\limits_{i \in J} M_i) \cap A = \emptyset$. Assume that $(\bigcup\limits_{i \in J} M_i) \cap A \neq \emptyset$. Then, there exists $a \in H$ such that $a \in (\bigcup\limits_{i \in J} M_i) \cap A \neq \emptyset$ and so $a \in \bigcup\limits_{i \in J} M_i$ and $a \in A$. Thus, $a \in M_i$ for some $i \in J$ and $a \in A$. So $M_i \cap A \neq \emptyset$. It is impossible. Hence, $(\bigcup\limits_{i \in J} M_i) \cap A = \emptyset$. Thus, $\bigcup\limits_{i \in J} M_i$ is the upper bound of $\{M_i\}$. By Zorn's Lemma the family τ_A contains a maximal element, M. We will show that M

is a semiprime left hyperideal of H. Let $f(X,X,X) \subseteq M$ where X is a left hyperideal of H. Assume that $X \nsubseteq M$. Then, there exists $x \in X$ such that $x \notin M$. We have $\langle a\rangle_l \cup M$ is a left hyperideal of H properly containing M such that $(\langle a\rangle_l \cup M) \cap A \neq \emptyset$. Let $c \in (\langle a\rangle_l \cup M) \cap A \Rightarrow c \in A$. Since A is a left p-system, there exist $c_1, c_2, c_3, c_4 \in H$ such that $f(c, c_1, c_2, c, c_3, c_4, c) \subseteq A$. Thus, $f(c, c_1, c_2, c, c_3, c_4, c) \subseteq f(\langle a\rangle_l \cup M, H, H, \langle a\rangle_l \cup M, H, H, \langle a\rangle_l \cup M)$. We have

Case I: If $c \in \langle a\rangle_l$, since $\langle a\rangle_l \subseteq X$, we have $c \in X$. Thus,

$$f(c, c_1, c_2, c, c_3, c_4, c) \subseteq f(X, f(H,H,X), f(H,H,X)) \subseteq f(X,X,X) \subseteq M.$$

Case II: If $c \in M$, then

$$f(c, c_1, c_2, c, c_3, c_4, c) \subseteq f(M, f(H,H,M), f(H,H,M)) \subseteq f(M,M,M) \subseteq M.$$

So, $A \cap M \neq \emptyset$, which is impossible. Hence, $X \subseteq M$. Thus, M is a semiprime left hyperideal of H. $\square$

5. On irreducible left hyperideals in ternary semihypergroups

In this section, we introduce and characterize irreducible left hyperideals in ternary semihypergroups. Some properties of them are investigated.

Definition 5.1. Let H be a ternary semihypergroup. A proper left hyperideal I of H is said to be irreducible, if for left hyperideals T and K of H, $T \cap K = I$ implies $I = T$ or $I = K$.

It can be easily shown that the above definition is equivalently with the following definition.

Definition 5.2. Let H be a ternary semihypergroup. A proper left hyperideal I of H is said to be irreducible, if for left hyperideals T and K of H, $T \cap K \subseteq I$ implies $T \subseteq I$ or $K \subseteq I$.

Proposition 5.3. Let H be a ternary semihypergroup, $a \in H$ and I a proper left hyperideal of H such that $a \notin I$. Then, there exists an irreducible left hyperideal T of H such that $I \subseteq T$ and $a \notin T$.

Proof. Let $\mathfrak{F} = \{J \text{ a left hyperideal of } H : I \subseteq J \text{ and } a \notin I\}$. It is clear that $\mathfrak{F} \neq \emptyset$ since $I \in \mathfrak{F}$. The set $\mathfrak{F}$ is partially ordered set under set inclusion. Let $\{T_i : i \in \Omega\}$ be a chain of left hyperideals in $\mathfrak{F}$. Then, $T = \bigcup_{i\in\Omega} T_i$ is a left hyperideal of H such that $a \notin T$. By Zorn's Lemma it follows that the collection $\mathfrak{F}$ contains a maximal element M. We have to show that M is an irreducible left hyperideal. Assume that $M = B \cap C$ where B and C are left hyperideal such that $I \subset B, C$. Since $M = B \cap C$, by the property of M, we have $a \in B$ and $a \in C$. Thus, $a \in B \cap C = M$, which is impossible. Thus, $M = B$ or $M = C$. Hence, M is an irreducible left hyperideal. $\square$

Proposition 5.4. Let H be a ternary semihypergroup. Every proper left hyperideal I of H is the intersection of all irreducible left hyperideals containing it.

Proof. It is straightforward. □

Proposition 5.5. Let H be a ternary semihypergroup and I be an irreducible semiprime left hyperideal of H. Then, I is a prime left hyperideal of H.

Proof. It is straightforward. □

Definition 5.6. Let H be a ternary semihypergroup. A nonemtpy subset A of H is called a left i-system if for $a, b \in A$, $\langle a\rangle_l \cap \langle b\rangle_l \cap A \neq \emptyset$.

Theorem 5.7. *Let H be a ternary semihypergroup. The following statements are equivalent:*

(1) *H is semisimple.*
(2) *For every three left hyperideals I, J, K of H, $I \cap J \cap K \subseteq f(I, J, K)$.*
(3) *Every left hyperideal of H is semiprime left.*
(4) *Every left hyperideal of H is the intersection of prime left hyperideal containing it.*

Proof. (1) ⇒ (2). Let H be semisimple and I, J, K be left hyperideals of H. Since $I \cap J \cap K$ is a left hyperideal of H, we have

$$I \cap J \cap K = f(I \cap J \cap K, I \cap J \cap K, I \cap J \cap K) \subseteq f(I, J, K).$$

(2) ⇒ (1). Taking $I = J = K$ we have $I \subseteq f(I, I, I)$ and since $f(I, I, I) \subseteq I$ it follows that $I = f(I, I, I)$.

(1) ⇒ (3). Let H be semisimple and I, J be left hyperideals of H such that $f(I, I, I) \subseteq J$. Since H is semisimple we have that $f(I, I, I) = I$. Thus, $I \subseteq J$. Hence, J is a semiprime left hyperideal of H.

(3) ⇒ (4). Suppose that every left hyperideal of H is a semiprime left hyperideal. By Proposition 4.4 and (3) we have that every left hyperideal of H is the intersection of all irreducible semiprime left hyperideals of H containing it. By proposition 4.5 every irreducible semiprime left hyperideal is prime left hyperideal. Thus, every left hyperideal I of H is the intersection of all prime left hyperideals containing it.

(4) ⇒ (1). Suppose that every left hyperideal of H is the intersection of all prime left hyperideals containing it. Let I be any left hyperideal of H. By (4) it is the intersection of all prime left hyperideals containing it. Since the intersection of prime hyperideals is a semiprime left hyperideal, we have that I is a semiprime left hyperideal. But since $f(I, I, I) \subseteq f(I, I, I)$, then $I \subseteq f(I, I, I)$ and $f(I, I, I) \subseteq I$ we have $I = f(I, I, I)$. Hence, H is semisimple. □

6. On hyperideal extensions in ternary semihypergroups

In this section we introduce the notion of hyperideal extension in ternary semihypergroups and characterize the properties of hyperideal extension in ternary semihypergroups.

Let H be a ternary semihypergroup, I be a hyperideal of H and $A \subseteq H$. Then, we have

$$< A, I >:= \{h \in H | f(A, H, h) \subseteq I\}.$$

contains I and is called the extension of I by A. It is clear that $f(A, H, < A, I >) \subseteq I$ and for $B \subseteq H, f(A, H, B) \subseteq I$ implies $B \subseteq < A, I >$. In fact, if H is commutative, $< A, I >$ is a hyperideal of H containing I. Since

$$\begin{aligned} f(A, H, f(< A, I >, H, H)) &= f(f(A, H, < A, I >), H, H) \\ &= f(I, H, H) \\ &= I, \end{aligned}$$

it follows that $f(< A, I >, H, H) \subseteq < A, I >$.

Lemma 6.1. *Let H be a ternary semihypergroup, I be a hyperideal of H and $A, B \subseteq H$. Then, the following statements hold:*

(1) *If $A \subseteq B$, then $< B, I > \subseteq < A, I >$.*
(2) *If $A \subseteq I$, then $< A, I >= H$.*
(3) *$< A, I > \subseteq < A \backslash I, I >$.*

Proof. (1). Let $A \subseteq B$. Then, we have

$$\begin{aligned} x \in < B, I > &\Rightarrow f(B, H, x) \subseteq I \\ &\Rightarrow f(A, H, x) \subseteq I \\ &\Rightarrow x \in < A, I > . \end{aligned}$$

Hence, $< B, I > \subseteq < A, I >$.

(2). Let $A \subseteq I$. It is clear that $< A, I > \subseteq H$. If $x \in H$, then we have

$$f(A, H, x) \subseteq f(I, H, x) \subseteq I.$$

Thus, $x \in < A, I >$, so $H \subseteq < A, I >$. Hence, $< A, I >= H$.

(3). Since $A \backslash I \subseteq A$, by (1) it follows that $< A, I > \subseteq < A \backslash I, I >$. □

Proposition 6.2. Let H be a ternary semihypergroup, I be a hyperideal of H and $A \subseteq H$. Then, for every subsets X, Y of H, we have

$$< A, I > \subseteq < f(X, H, A), I > \subseteq < f(X, Y, A), I >.$$

Proof. Since

$$\begin{aligned} f(f(X, H, A), H, < A, I >) &= f(X, H, f(A, H, < A, I >)) \\ &\subseteq f(X, H, I) \\ &\subseteq I \end{aligned}$$

and

$$\begin{aligned} f(f(X,Y,A),H,< f(X,H,A),I >) &\subseteq f(f(X,H,A),H,< f(X,H,A),I >) \\ &\subseteq I, \end{aligned}$$

it follows that

$$< A, I > \subseteq < f(X,H,A), I > \text{ and } < f(X,H,A), I > \subseteq < f(X,Y,A), I > .$$

. □

Proposition 6.3. Let H be a ternary semihypergroup, I and I_i be hyperideals of H and $A, A_i \subseteq H$ for all $i \in \Omega$. Then, we have

(1) $< A, \bigcap\limits_{i\in\Omega} I_i >= \bigcap\limits_{i\in\Omega} < A, I_i >$.

(2) $< \bigcup\limits_{i\in\Omega} A_i, I >= \bigcap\limits_{i\in\Omega} < A_i, I >$.

Proof. (1). For $x \in H$, we have

$$\begin{aligned} x \in < A, \bigcap_{i\in\Omega} I_i > &\Leftrightarrow f(A,H,x) \subseteq \bigcap_{i\in\Omega} I_i \\ &\Leftrightarrow f(A,H,x) \subseteq I_i, \forall i \in \Omega \\ &\Leftrightarrow x \in < A, I_i >, \forall i \in \Omega \\ &\Leftrightarrow x \in \bigcap_{i\in\Omega} < A, I_i > . \end{aligned}$$

Hence, $< A, \bigcap\limits_{i\in\Omega} I_i >= \bigcap\limits_{i\in\Omega} < A, I_i >$.

(2). For $x \in H$, we have

$$\begin{aligned} x \in < \bigcup_{i\in\Omega} A_i, I > &\Leftrightarrow f(\bigcup_{i\in\Omega} A_i, H, x) \subseteq I \\ &\Leftrightarrow f(A_i,H,x) \subseteq I, \forall i \in \Omega \\ &\Leftrightarrow x \in < A_i, I >, \forall i \in \Omega \\ &\Leftrightarrow x \in \bigcap_{i\in\Omega} < A_i, I > . \end{aligned}$$

Hence, $< \bigcup\limits_{i\in\Omega} A_i, I >= \bigcap\limits_{i\in\Omega} < A_i, I >$. □

Proposition 6.4. Let H be a ternary semihypergroup, I be a hyperideal of H. Then, I is a prime hyperideal of H if and only if $< A, I >= I$ for all $A \subseteq H$ with $A \nsubseteq I$.

Proof. Let I be a prime hyperideal of H and let $A \nsubseteq I$. Since $f(A, H, < A, I >) \subseteq I, A \nsubseteq I$ and I is a prime hyperideal of H, it follows that $< A, I > \subseteq I$ which implies that $< A, I >= I$.

Conversely, assume that $< A, I >= I$ for all $A \nsubseteq I$. Let $A, B \subseteq H$ such that $f(A, H, B) \subseteq I$ and $A \nsubseteq I$. Then, $B \subseteq < A, I >= I$. Hence, I is a prime hyperideal of H. □

The following proposition is an immediate corollary of Lemma 5.1(2) and Proposition 5.4.

Corollary 6.5. Let H be a commutative ternary semihypergroup. If I is a prime hyperideal of H and $A \subseteq H$, then $< A, I >$ is a prime hyperideal of H.

Corollary 6.6. Let H be a commutative ternary semihypergroup and $A \subseteq H$. If $\{I_i | i \in \Omega\}$ is a collection of prime hyperideals of H such that $\bigcap\limits_{i\in\Omega} I_i \neq \emptyset$, then $< A, \bigcap\limits_{i\in\Omega} I_i >$ is a semiprime hyperideal of H.

Proof. By Corollary 5.5 we have that $< A, I_i >$ is a prime hyperideal of H for all $i \in \Omega$. By Proposition 5.3(1) we have $< A, \bigcap\limits_{i\in\Omega} I_i = \bigcap\limits_{i\in\Omega} < A, I_i >$. By the obviously fact that a nonempty intersection of prime hyperideals of H is a semiprime hyperideal of H, it follows that $< A, \bigcap\limits_{i\in\Omega} I_i >$ is a semiprime hyperideal. □

Proposition 6.7. Let H be a commutative ternary semihypergroup and $A, B \subseteq H$. If $\langle A \rangle \subseteq \langle B \rangle$, then for every hyperideal I of H, $< B, I > \subseteq < A, I >$.

Proof. Let $\langle A \rangle \subseteq \langle B \rangle$ and I be a hyperideal of H. If $x \in < B, I >$, then $f(B, H, x) \subseteq I$. By Lemma 5.1(1), since $A \subseteq \langle B \rangle$, it follows that $< \langle B \rangle, I > \subseteq < A, I >$. By Lemma 5.1(1), Proposition 5.2 and Proposition 5.3(2), we get

$$\begin{aligned} < B, I > &\subseteq < B, I > \cap < f(H, H, B), I > \\ &= < B \cup f(H, H, B), I > \\ &= < \langle B \rangle, I > \\ &\subseteq < A, I > . \end{aligned}$$

□

Acknowledgments

The first author wish to express his warmest thanks to Professor Apostolos Thoma and Department of Mathematics, University of Ioannina (Greece) for their hospitality during the author's postdoctoral scholarship supported by The State Scholarships Foundation of Hellenic Republic.

References

1. P. Corsini, *Prolegomena of hypergroup theory*, Second edition, Aviani editor, 1993.
2. B. Davvaz, Groups in polygroups, *Iranian Journal of Mathematical Sciences and Informatics*, **1**(1), (2006), 25-31.
3. B. Davvaz, *Polygroup Theory and Related Systems*, World Scientific, 2013.
4. B. Davvaz and V. Leoreanu, Binary relations on ternary semihypergroups, *Communications in Algebra*, **38**(10), (2010), 3621-3636.
5. V. N. Dixit and S. Dewan, A note on quasi and bi-ideals in ternary semigroups, *Int. J. Math. and Math. Sci.*, **18**(3), (1995), 501-508.
6. T. K. Dutta and S. Kar, On prime ideals and prime radical of ternary semiring, *Bull. Cal. Math. Soc.*, **97**(5), (2005), 445-454.

7. T. K. Dutta and S. Kar, On semiprime ideals and irreducible ideals of ternary semiring, *Bull. Cal. Math. Soc.*, **97**(5), (2005), 467-476.
8. J. M. Howie, *Fundamentals of semigroup theory*, Oxford University Press, New York, 1995.
9. M. Jafarpoura, V. Leoreanu-Fotea and A. Zolfagharic, Weak complete parts in semihypergroups, *Iranian Journal of Mathematical Sciences and Informatics*, **8**(2), (2013), 101-109.
10. J. Los, On the extending of models I, *Fundamenta Mathematicae*, **42**, (1995), 38-54.
11. F. Marty, *Sur une Generalization de la Notion de Groupe*, 8th Congress Math. Scandenaves, Stockholm, Sweden, (1934) 45-49.
12. M. Shabir and Sh. Bashir, Prime ideals in ternary semigroups, *Asian-European Journal of Mathematics*, **2**(1), (2009), 141-154.
13. F. M. Sioson, Ideal theory in ternary semigroups, *Math. Japonica*, **10**, (1965), 63-64.