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Mangasarian-Fromovitz and Zangwill Conditions for Non-Smooth Infinite Optimization Problems in Banach Spaces

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Abstract. In this paper we study optimization problems with infinity many inequality constraints on a Banach space where the objective function and the binding constraints are Lipschitz near the optimal solution. Necessary optimality conditions and constraint qualifications in terms of Michel-Penot subdifferential are given.

Keywords: Infinite programming, Constraint qualification, Optimality conditions, Michel-Penot subdifferential.

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1. Introduction

We consider the following optimization programming problem:

$$(P) \qquad \begin{cases} \text{Minimize } f(x), \\ \text{subject to } f_t(x) \leq 0, \quad \forall t \in T, \end{cases}$$

where T is an arbitrary set, and all emerging functions f and f_t for $t \in T$ are extended real-valued locally Lipschitz from the Banach space $\mathbb{X}$.

If $|T| < \infty$, necessary conditions of Karusk-Kuhn-Tucker (KKT, shortly) type for optimality can be established under various constraint qualifications

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(CQ, briefly). In order to study and compare these CQs in smooth and non-smooth cases, see the books [4, 6, 10, 18] and the papers [1, 22, 24, 25].

If T is arbitrary index set and $\mathbb{X} = \mathbb{R}^n$, the KKT necessary optimality conditions have been studied by many authors who have used term semi-infinite programming problem; see for example [7, 14] in linear case, [5, 15] in convex case, [8] in smooth case, and [11, 12, 13, 27] in locally Lipshitz case.

If T is an infinite index set and $\mathbb{X}$ has infinite dimension, The problem (P) is called *infinite problem*. Several papers studied infinite problems and gave the KKT necessary conditions (see e.g., [3, 19, 20, 21] and their references). In these papers, three kinds of CQs are usually considered including "Farkas-Minkowski CQ" and "closedness CQ", using basic/limiting subdifferential or convex ones.

This paper focuses mainly on some kinds of CQs for infinite problem (P) which are based on Michel-Penot subdifferential, their interrelations, and their applications to KKT necessary optimality conditions.

The remainder of the present paper is organized as follows. In Section 2, basic notations and preliminary results are reviewed. In Section 3, we introduce the Zangwill CQ, first Mangasarian-Fromovitz CQ, second Mangasarian-Fromovitz CQ, and linear independence CQ for the problem (P). In Section 4 we present first-order necessary optimality conditions for the problem (P) under the constraint qualifications introduced in section 3.

2. Notations and Preliminaries

Let $\mathbb{X}^*$ be the (continuous) dual space of $\mathbb{X}$, and let $\langle x^*, x\rangle$ denotes the value of the function $x^* \in \mathbb{X}^*$ at $x \in \mathbb{X}$. If $A^* \subseteq \mathbb{X}^*$, set $\langle A^*, x\rangle := \{\langle a^*, x\rangle \mid a^* \in A^*\}$. When we write $B \leq 0$ for some $B \subseteq \mathbb{R}$, means $b \leq 0$ for all $b \in B$. The symbols $\overline{B}$, $conv(B)$, and $cone(B)$ denote the closure, the convex hull, and the convex cone (containing zero) of $B \subseteq \mathbb{X}$ respectively.

Let $\hat{x} \in \mathbb{X}$ and let $\varphi : \mathbb{X} \to \mathbb{R}$ be any function. The Michel-Penot (M-P, briefly) directional derivative of φ at $\hat{x}$ in the direction $v \in \mathbb{X}$ introduced in [16] is given by

$$\varphi^{\diamond}(\hat{x}; v) := \sup_{w\in\mathbb{X}} \limsup_{\alpha\downarrow 0} \frac{\varphi(\hat{x} + \alpha v + \alpha w) - \varphi(\hat{x} + \alpha w)}{\alpha},$$

and the M-P subdifferential of φ at $\hat{x}$ is given by the set

$$\partial^{\diamond}\varphi(\hat{x}) := \big\{\xi \in \mathbb{X}^* \mid \langle \xi, v\rangle \leq \varphi^{\diamond}(\hat{x}; v) \qquad \text{for all } v \in \mathbb{X}\big\}.$$

The M-P subdifferential is a natural generalization of the Gâteaux derivative (see [16, Proposition 1.3]). Moreover when a function φ is convex, the M-P subdifferential coincides with the subdifferential in the sense of convex analysis, denoted by ∂.

In the following theorem we summarize some important properties of the M-P directional derivative and the M-P subdifferential from [16, 17] which are widely used in what follows.

Theorem 2.1. *Let φ and ϕ be functions from $\mathbb{X}$ to $\mathbb{R}$ which are Lipschitz near $\hat{x}$. Then, the following assertions hold:*

(i)

$$\varphi^{\diamond}(\hat{x};v) = \max\left\{ \langle \xi, v\rangle \mid \xi \in \partial^{\diamond}\varphi(\hat{x})\right\}, \tag{2.1}$$

$$\partial^{\diamond}\big(\max\{\varphi,\phi\}\big)(x) \subseteq conv\big(\partial^{\diamond}\varphi(x) \cup \partial^{\diamond}\phi(x)\big), \tag{2.2}$$

$$\partial^{\diamond}(\varphi+\phi)(\hat{x}) \subseteq \partial^{\diamond}\varphi(\hat{x}) + \partial^{\diamond}\phi(\hat{x}). \tag{2.3}$$

(ii) *The function $v \to \varphi^{\diamond}(\hat{x};v)$ is finite, positively homogeneous, and subadditive on $\mathbb{X}$, and*

$$\partial\big(\varphi^{\diamond}(\hat{x};.)\big)(0) = \partial^{\diamond}\varphi(\hat{x}). \tag{2.4}$$

(iii) *$\partial^{\diamond}\varphi(\hat{x})$ is nonempty, convex, and weak*-compact subset of $\mathbb{X}^*$.*

3. Qualification Conditions

In this section, we present several constraint qualifications for problem (P), and investigate the relationships with them. As a starting, we denote the feasible set of problem (P) with

$$\Omega := \big\{x \in \mathbb{X} \mid f_t(x) \leq 0 \qquad \forall t \in T\big\}.$$

The feasible directions cone of Ω at $\hat{x} \in \Omega$ is defined as

$$D_{\Omega}(\hat{x}) := \Big\{z \in \mathbb{X} \mid \exists\varepsilon > 0, \text{ such that } \hat{x} + \alpha z \in \Omega \quad \forall \alpha \in (0,\varepsilon)\Big\}.$$

For a given $\hat{x} \in \Omega$, let $T(\hat{x})$ denotes the index set of all active constraints at $\hat{x}$, i.e.,

$$T(\hat{x}) := \{t \in T \mid f_t(\hat{x}) = 0\}.$$

Based on the above notations and the Michel-Penot subdifferential, we extend the Zangwill CQ to nondifferentiable infinite problem (P).

Definition 3.1. Let $\hat{x} \in \Omega$. We say that The Zangwill CQ holds at $\hat{x}$ if

$$\Big\{v \in \mathbb{X} \mid \Big\langle \bigcup_{t \in T(\hat{x})} \partial^{\diamond} f_t(\hat{x}), v\Big\rangle \leq 0\Big\} \subseteq \overline{D_{\Omega}(\hat{x})}.$$

Set

$$F(x) := \sup_{t \in T} f_t(x), \qquad \forall x \in \Omega.$$

One reason for difficulty of extending the results from a finite problem (i.e., $|T| < \infty$) to problem (P) is that in the finite case $F(.)$ is locally Lipschitz and we have (using (2.2) and mathematical induction):

$$\partial^{\diamond} F(x) \subseteq conv\Big(\bigcup_{t \in T(x)} \partial^{\diamond} f_t(x)\Big), \qquad \forall x \in \Omega, \tag{3.1}$$

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but in general, (3.1) does not hold for infinite problem (P).

At this point, we recall from differentiable finite programming theory (i.e., $T=\{1,2,\ldots,m\}$) that the Mangasarian-Fromovitz CQ holds at $\hat{x}$, if there exists an $\hat{u}\in\mathbb{X}$ such that $\big\langle \nabla f_t(\hat{x}),\hat{u}\big\rangle<0$ for all $t\in T(\hat{x})$. It is easy to see that the Mangasarian-Fromovitz CQ in differentiable finite problem is equivalent to the following implication (see, e.g., [2]):

$$\sum_{t\in T(\hat{x})}\lambda_t\nabla f_t(\hat{x})=0,\quad \lambda_t\geq 0\quad \forall t\in T(\hat{x})\quad\Longrightarrow\ \lambda_t=0\quad\ \forall t\in T(\hat{x}).$$

We now extend the Mangasarian-Fromovitz CQ for problem (P) in two different forms.

Definition 3.2. We say that the first Mangasarian-Fromovitz CQ holds at $\hat{x}$ if the following assertions satisfy:

(A): $F(.)$ is Lipschitz continuous around $\hat{x}$.
(Á): $\partial^\diamond F(\hat{x})\subseteq conv\Big(\bigcup_{t\in T(\hat{x})}\partial^\diamond f_t(\hat{x})\Big)$.
(À): $\Big\{v\in\mathbb{X}\mid\ \Big\langle\bigcup_{t\in T(\hat{x})}\partial^\diamond f_t(\hat{x}),v\Big\rangle<0\Big\}\neq\emptyset$.

Remark 3.3. An interesting sufficient condition ensuring the Lipschitzian property of F around $\hat{x}$ in finite dimensional space can be found in [23, Theorem 9.2]. The condition Á was called the Pshenichnyi-Levin-Valadire property for convex infinite problems in [26].

We observe that there is no relation of implication between the Á and the À in Definition 3.2. Indeed, for any finite T the À is true, but it may not satisfy the Á; while in the following example the problem actually satisfies the À at $\hat{x}:=0$, but the Á does not hold at this point.

Example 3.4. Consider the following problem:

$$\begin{aligned} &\inf f(x):=|x|\\ \text{s.t.}\quad &f_t(x)\leq 0,\quad t\in T:=\mathbb{N}\\ &x\in\mathbb{R},\end{aligned}$$

where

$$f_t(x):=\begin{cases}5x-\frac{2}{t+1} & \text{if}\quad t \text{ is odd}\\ 6x & \text{if}\ \ t=2\\ 7x-\frac{2}{t} & \text{if}\quad t\geq 4 \text{ and } t \text{ is even.}\end{cases}$$

If we consider the point $\hat{x}:=0$, then $T(\hat{x})=\{2\}$. This implies

$$\Big\{v\in\mathbb{X}\mid\ \Big\langle\bigcup_{t\in T(\hat{x})}\partial^\diamond f_t(\hat{x}),v\Big\rangle<0\Big\}=\Big\{v\in\mathbb{R}\mid\ \big\langle\partial^\diamond f_2(\hat{x}),v\big\rangle<0\Big\}=$$
$$\Big\{v\in\mathbb{R}\mid\big\langle 6,v\big\rangle<0\Big\}=(-\infty,0)\neq\emptyset.$$

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Thus À satisfies. On the other hand, a short calculation shows that

$$F(x) = \begin{cases} 7x & \text{if} \quad x \geq 0 \\ 5x & \text{if} \quad x < 0 \end{cases}, \qquad \Longrightarrow$$
$$\partial^\diamond F(\hat{x}) = [5,7] \nsubseteq \{6\} = conv\Big(\bigcup_{t\in T(\hat{x})} \partial^\diamond f_t(\hat{x})\Big).$$

Hence, Á is not satisfy.

Theorem 3.5. *The first Mangasarian-Fromovitz CQ at $\hat{x} \in \Omega$ implies the Zangwill CQ at this point.*

Proof. By À, let $\hat{v}$ be an element of $\Big\{v \in \mathbb{X} \mid \Big\langle \bigcup_{t\in T(\hat{x})} \partial^\diamond f_t(\hat{x}), v\Big\rangle < 0\Big\}$. It is easy to show that

$$\hat{v} \in \Big\{v \in \mathbb{X} \mid \Big\langle conv\Big(\bigcup_{t\in T(\hat{x})} \partial^\diamond f_t(\hat{x})\Big), v\Big\rangle < 0\Big\}.$$

On the other hand, Á leads to

$$\Big\{v \in \mathbb{X} \mid \Big\langle conv\Big(\bigcup_{t\in T(\hat{x})} \partial^\diamond f_t(\hat{x})\Big), v\Big\rangle < 0\Big\} \subseteq \Big\{v \in \mathbb{X} \mid \Big\langle \partial^\diamond F(\hat{x}), v\Big\rangle < 0\Big\}.$$

The above two relations imply $\langle \partial^\diamond F(\hat{x}), \hat{v}\rangle < 0$. From this inequality and (2.1), we obtain $F^\diamond(\hat{x};\hat{v}) < 0$. Now, from the definition of M-P subdifferential we get

$$\limsup_{\alpha\downarrow 0} \frac{F(\hat{x}+\alpha\hat{v}) - F(\hat{x})}{\alpha} \leq F^\diamond(\hat{x};\hat{v}) < 0,$$

and consequently, there exists a scalar $\varepsilon > 0$ such that:

$$F(\hat{x}+\alpha\hat{v}) < F(\hat{x}) \leq 0, \qquad \forall\, \alpha \in (0,\varepsilon).$$

Thus, for all $t \in T$ and for all $\alpha \in (0,\varepsilon)$, we conclude $f_t(\hat{x}+\alpha\hat{v}) < 0$, which implies

$$\hat{x}+\alpha\hat{v} \in \Omega, \qquad \forall\, \alpha \in (0,\varepsilon).$$

Therefore, we have proved

$$\Big\{v \in \mathbb{X} \mid \Big\langle \bigcup_{t\in T(\hat{x})} \partial^\diamond f_t(\hat{x}), v\Big\rangle < 0\Big\} \subseteq D_\Omega(\hat{x}).$$

Hence, we obtain that:

$$\Big\{v \in \mathbb{X} \mid \Big\langle \bigcup_{t\in T(\hat{x})} \partial^\diamond f_t(\hat{x}), v\Big\rangle \leq 0\Big\} = \overline{\Big\{v \in \mathbb{X} \mid \Big\langle \bigcup_{t\in T(\hat{x})} \partial^\diamond f_t(\hat{x}), v\Big\rangle < 0\Big\}} \subseteq \overline{D_\Omega(\hat{x})},$$

and the proof is complete. $\square$

Definition 3.6. We say that the problem (P) satisfies in the second Mangasarian-Fromovitz CQ at $\hat{x} \in \Omega$, if the following assertions hold:

(A): $F(.)$ is Lipschitz continuous around $\hat{x}$.

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(Á): $\partial^\diamond F(\hat{x}) \subseteq conv\Big(\bigcup_{t\in T(\hat{x})} \partial^\diamond f_t(\hat{x})\Big)$.

(À$_1$): for each finite index set $\hat{T} \subseteq T(\hat{x})$, the next implication is true:

$$0 \in \sum_{t\in\hat{T}} \lambda_t \partial^\diamond f_t(\hat{x}), \qquad \lambda_t \geq 0 \quad \forall t \in \hat{T} \implies \lambda_t = 0 \quad \forall t \in \hat{T}.$$

Theorem 3.7. *The first Mangasarian-Fromovitz CQ at $\hat{x} \in \Omega$ implies the second Mangasarian-Fromovitz CQ at this point.*

Proof. It is enough to establish (À) $\Longrightarrow$ (À$_1$). Suppose that (À) holds. Then there exists an element $\hat{v} \in \mathbb{X}$ such that

$$\Big\langle \bigcup_{t\in T(\hat{x})} \partial^\diamond f_t(\hat{x}), \hat{v} \Big\rangle < 0. \tag{3.2}$$

If $\hat{T} \subseteq T$ is a finite index set and $\lambda_t,\ t \in \hat{T}$ are non-negative scalars satisfying

$$0 \in \sum_{t\in\hat{T}} \lambda_t \partial^\diamond f_t(\hat{x}),$$

then, we conclude

$$0 = \langle 0, \hat{v} \rangle \in \sum_{t\in\hat{T}} \lambda_t \langle \partial^\diamond f_t(\hat{x}), \hat{v} \rangle.$$

By (3.2) and sign of λ_ts, the last inclusion is fulfill only if $\lambda_t = 0$ for all $t \in \hat{T}$, as request. $\square$

To establish the converse of Theorem 3.7, we require the following definition from [13].

Definition 3.8. Let Γ is an arbitrary index set, and the function φ_ω for each $\omega \in \Gamma$ is defined from $\mathbb{X}$ to $\mathbb{R}$. We say that the system

$$\big\{\varphi_\omega(x) < 0 \mid \omega \in \Gamma\big\},$$

is compactable, when the following proposition holds:

" *If $\big\{\varphi_\omega(x) < 0 \mid \omega \in \Gamma_0\big\}$ has solution for each finite index set $\Gamma_0 \subseteq \Gamma$, then $\big\{\varphi_\omega(x) < 0 \mid \omega \in \Gamma\big\}$ has solution in $\mathbb{X}$.*"

Theorem 3.9. *The second Mangasarian-Fromovitz CQ at $\hat{x} \in \Omega$ implies the first Mangasarian-Fromovitz CQ at this point, if $\big\{f_t^\diamond(\hat{x}; .) < 0 \mid t \in T(\hat{x})\big\}$ is a compactable system.*

Proof. It is enough to establish (À$_1$) $\Longrightarrow$ (À). We first prove that for any given $t_1 \in T(\hat{x})$, there exists $\hat{v} \in \mathbb{X}$ such that

$$f_{t_1}^\diamond(\hat{x}, \hat{v}) < 0. \tag{3.3}$$

If, on contrary, the above inequality has no solution with respect to $\hat{v}$, then $v_0 := 0$ is a solution to the following optimization problem

$$\begin{aligned} &\min\ f_{t_1}^\diamond(\hat{x}, v) \\ \text{s.t.} \qquad &v \in \mathbb{X}. \end{aligned}$$

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Since the objective function is convex, by the Lagrange multiplier rule and virtue of (2.4), we obtain that

$$0 \in \partial\big(f^{\diamond}_{t_1}(\hat{x}, .)\big)(0) = \partial f_{t_1}(\hat{x}),$$

which contracts ($\grave{\mathrm{A}}_1$). Now, establish that for any two given $t_1, t_2 \in T(\hat{x})$, there exists $\hat{v} \in \mathbb{X}$ such that

$$f^{\diamond}_{t_1}(\hat{x}, \hat{v}) < 0,$$
$$f^{\diamond}_{t_2}(\hat{x}, \hat{v}) < 0.$$

On the contrary, suppose that the above system does not have a solution. Then $f^{\diamond}_{t_2}(\hat{x}, \hat{v}) \geq 0$ for all $\hat{v}$ satisfying the (3.3), which implies that $v_0 := 0$ is a solution to the following optimization problem with convex objective and constraints:

$$\begin{aligned} &\min\ f^{\diamond}_{t_2}(\hat{x}, v) \\ \text{s.t.} \qquad &f^{\diamond}_{t_1}(\hat{x}, v) \leq 0. \end{aligned}$$

Indeed, let v^* be any feasible solution of the above problem and let u^* be a solution of (3.3); then by Theorem 2.1(ii), $v^* + \alpha u^*$ ia a solution of (3.3) for any $\alpha > 0$, and hence $f^{\diamond}_{t_2}(\hat{x}, v^* + \alpha u^*) \geq 0$ by the assumption, which implies that $f^{\diamond}_{t_2}(\hat{x}, v^*) \geq 0$ after taking limits as $\alpha \to 0$. By the Lagrange multiplier rule, there must exist $\lambda_{t_1}, \lambda_{t_2} \geq 0$ such that

$$0 \in \lambda_{t_2}\partial^{\diamond} f_{t_2}(\hat{x}) + \lambda_{t_1}\partial^{\diamond} f_{t_1}(\hat{x}), \quad \text{and} \quad (\lambda_{t_1}, \lambda_{t_2}) \neq (0, 0),$$

which contradicts ($\grave{\mathrm{A}}_1$). It follows by the mathematical induction that for each finite set $\hat{T} \subseteq T(\hat{x})$, we can find a $\hat{v} \in \mathbb{X}$ such that:

$$f^{\diamond}_{\hat{t}}(\hat{x}, \hat{v}) < 0, \qquad \text{for all } \ \hat{t} \in \hat{T}.$$

Now, the compactable assumption implies that there is a $\hat{v} \in \mathbb{X}$, such that

$$f^{\diamond}_{t}(\hat{x}, \hat{v}) < 0, \qquad \text{for all } \ t \in \hat{T}(\hat{x}).$$

Hence, by (2.1), the proof is complete. □

Definition 3.10. We say that the linear independence CQ is satisfied at $\hat{x} \in \Omega$, if the following assertions hold:

- **(A):** $F(.)$ is Lipschitz continuous around $\hat{x}$.
- **(Á):** $\partial^{\diamond} F(\hat{x}) \subseteq conv\Big(\bigcup_{t\in T(\hat{x})} \partial^{\diamond} f_t(\hat{x})\Big)$.
- **($\grave{\mathbf{A}}_2$):** $\big\{\xi_t \mid t \in \hat{T}\big\}$ is linear independent for each finite index set $\hat{T} \subseteq T$ and for each $\xi_t \in \partial^{\diamond} f_t(\hat{x})$.

Theorem 3.11. *The linear independence CQ at $\hat{x} \in \Omega$ implies the second Mangasarian-Fromovitz CQ at this point.*

Proof. It follows that ($\grave{\mathrm{A}}_2$) $\Longrightarrow$ ($\grave{\mathrm{A}}_1$), and the result is immediate. □

By Theorems 3.5 & 3.7, and 3.11, the relationships between the various constraint qualifications are given in the following diagram:

$$\begin{array}{ccc} & & \text{L.I. CQ} \\ & & \Downarrow \\ \text{M.F. CQ}_1 & \Rightarrow & \text{M.F. CQ}_2 \\ \Downarrow & & \\ \text{Z. CQ} & & \end{array} \tag{3.4}$$

4. Necessary Conditions

The first theorem in this section gives a KKT type necessary condition for optimal solution of problem (P) under the second Mangasarian-Fromovitz CQ.

Theorem 4.1. *Suppose that $\hat{x}$ is an optimal solution for problem (P), and the second Mangasarian-Fromovitz CQ holds at $\hat{x}$. Then, there exist $\lambda_t \geq 0,\ t \in T(\hat{x})$, where $\lambda_t \neq 0$ for finitely many $t \in T(\hat{x})$, such that*

$$0 \in \partial^\diamond f(\hat{x}) + \sum_{t \in T(\hat{x})} \lambda_t \partial^\diamond f_t(\hat{x}).$$

Proof. Observe that

$$\Omega = \big\{ x \in \mathbb{X} \mid F(x) \leq 0 \big\},$$

and hence, $\hat{x}$ is a solution of the following optimization problem:

$$\begin{array}{ll} & \min\ f(x) \\ \text{s.t.} & F(x) \leq 0. \end{array}$$

Since the objective and the constraint functions of above problem are Lipschitz near $\hat{x}$, by the Fritz-John multiplier rule and $(\acute{A})$, we find non-negative scalars β_0, β_1 such that $\beta_0 + \beta_1 = 1$ and

$$0 \in \beta_0 \partial^\diamond f(\hat{x}) + \beta_1 \partial^\diamond F(\hat{x}) \subseteq \beta_0 \partial^\diamond f(\hat{x}) + \beta_1 conv\Big(\bigcup_{t \in T(\hat{x})} \partial^\diamond f_t(\hat{x}) \Big).$$

Therefore, there are a finite index set $\hat{T} \subseteq T(\hat{x})$ and $\gamma_t \geq 0$ for $t \in \hat{T}$ such that $\sum_{t \in \hat{T}} \gamma_t = 1$ and

$$0 \in \beta_0 \partial^\diamond f(\hat{x}) + \beta_1 \sum_{t \in \hat{T}} \gamma_t \partial^\diamond f_t(\hat{x}).$$

If $\beta_0 = 0$, then $\beta_1 = 1$ by $\beta_0 + \beta_1 = 1$. Thus the above inclusion and $(\grave{A}_1)$ imply $\gamma_t = 0$ for all $t \in \hat{T}$ which is a contradiction. Hence, $\beta_0 \neq 0$, and the result is verified with taking $\lambda_t := \frac{\beta_1 \gamma_t}{\beta_0}$ for each $t \in \hat{T}$. □

Before proving the next theorems, we give a lemma, which will be useful.

Lemma 4.2. *Let $\hat{x}$ be an optimal solution of problem (P), and $v^* \in \overline{\big(D_\Omega(\hat{x})\big)}$. Then one has $f^\diamond(\hat{x}; v^*) \geq 0$*

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Proof. We first claim that each $\hat{v} \in D_\Omega(\hat{x})$ satisfying $f^\diamond(\hat{x};\hat{v}) \geq 0$. On the contrary, suppose there exists $\hat{v} \in D_\Omega(\hat{x})$ such that $f^\diamond(\hat{x};\hat{v}) < 0$. Then

$$\limsup_{\alpha\downarrow 0} \frac{f(\hat{x}+\alpha\hat{v}) - f(\hat{x})}{\alpha} \leq f^\diamond(\hat{x};\hat{v}) < 0,$$

which implies that there exists $\varepsilon_1 > 0$ such that:

$$f(\hat{x}+\alpha\hat{v}) - f(\hat{x}) < 0 \qquad \forall \alpha \in (0,\varepsilon_1).$$

By the definition of $D_\Omega(\hat{x})$, there exists $\varepsilon_2 > 0$ such that

$$\hat{x}+\alpha\hat{v} \in \Omega \qquad \forall \alpha \in (0,\varepsilon_2).$$

By the above two relations, for each $\alpha \in (0,\varepsilon)$ with $\varepsilon := \min\{\varepsilon_1,\varepsilon_2\}$, we have

$$hx+\alpha\hat{v} \in \Omega \qquad \text{and} \qquad f(\hat{x}+\alpha\hat{v}) < f(\hat{x}).$$

But this contradicts the fact that $\hat{x}$ is an optimal solution of (P), and hence our claim holds.

Now, let $v^* \in \overline{\big(D_\Omega(\hat{x})\big)}$. Then, there exists a sequence $\{\hat{v}_l\}_{l=1}^\infty$ in $D_\Omega(\hat{x})$ converging to v^*. Taking into consideration the continuity of $f^\diamond(\hat{x};.)$, and $f^\diamond(\hat{x};\hat{v}_l) \geq 0$ for all $l \in \mathbb{N}$, it follows that $f^\diamond(\hat{x};v^*) \geq 0$, as required. □

Theorem 4.3. *Suppose that $\hat{x}$ is an optimal solution of problem (P), and the Zangwill CQ is satisfied at $\hat{x}$. Then the following inclusion holds:*

$$0 \in \partial^\diamond f(\hat{x}) + \overline{cone}\Big(\bigcup_{t\in T(\hat{x})} \partial^\diamond f_t(\hat{x})\Big).$$

Proof. Let $\hat{v}$ is an element of $\mathbb{X}$ satisfying $\Big\langle \bigcup_{t\in T(\hat{x})} \partial^\diamond f_t(\hat{x}), \hat{v}\Big\rangle \leq 0$. By the Guignard CQ and Lemma 4.2 we conclude $f^\diamond(\hat{x};\hat{v}) \geq 0$. Thus, we obtain

$$f^\diamond(\hat{x};\hat{v}) \geq 0, \qquad \text{for all } \hat{v} \in \Big\{v \in \mathbb{X} \mid \Big\langle \overline{cone}\Big(\bigcup_{t\in T(\hat{x})} \partial^\diamond f_t(\hat{x})\Big), v\Big\rangle \leq 0\Big\},$$

in view of

$$\Big\{v \in \mathbb{X} \mid \Big\langle \overline{cone}\Big(\bigcup_{t\in T(\hat{x})} \partial^\diamond f_t(\hat{x})\Big), v\Big\rangle \leq 0\Big\} = \Big\{v \in \mathbb{X} \mid \Big\langle \bigcup_{t\in T(\hat{x})} \partial^\diamond f_t(\hat{x}), v\Big\rangle \leq 0\Big\}.$$

The above result implies that $v^* := 0$ is a global minimizer of the convex function $v \to H(v) := f^\diamond(\hat{x};v) + \Theta(v)$, where $\Theta(.)$ denotes the indicator function of set $\Big\{v \in \mathbb{X} \mid \Big\langle \overline{cone}\Big(\bigcup_{t\in T(\hat{x})} \partial^\diamond f_t(\hat{x})\Big), v\Big\rangle \leq 0\Big\}$; i.e., $\Theta(v) = 0$ if $\Big\langle \overline{cone}\Big(\bigcup_{t\in T(\hat{x})} \partial^\diamond f_t(\hat{x})\Big), v\Big\rangle \leq 0$, and $\Theta(v) = +\infty$ otherwise.

Now, by necessary condition for convex optimization problems (see e.g., [9]), and by the sum rule formula (2.3) (which equality holds there for convex functions), one has

$$0 \in \partial\big(f^\diamond(\hat{x};.)\big)(0) + \partial\Theta(0),$$

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where $\partial\varphi$ denotes the subdifferential of convex function φ in the sense of convex analysis. Finally, the virtue of (2.4), and the fact that $\partial\Theta(0) = \overline{cone}\Big(\bigcup_{t\in T(\hat{x})}\partial^\diamond f_t(\hat{x})\Big)$, conclude that

$$0 \in \partial^\diamond f(\hat{x}) + \overline{cone}\Big(\bigcup_{t\in T(\hat{x})}\partial^\diamond f_t(\hat{x})\Big),$$

as required. □

Now, we can formulate our main result of this section.

Theorem 4.4. *Suppose that $\hat{x}$ is an optimal solution of problem (P), and one of the following conditions holds:*

(1) *Zangwill CQ at $\hat{x}$, and closedness of $cone\Big(\bigcup_{t\in T(\hat{x})}\partial^\diamond f_t(\hat{x})\Big)$.*
(2) *First Mangasarian-Fromovitz CQ at $\hat{x}$.*
(3) *Second Mangasarian-Fromovitz CQ at $\hat{x}$.*
(4) *Linear independence CQ at $\hat{x}$.*

Then, there exist $\lambda_t \geq 0,\ t \in T(\hat{x})$, where $\lambda_t \neq 0$ for finitely many $t \in T(\hat{x})$, such that

$$0 \in \partial^\diamond f(\hat{x}) + \sum_{t\in T(\hat{x})}\lambda_t\partial^\diamond f_t(\hat{x}).$$

Proof. By Theorems 4.1 & 4.3, diagram (3.4), and the following fact for convex sets $A_\gamma,\ \gamma\in\Gamma$ (see e.g., [9]):

$$cone(\bigcup_{\gamma\in\Gamma}A_\gamma) = \Big\{\sum_{\gamma\in\Gamma_0}\tau_\gamma a_\gamma \mid \Gamma_0 \text{ is finite subset of } \Gamma,\ a_\gamma\in A_\gamma,\ \tau_\gamma\geq 0\Big\},$$

the result is immediate. □

Note that $cone\Big(\bigcup_{t\in T(\hat{x})}\partial^\diamond f_t(\hat{x})\Big)$ is assumed to be closed in part 1 of Theorem 4.4. The following example shows that this assumption can not be waived, even when $\mathbb{X}$ has finite dimension and f_ts are convex.

Example 4.5. For all $t\in T := \mathbb{N}$, take $A_t := \big\{(a_1,a_2)\in\mathbb{R}^2 \mid a_1^2 + a_2^2 - 2ta_2 \leq 0\big\}$. Set $f(x_1,x_2) := -x_1$ and

$$f_t(x_1,x_2) := \sup_{(a_1,a_2)\in A_t}(a_1x_1 + a_2x_2).$$

It is easy to see that $\Omega := (-\infty,0]\times(-\infty,0]$ and $\hat{x} := (0,0)$ are respectively the feasible solution set and the optimal solution of the following problem:

$$\inf\big\{f(x_1,x_2) \mid f_t(x_1,x_2)\leq 0, \quad t\in T\big\}.$$

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We observe that $T(\hat{x}) = T$. Since f_t is support function of A_t, we obtain $\partial^\diamond f_t(\hat{x}) = A_t$, and hence

$$cone\Big(\bigcup_{t\in T(\hat{x})} \partial^\diamond f_t(\hat{x})\Big) = \Big([0,+\infty)\times[0,+\infty)\Big)\cup\{(0,0)\},$$

$$\Big\{v\in\mathbb{X} \mid \Big\langle \bigcup_{t\in T(\hat{x})} \partial^\diamond f_t(\hat{x}), v\Big\rangle \le 0\Big\} = \Omega.$$

By $K_\Omega(\hat{x}) = \Omega$ and convexity of Ω we conclude that the Zangwill CQ holds at $\hat{x}$. Note that $cone\Big(\bigcup_{t\in T(\hat{x})} \partial^\diamond f_t(\hat{x})\Big)$ is not closed. It is easy to see that there is no sequence of scalars satisfying Theorem 4.4. Moreover, it can show that

$$0\in\partial^\diamond f(\hat{x}) + \overline{cone}\Big(\bigcup_{t\in T(\hat{x})} \partial^\diamond f_t(\hat{x})\Big).$$

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References

1. M. Abbasi, KH. Pourbemani, The System of Vector Variational-like Inequalities with Weakly $(\eta_\gamma - \alpha_\gamma)$ Pseudomonotone Mapping in Banach Space, *Iranian Journal of Mathematical Sciences and Informatics*, **2**, (2010), 1-11.
2. J. M. Borwein, A. S. Lewis, *Convex Analysis and Nonlinear Optimization: Theory and Examples*, Springer, New York, 2000.
3. J. M. Borwein, Q. J. Zhu, A Survey of Subdifferential Calculus with Applications, *Nonlinear Anal.*, **38**, (1999), 687-773.
4. Du. DingZhu, M. Panos, P. M. Pardalos, Wu. Weili, *Mathematical Theory of Optimization*, Kluwer Academic Publishers, 2001.
5. M. D. Fajardo, M. A. López, Locally Farkas-Minkowski Systems in Convex Semi-infinite programming, *J. Optim. Theory Appl.*, **103**, (1999), 313-335.
6. J. Giorgi, A. Gwirraggio, J. Thierselder, *Mathematics of Optimization, Smooth and Nonsmooth Cases*, Elsivier, 2004.
7. M. A. Goberna, M. A. López, *Linear Semi-infinite Optimazation*, Wiley, Chichester, 1998.
8. R. Hettich, O. Kortanek, Semi-infinite programming: Theory, Methods, and Applications, *Siam Riview*, **35**, (1993), 380-429.
9. J. B. Hiriart- Urruty, C. Lemarechal, *Convex Analysis and Minimization Algorithms, I & II*, Springer, Berlin, Heidelberg, 1991.
10. R. Horst, P. M. Pardalos, *Handbook of Global Optimization*, Kluwer Academic Publishers, 1995.
11. N. Kanzi, Necessary Optimality Conditions for Nonsmooth Semi-infinite Programming Problems, *J. Global Optim.*, **49**, (2011), 713-725.
12. N. Kanzi, S. Nobakhtian, Optimality Conditions for Nonsmooth Semi-infinite Programming, *Optimization*, **59**, (2008), 717-727.
13. N. Kanzi, S. Nobakhtian, Nonsmooth Semi-infinite Programming Problems with Mixed Constraints, *J. Math. Anal. Appl.*, **351**, (2008), 170-181.
14. M. A. López, G. Still, Semi-infinite Programming, *European J. Opera. Res.*, **180**, (2007), 491-518.

15. M. A. López, E. Vercher, Optimality Conditions for Nondifferentiable Convex Semi-infinite Programming, *Math. Programming*, **27**, (1983), 307-319.
16. P. Michel, J. P. Penot, Calcul Sous- differentiel Pour des Fonctions Lipschitziennes et non Lipschitziennes, *C.R. Acad. Sci. Paris sér. I Math.*, **12**, (1984), 269-272.
17. P. Michel, J. P. Penot, A Generalized Derivative for Calm and Stable Functions, *Diff. Integral Equa.*, **5**, (1992), 433- 454.
18. B. S. Mordukhovich, *Variational Analysis and Generalized Differentiation*, Springer-Verlag, Berlin, 2006.
19. B. S. Mordukhovich, T. T. A. Nghia, Constraint Qualification and Optimality Conditions in Semi-infinite and Infinite Programming, *Math. Program.*, to appear, (2012).
20. B. S. Mordukhovich, T. T. A. Nghia, Nonsmooth Cone-constrained Optimization with Applications to Semi-infinite Programming, *Optimization*, online 3/3396, (2012).
21. B. S. Mordukhovich, T. T. A. Nghia, Subdifferentials of Nonconvex Supremum Functions and Their Applications to Semi-infinite and Infinite Programs with Lipschitzian Data, *Optimization*, online 12/3261, (2011).
22. KH. Pourbemani, M. Abbasi, On the Vector Variational-like Inequalities with $\eta - \alpha$ Pseudomonotone Mapping, *Iranian Journal of Mathematical Scinces and Informatics*, **1**, (2009), 37-42.
23. R. T. Rockafellar, J. B. Wets, *Variational analysis*, Springer-Verlag, 1998.
24. O. Stein, On Constraint Qualifications in Nonsmooth Optimization, *J. Optim. Theory. Apll.*, **121**, (2004), 647-671.
25. J. Ye, Nondifferentiable Multiplier Rules For Optimization and Bilevel Optimization Problems, *Siam J. Optim.*, **15**, (2004), 252-274.
26. X. T. Ye, C. Li, On Basic Constraint Qualifications for Infinite System of Convex Inequalities in Banach Spaces, *Acta Mathematica Sinica, English Series*, **23**, (2006), 65-76.
27. X. Y. Zheng, X. Yang, Lagrange Multipliers in Nonsmooth Semi-infinite Optimization Problems, *J. Oper. Res.*, **32**, (2007), 168-181.

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