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A Generalized Singular Value Inequality for Heinz Means

Alemeh Sheikh Hosseini

Department of Mathematics, Shahid Bahonar University of Kerman, Kerman, Iran.

E-mail: alemehsheikhhoseiny@yahoo.com

ABSTRACT. In this paper we will generalize a singular value inequality that was proved before. In particular we obtain an inequality for numerical radius as follows:

$$2\sqrt{t(1-t)}\omega(tA^{\nu}B^{1-\nu} + (1-t)A^{1-\nu}B^{\nu}) \le \omega(tA + (1-t)B),$$

where, A and B are positive semidefinite matrices, $0 \leq t \leq 1$ and $0 \leq \nu \leq \frac{3}{2}.$

Keywords: Matrix monotone functions, Numerical radius, Singular values, Unitarily invariant norms.

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1. Introduction

Let \mathbb{M}_n be the algebra of all $n \times n$ complex matrices. A norm $\|\|.\|\|$ on \mathbb{M}_n is said to be unitarily invariant if $\|\|UAV\|\| = \|\|A\|\|$ for all $A \in \mathbb{M}_n$ and all unitary $U, V \in \mathbb{M}_n$. Special examples of such norms are the "Ky Fan norms"

$$||A||_{(k)} = \sum_{j=1}^{k} s_j(A), \qquad 1 \le k \le n.$$

Note that the operator norm, in this notation, is $||A|| = ||A||_{(1)} = s_1(A)$; see [4] and [9] for more information.

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^{*}Corresponding Author

If $\|A\|_{(k)} \leq \|B\|_{(k)}$ for $1 \leq k \leq n$, then $\|A\| \leq \|B\|$ for all unitary invariant norms. This is called the "Fan dominance theorem." If A is a Hermitian element of \mathbb{M}_n , then we arrange its eigenvalues in decreasing order as $\lambda_1(A) \geq \lambda_2(A) \geq \cdots \geq \lambda_n(A)$. If A is arbitrary, then its singular values are enumerated as $s_1(A) \geq s_2(A) \geq \cdots \geq s_n(A)$. These are the eigenvalues of the positive semidefinite matrix $|A| = (A^*A)^{1/2}$. If A and B are Hermitian matrices, and A - B is positive semidefinite, then we say that $B \leq A$.

Weyl's monotonocity theorem [4, p. 63] says that $B \leq A$ implies $\lambda_j(A) \leq \lambda_j(B)$, for all j = 1, ..., n. Let f be a real valued function on an interval I. Then f is said to be matrix monotone if $A, B \in \mathbb{M}_n$ are Hermitian matrices with all their eigenvalues in I and $A \geq B$, then $f(A) \geq f(B)$ and also, f is said to be matrix convex if

$$f(tA + (1-t)B) \le tf(A) + (1-t)f(B), \ 0 \le t \le 1$$

and matrix concave if

$$f(tA + (1-t)B) \ge tf(A) + (1-t)f(B), \ 0 \le t \le 1.$$

In response to a conjecture by Zhan [13], Audenaert [2] has proved that if $A, B \in \mathbb{M}_n$ are positive semidefinite, then the inequality

$$s_j(A^{\nu}B^{1-\nu} + A^{1-\nu}B^{\nu}) \le s_j(A+B), \quad 1 \le j \le n$$

holds, for all $0 \le \nu \le 1$. In this paper we generalize this inequality as follows: If $A, B \in \mathbb{M}_n$ are positive semidefinite matrices, then for all $0 \le t \le 1$ and $0 \le \nu \le \frac{3}{2}$

$$2\sqrt{t(1-t)}s_j(tA^{\nu}B^{1-\nu} + (1-t)A^{1-\nu}B^{\nu}) \le s_j(tA + (1-t)B).$$

For more details about inequalities and their generalizations with their history of origin, the reader may refer to [1, 5, 6, 11, 12, 13].

2. Main Results

Lemma 2.1. [14] If
$$X = \begin{bmatrix} A & C \\ C^* & B \end{bmatrix}$$
 is positive, then $2s_j(C) \leq s_j(X)$ for all $1 \leq j \leq n$.

Theorem 2.2. Let f be a matrix monotone function on $[0, \infty)$ and A and B be positive semidefinite matrices. Then

$$tAf(A) + (1-t)Bf(B) \ge (tA + (1-t)B)^{1/2} (tf(A) + (1-t)f(B))(tA + (1-t)B)^{1/2}$$
(2.1)

for all $0 \le t \le 1$.

Proof. The function f is also matrix concave, and g(x) = xf(x) is matrix convex. (See [4]). The matrix convexity of g implies the inequality

$$(tA + (1-t)B)f(tA + (1-t)B) \le tAf(A) + (1-t)Bf(B), \quad 0 \le t \le 1.$$
 (2.2)

Since the matrix tA + (1 - t)B is positive semidefinite, in view of the spectral decomposition theorem, it is easy to see that for all $0 \le t \le 1$,

$$(tA+(1-t)B)f(tA+(1-t)B) = (tA+(1-t)B)^{1/2}f(tA+(1-t)B)(tA+(1-t)B)^{1/2}.$$
(2.3)

Also, the matrix concavity of f implies that

$$tf(A) + (1-t)f(B) \le f(tA + (1-t)B), \quad 0 \le t \le 1.$$
 (2.4)

Combining the relations (2.2), (2.3) and (2.4), we get (2.1).

Theorem 2.3. Let $A, B \in \mathbb{M}_n$ be positive semidefinite matrices. Then for all $0 \le t \le 1$ and $0 \le \nu \le \frac{3}{2}$

$$2\sqrt{t(1-t)}s_j(tA^{\nu}B^{1-\nu} + (1-t)A^{1-\nu}B^{\nu}) \le s_j(tA + (1-t)B). \tag{2.5}$$

Proof. The proof depends on the fact that the matrices XY and YX have the same eigenvalues. Let $f(x) = x^r, 0 \le r \le 1$. This function is matrix monotone on $[0, \infty)$. Hence from (2.1) and Weyl's monotonocity theorem we have

$$\lambda_j(tA^{r+1} + (1-t)B^{r+1}) \ge \lambda_j((tA + (1-t)B)(tA^r + (1-t)B^r)). \tag{2.6}$$

Except for trivial zeroes the eigenvalues of $(tA + (1-t)B)(tA^r + (1-t)B^r)$ are the same as those of the matrix

$$\begin{bmatrix} tA+(1-t)B & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \sqrt{t}A^{r/2} & \sqrt{1-t}B^{r/2} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \sqrt{t}A^{r/2} & 0 \\ \sqrt{1-t}B^{r/2} & 0 \end{bmatrix}$$

and in turn, these are the same as the eigenvalues of

$$\begin{bmatrix} \sqrt{t}A^{r/2} & 0 \\ \sqrt{1-t}B^{r/2} & 0 \end{bmatrix} \begin{bmatrix} tA + (1-t)B & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \sqrt{t}A^{r/2} & \sqrt{1-t}B^{r/2} \\ 0 & 0 \end{bmatrix}$$

$$=\begin{bmatrix} tA^{r/2}(tA+(1-t)B)A^{r/2} & \sqrt{t(1-t)}A^{r/2}(tA+(1-t)B)B^{r/2} \\ \sqrt{t(1-t)}B^{r/2}(tA+(1-t)B)A^{r/2} & (1-t)B^{r/2}(tA+(1-t)B)B^{r/2} \end{bmatrix}.$$

So, Lemma 2.1 and inequality (2.6) together give

$$\lambda_i(tA^{r+1} + (1-t)B^{r+1}) \ge 2\sqrt{t(1-t)}s_i(A^{r/2}(tA + (1-t)B)B^{r/2})$$

$$=2\sqrt{t(1-t)}s_i(tA^{1+\frac{r}{2}}B^{r/2}+(1-t)A^{r/2}B^{1+\frac{r}{2}}).$$

Replacing A and B by $A^{1/r+1}$ and $B^{1/r+1}$, respectively, we get from this

$$s_{i}(tA+(1-t)B) \geq 2\sqrt{t(1-t)}s_{i}(tA^{\frac{r+2}{2r+2}}B^{\frac{r}{2r+2}}+(1-t)A^{\frac{r}{2r+2}}B^{\frac{2+r}{2r+2}}),\ 0 \leq r,t \leq 1.$$

Now, if we put $\nu = \frac{r+2}{2r+2}$, then trivially, we get

$$s_j(tA + (1-t)B) \ge 2\sqrt{t(1-t)}s_j(tA^{\nu}B^{1-\nu} + (1-t)A^{1-\nu}B^{\nu})$$

for all $0 \le t \le 1$ and $\frac{1}{2} \le \nu \le \frac{3}{2}$ and we have proved (2.5) for this special range.

Symmetry, if we put $\nu = \frac{r}{2r+2}$, then it is easy to see that the inequality (2.5) holds for all $0 \le t \le 1$ and $0 \le \nu \le \frac{1}{2}$. Hence the proof is complete. \square

If in Theorem 2.3, we put $t = \frac{1}{2}$, then we have the following corollary, which obtained by Audenaert in [2] and by Bhatia and Kittaneh in [6].

Corollary 2.4. Let $A, B \in \mathbb{M}_n$ be positive semidefinite matrices. Then for all $0 \le \nu \le 1$

$$s_j(A^{\nu}B^{1-\nu} + A^{1-\nu}B^{\nu}) \le s_j(A+B).$$

Corollary 2.5. Let $A, B \in \mathbb{M}_n$ be positive semidefinite matrices. Then for all $0 \le t \le 1$ and $0 \le \nu \le \frac{3}{2}$

$$2\sqrt{t(1-t)} \left\| \left\| tA^{\nu}B^{1-\nu} + (1-t)A^{1-\nu}B^{\nu} \right\| \right\| \le \left\| \left\| tA + (1-t)B \right\| \right\|.$$

For $A \in \mathbb{M}_n$, the numerical radius of A is defined and denoted by

$$\omega(A) = \max\{|x^*Ax| : x \in \mathbb{C}^n, x^*x = 1\}.$$

The quantity $\omega(A)$ is useful in studying perturbations, convergence, stability, approximation problems, iterative method, etc. For more information see [3, 7]. It is known that $\omega(.)$ is a vector norm on \mathbb{M}_n , but is not unitarily invariant. We recall the following results about the numerical radius of matrices which can be found in [8] (see also [10, Chapter 1]).

Lemma 2.6. Let $A \in \mathbb{M}_n$ and $\omega(.)$ be the numerical radius. Then the following assertions are true:

- (i) $\omega(U^*AU) = \omega(A)$, where U is unitary;
- (ii) $\frac{1}{2} ||A|| \le \omega(A) \le ||A||$;
- (iii) $\omega(A) = ||A||$ if (but not only if) A is normal.

Utilizing Lemma 2.6 (parts (ii) and (iii)) and by Corollary 2.5 we obtain the following corollary.

Corollary 2.7. Let $A, B \in \mathbb{M}_n$ be positive semidefinite matrices. Then for all $0 \le t \le 1$ and $0 \le \nu \le \frac{3}{2}$

$$2\sqrt{t(1-t)}\omega(tA^{\nu}B^{1-\nu} + (1-t)A^{1-\nu}B^{\nu}) \le \omega(tA + (1-t)B).$$

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References

- T. Ando, Majorization and Inequalities in Matrix Theory, Linear Algebra Appl., 199, (1994), 17-67.
- K. Audenaert, A Singular Value Inequality for Heinz Means, Linear Algebra Appl., 422, (2007), 279-283.
- O. Axelsson, H. Lu, B. Polman, On the Numerical Radius of Matrices and Its Application to Iterative Solution Methods, *Linear Multilinear Algebra*, 37, (1994), 225-238.
- 4. R. Bhatia, Matrix Analysis, Springer-Verlag, New York, 1997.
- 5. R. Bhatia, Positive Definite Matrices, Princeton University Press, 2007.
- R. BhatiaF. Kittaneh, The Matrix Arithmetic-Geometric Mean Inequality Revisited, Linear Algebra Appl., 428, (2008), 2177-2191.
- M. Eiermann, Field of Values and Iterative Methods, Linear Algebra Appl., 180, (1993), 167-197.
- 8. K. E. Gustafson, D. K. M. Rao, Numerical Range, Springer-Verlag, New York, 1997.
- R. A. Horn, C. R. Johnson, Matrix Analysis, Cambridge University Press, Cambridge, 1985.
- R. A. Horn, C. R. Johnson, Topics in Matrix Analysis, Cambridge University Press, Cambridge, 1991.
- K. M. Nagaraja, P. Siva Kota Reddy, Sudhir Kumar Sahu, Generalization of -Centroidal Mean and Its Dual, *Iranian Journal of Mathematical Sciences and Informatics*, 8(2), (2013), 39-47.
- M. Z. Sarikaya, A. Saglam, H. Yildirim, On Generalization of Cebysev Type Inequalities, Iranian Journal of Mathematical Sciences and Informatics, 5(1), (2010), 41-48.
- X. Zhan, Matrix Inequalities, Lecture Notes in Mathematics, 1790, Springer-Verlag, Berlin, 2002.
- Y. Tao, More Results on Singular Value Inequalities, Linear Algebra Appl., 416, (2006), 724-729.