p-Analog of the Semigroup Fourier-Steiltjes Algebras

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Abstract. In this paper we define the p-analog of the restricted representations and the p-analog of the Fourier–Stieltjes algebras on inverse semigroups. Also we improve some results about Herz algebras on Clifford semigroups and we give a necessary and sufficient condition for amenability of these algebras on Clifford semigroups.

Keywords: Restricted fourier–Stieltjes algebras, Restricted inverse semigroup, Restricted representations, \( QSL_p \)-spaces, p-Analog of the Fourier–Stieltjes algebras.


1. Introduction and Preliminaries

An inverse semigroup \( S \) is a discrete semigroup such that for each \( s \in S \) there exists a unique element \( s^* \in S \) such that \( ss^*s = s, s^*ss^* = s^* \). The set \( E(S) \) of idempotents of \( S \) consists of elements of the form \( ss^* \), \( s \in S \). Actually for each abstract inverse semigroup \( S \) there is a \( * \)-semigroup homomorphism from \( S \) into the inverse semigroup of partial isometries on some Hilbert space[18].

Dunkl and Ramirez in [8] and T. M. Lau in [15] attempted to define a suitable substitution for Fourier and Fourier–Stieltjes algebras on semigroups. Each definition has its own difficulties. Amini and Medghalchi introduced and

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extensively studied the theory of restricted semigroups and restricted representations and restricted Fourier and Fourier–Stieltjes algebras, \( A_{r,e}(S) \), \( B_{r,e}(S) \) in [2] and [3]. Also they studied the spectrum of the Fourier Stieltjes algebra for a unital foundation topological \(*\)-semigroup in [4]. In this section we mention some of their results.

Throughout this paper \( S \) is an inverse semigroup. Given \( x,y \in S \), the restricted product of \( x,y \) is \( xy \) if \( x^{*}x = yy^{*} \), and undefined, otherwise. The set \( S \) with its restricted product forms a groupoid [16, 3.1.4] which is called the associated groupoid of \( S \). If we adjoin a zero element 0 to this groupoid, and put \( 0^{*} = 0 \), we will have an inverse semigroup \( S_{r} \) with the multiplication rule

\[
x \cdot y = \begin{cases} xy & \text{if } x^{*}x = yy^{*}, \\ 0 & \text{otherwise} \end{cases}
\]

for \( x,y \in S \cup \{0\} \), which is called the restricted semigroup of \( S \). A restricted representation \( \{\pi, H_{\pi}\} \) of \( S \) is a map \( \pi : S \rightarrow B(H_{\pi}) \) such that \( \pi(x^{*}) = \pi(x)^{*} \) (\( x \in S \)) and

\[
\pi(x)\pi(y) = \begin{cases} \pi(xy) & \text{if } x^{*}x = yy^{*}, \\ 0 & \text{otherwise} \end{cases}
\]

for \( x,y \in S \). Let \( \Sigma_{r} = \Sigma_{r}(S) \) be the family of all restricted representations \( \pi \) of \( S \) with \( \|\pi\| \leq 1 \). Now it is clear that, via a canonical identification, \( \Sigma_{r}(S) = \Sigma_{0}(S_{r}) \), consist of all \( \pi \in \Sigma(S_{r}) \) with \( \pi(0) = 0 \), where the notation \( \Sigma \) has been used for all \(*\)-homomorphism from \( S \) into \( B(H) \) [2]. One of the central concepts in the analytic theory of inverse semigroups is the left regular representation \( \lambda : S \rightarrow B(\ell^{2}(S)) \) defined by

\[
\lambda(x)\xi(y) = \begin{cases} \xi(x^{*}y) & \text{if } xx^{*} \geq yy^{*}, \\ 0 & \text{otherwise} \end{cases}
\]

for \( \xi \in \ell^{2}(S), x,y \in S \). The restricted left regular representation \( \lambda_{r} : S \rightarrow B(\ell^{2}(S)) \) is defined in [2] by

\[
\lambda_{r}(x)\xi(y) = \begin{cases} \xi(x^{*}y) & \text{if } xx^{*} = yy^{*}, \\ 0 & \text{otherwise} \end{cases}
\]

for \( \xi \in \ell^{2}(S), x,y \in S \). The main objective of [2] is to change the convolution product on the semigroup algebra to restore the relation with the left regular representation.

For each \( f,g \in \ell^{1}(S) \), define

\[
(f \ast g)(x) = \sum_{x^{*}x = yy^{*}} f(xy)g(y^{*}) \quad (x \in S),
\]

and for all \( x \in S \), \( \hat{f}(x) = \frac{f(x)}{\|f\|} \), \( \ell^{1}_{r}(S) := (\ell^{1}(S), \ast, \cdot) \) is a Banach \(*\)-algebra with an approximate identity . The left regular representation \( \lambda_{r} \) lifts to a faithful representation \( \hat{\lambda} \) of \( \ell^{1}_{r}(S) \). We call the completion \( C_{\hat{\lambda}_{r}}(S) \) of \( \ell^{1}_{r}(S) \) with the
norm $\| \cdot \|_{\lambda_r} := \| \tilde{\lambda}_r(\cdot) \|$ which is a $C^*$-norm on $\ell_1^r(S)$, the restricted reduced $C^*$-algebra and its completion with the norm $\| \cdot \|_{\Sigma_r} := \sup\{\|\tilde{\pi}(\cdot)\|, \pi \in \Sigma(S_r)\}$ the restricted full $C^*$-algebra and show it by $C^*_r(S)$. The dual space of $C^*$-algebra $C^*_r(S)$ is a unital Banach algebra which is called the restricted Fourier-Stieltjes algebra and is denoted by $B_{r,e}(S)$. The closure of the set of finitely support functions in $B_{r,e}(S)$ is called the restricted Fourier algebra and is denoted by $A_{r,e}(S)$.

In [10], Figà-Talamanca introduced a natural generalization of the Fourier algebra, for a compact abelian group $G$, by replacing $L_2(G)$ by $L_p(G)$. In [11], Herz extended the notion to an arbitrary group, to get the commutative Banach algebra $A_p(G)$, called the Figà–Talamanca–Herz algebra. Figà–Talamanca–Herz algebra and Eymard’s Fourier algebra have very similar behavior. For example, Leptin’s theorem is valid: $G$ is amenable if and only if $A_p(G)$ has a bounded approximate identity [12]. The $p$-analog, $B_p(G)$ of the Fourier–Stieltjes algebra is defined as the multiplier algebra of $A_p(G)$, by some authors, as mentioned in [5] and [19]. Runde in [20] defined and studied $B_p(G)$, the $p$-analog of the Fourier–Stieltjes algebra on the locally compact group $G$. He developed the theory of representations and defined the suitable coefficient functions on them.

For $p \in (1, \infty)$, Medghalchi and Pourmahmood Aghababa developed the theory of restricted representations on $\ell_p(S)$ and defined the Banach algebra of $p$-pseudomeasures $PM_p(S)$ and the Figà–Talamanca–Herz algebras $A_p(S)$. They showed that $A_p(S)^* = PM_p(S)$ for dual pairs $p, q$. They characterized $PM_p(S)$ and $A_p(S)$ for Clifford semigroups, in the sense of $p$-pseudomeasures and Figà–Talamanca–Herz algebras of maximal semigroups of $S$, respectively [17].

Amini also worked on quantum version of Fourier transforms in [1].

In this paper we will combine what Medghalchi–Pourmahmood Aghababa and Runde have done. We will define the restricted representations on $QSL_p$-spaces and the $p$-analog of the Fourier–Stieltjes algebra on the restricted inverse semigroup.

Section 2 is a review of the theory of $QSL_p$-spaces. In Section 3 we define the restricted representations on $QSL_p$-spaces and study their tensor product. In Sections 4 and 5 we construct the $p$-analog of the restricted Fourier–Stieltjes algebra and study its order structure. The last section will be about Clifford semigroups and the $p$-analog of their restricted Fourier–Stieltjes algebra. Some new results which improves the results of [17] and [22] will be given in Section 6.

2. REVIEW OF THE THEORY OF $QSL_p$-SPACES

This section is a review of the paper of Runde [20].

Definition 2.1. A Banach space $\mathcal{E}$ is called
(i) an $L_p$-space if it is of the form $L_p(X)$, for some measure space $X$.

(ii) a QSL$_p$-space if it is isometrically isomorphic to a quotient of a subspace of an $L_p$-space (or equivalently, a subspace of a quotient of an $L_p$-space [20, Section 1, Remark 1]).

If $E$ is a QSL$_p$-space and if $p' \in (1, \infty)$ is such that $\frac{1}{p} + \frac{1}{p'} = 1$, the dual space $E^*$ is an QSL$_{p'}$-spaces. In particular, every QSL$_p$-space is reflexive.

By [14, Theorem 2], the QSL$_p$-spaces are precisely the $p$-spaces in the sense of [11], i.e. those Banach spaces $E$ such that for any two measure spaces $X$ and $Y$ the amplification map

$$B(L_p(X), L_p(Y)) \rightarrow B(L_p(X,E), L_p(Y,E)), T \rightarrow T \otimes id_E$$

is an isometry. In particular, an $L_q$-space is a QSL$_p$-space if and only if $2 \leq q \leq p$ or $p \leq q \leq 2$. Consequently, if $2 \leq q \leq p$ or $p \leq q \leq 2$, then every QSL$_q$-space is a QSL$_p$-space.

Runde equipped the algebraic tensor product of two QSL$_p$-spaces with a suitable norm, which comes in the following.

**Theorem 2.2.** [20, Theorem 3.1] Let $E$ and $F$ be QSL$_p$-spaces. Then there exists a norm $\| \cdot \|_p$ on the algebraic tensor product $E \otimes F$ such that:

(i) $\| \cdot \|_p$ dominates the injective norm;

(ii) $\| \cdot \|_p$ is a cross norm;

(iii) the completion $E \hat{\otimes}_p F$ of $E \otimes F$ with respect to $\| \cdot \|_p$ is a QSL$_p$-space.

The Banach space $E \hat{\otimes}_p F$ will be called $p$-projective tensor product of $E$ and $F$.

### 3. Restricted Representation on a Banach space

In this section we give an analog of the theory of group representations on a Hilbert space for the restricted representations for an inverse semigroup on a QSL$_p$-space.

**Definition 3.1.** A representation of a discrete inverse semigroup $S$ on a Banach space $E$ is a pair $(\pi, E)$ consisting of a map $\pi : S \rightarrow B(E)$ satisfying $\pi(x)\pi(y) = \pi(xy)$, for $x, y \in S$ and $\|\pi(x)\| \leq 1$, for all $x \in S$.

**Definition 3.2.** A restricted representation of a discrete inverse semigroup $S$ on a Banach space $E$ is a pair $(\pi, E)$ consisting of a map $\pi : S \rightarrow B(E)$ satisfying

$$\pi(x)\pi(y) = \begin{cases} \pi(xy) & \text{if } x^*x = y^*y, \\ 0 & \text{otherwise} \end{cases}$$

for $x, y \in S$, and $\|\pi(x)\| \leq 1$, for all $x \in S$.

**Definition 3.3.** Let $S$ be an inverse semigroup, and let $(\pi, E)$ and $(\rho, F)$ be restricted representations of $S$, then these restricted representations are said to be equivalent if there exists a surjective isometry $T : E \rightarrow F$ such that

$$T\pi(x)T^{-1} = \rho(x), \quad (x \in S).$$
For any inverse semigroup $S$ and $p \in (1, \infty)$, we denote by $\Sigma_{p,r}(S)$ the collection of all (equivalence classes) of restricted representations of $S$ on a $QSL_p$-space.

**Remark 3.4.** By [17] for $p \in (1, \infty)$ the restricted left regular representation $\lambda_p : S \rightarrow B(\ell^p(S))$

$$\lambda_p(s)(\delta_t) = \begin{cases} 
\delta_{st} & \text{if } s^*s = tt^*, \\
0 & \text{otherwise}
\end{cases}$$

for $s, t \in S$ is a restricted representation so it belongs to $\Sigma_{p,r}(S)$.

The following propositions are easy to check, similar to [2].

**Proposition 3.5.** For an inverse semigroup $S$ and its related restricted semigroup $S_r$, each restricted representation of $S$ on a Banach space is a representation on $S_r$ which is zero on $0 \in S_r$, i.e. it is multiplicative with respect to the restricted multiplication.

**Proposition 3.6.** For an inverse semigroup $S$, each restricted representation $\pi$ of $S$ on a Banach space lifts to a representation of $\ell^1(S)$, via

$$\hat{\pi}(f) = \sum_{x \in S} f(x)\pi(x).$$

4. **Banach Algebra $B_{p,r}(S)$**

In this section we define the $p$-analog of the Fourier–Stieltjes algebra on an inverse semigroup. We show that for $p = 2$ we get the known algebra $B_{r,e}(S)$, defined in [2].

**Theorem 4.1.** Let $(\pi, E), (\rho, F) \in \Sigma_{p,r}(S)$ then $(\pi \otimes \rho, E \otimes_p F) \in \Sigma_{p,r}(S)$.

**Proof.** By the definition of $\pi \otimes \rho$ we have $\pi \otimes \rho(x)(\xi \otimes \eta) = \pi(x)\xi \otimes \rho(x)\eta$. For $x, y \in S$, $x^*x = yy^*$,

$$\pi \otimes \rho(xy)(\xi \otimes \eta) = \pi(xy)\xi \otimes \rho(xy)\eta$$

$$= \pi(x)\pi(y)\xi \otimes \rho(x)\rho(y)\eta$$

$$= \pi(x)(\pi(y)\xi) \otimes \rho(x)(\rho(y)\eta)$$

$$= \pi \otimes \rho(x)(\pi(y)\xi \otimes \rho(y)\eta)$$

$$= \pi \otimes \rho(x)\pi \otimes \rho(y)(\xi \otimes \eta)$$

when $x^*x \neq yy^*$

$$\pi \otimes \rho(x)\pi \otimes \rho(y)(\xi \otimes \eta) = \pi \otimes \rho(x)(\pi(y)\xi \otimes \rho(y)\eta)$$

$$= \pi(x)(\pi(y)\xi) \otimes \rho(x)(\rho(y)\eta)$$

which is equal to zero. Now it is enough to show that $\pi(x) \in B(E)$ and $\rho(y) \in B(F)$, $\pi(x) \otimes \rho(y)$ could be extend to $E \otimes_p F$. This is shown as in the group case [20, Theorem 3.1]. □
Definition 4.2. Let $S$ be an inverse semigroup, and let $(\pi, E) \in \Sigma_{p,r}(S)$. A coefficient function of $(\pi, E)$ is a function $f : S \to \mathbb{C}$ of the form

$$f(x) = \langle \pi(x)\xi, \phi \rangle \quad (x \in S),$$

where $\xi \in E$ and $\phi \in E^*$.

Definition 4.3. Let $S$ be an inverse semigroup, let $p \in (1, \infty)$, and let $q \in (1, \infty)$ be the dual scalar to $p$, i.e. $\frac{1}{p} + \frac{1}{q} = 1$. We define

$$B_{p,r}(S) := \{ f : S \to \mathbb{C} : f \text{ is a coefficient function of some } (\pi, E) \in \Sigma_{q,r}(S) \}.$$

Proposition 4.4. Let $S$ be an inverse semigroup, let $p \in (1, \infty)$, and let $q \in (1, \infty)$ be the dual scalar to $p$, i.e. $\frac{1}{p} + \frac{1}{q} = 1$, and let $f : S \to \mathbb{C}$ defined by

$$f(x) = \sum_{n=1}^{\infty} \langle \pi_n(x)\xi_n, \phi_n \rangle \quad (x \in S),$$

where $((\pi_n, E_n))_{n=1}^{\infty}, (\xi_n)_{n=1}^{\infty},$ and $(\phi_n)_{n=1}^{\infty}$ are sequences with $(\pi_n, E_n) \in \Sigma_{q,r}(S)$, $\xi_n \in E_n$, and $\phi_n \in E_n^*$, for $n \in \mathbb{N}$ such that

$$\sum_{n=1}^{\infty} \|\xi_n\|\|\phi_n\| < \infty.$$

Then $f$ lies in $B_{p,r}(S)$.

Proof. The proof is similar to [20]. Without loss of generality, we may suppose that

$$\sum_{i=1}^{\infty} \|\xi_i\|_q < \infty, \quad \text{and} \quad \sum_{i=1}^{\infty} \|\phi_i\|_p < \infty.$$

Then $E := \ell_q - \oplus_{n=1}^{\infty} E_n$ is a QSLq-space and for $\xi := (\xi_1, \xi_2, \ldots)$ and $\phi := (\phi_1, \phi_2, \ldots)$, we have $\xi \in E$ and $\phi \in E^*$. Now the map $\pi : S \to B(E)$ with $\pi(x) \eta = (\pi_1(x)\eta, \pi_2(x)\eta, \ldots)$ is a restricted representation of $S$ on $E$, and $f$ is the coefficient function of $\pi$. \qed

Definition 4.5. [17, Definition 3.1]. Let $S$ be an inverse semigroup and let $p, q \in (1, \infty)$ be dual pairs. The space $A_q(S)$ consists of those $u \in c_0(S)$ such that there exist sequences $(f_n)_{n=1}^{\infty} \subseteq \ell_q(S)$ and $(g_n)_{n=1}^{\infty} \subseteq \ell_p(S)$ with $\sum_{n=1}^{\infty} \|f_n\|_q \|g_n\|_q \leq \infty$ and $u = \sum_{n=1}^{\infty} f_n \bullet g_n$. For $u \in A_q(S)$, let

$$\|u\| = \inf \left\{ \sum_{n=1}^{\infty} \|f_n\|_q \|g_n\|_p : u = \sum_{n=1}^{\infty} f_n \bullet g_n \right\}.$$

Proposition 4.6. [17, Proposition 3.2]. Let $S$ be an inverse semigroup and let $p \in (1, \infty)$, then $A_p(S)$ is a Banach space and is the closure of finite support functions on $S$. 
Proposition 4.7. Let $S$ be an inverse semigroup, let $p \in (1, \infty)$. Then $B_{p,r}(S)$ is a linear subspace of $c_0(S)$ containing $A_p(S)$. Moreover, if $2 \leq q \leq p$ or $p \leq q \leq 2$, we have $B_{q,r}(S) \subseteq B_{p,r}(S)$.

Proof. Every thing is easy to check, and is similar to [20].

Definition 4.8. Let $S$ be an inverse semigroup, and let $(\pi, E)$ be a restricted representation of $S$ on the Banach space $E$. Then $(\pi, E)$ is called cyclic if there exists $x \in E$ such that $\pi(\ell^1_n(S))x$ is dense in $E$. For $p \in (1, \infty)$, we set $\text{Cyc}_{p,r}(S) := \{(\pi, E) : (\pi, E) \text{ is a cyclic restricted representation on a } \mathbb{Q}SL_{p^*}\text{-space } E\}$.

Definition 4.9. Let $S$ be an inverse semigroup, let $p, q \in (1, \infty)$ be the dual scalars, and let $f \in B_{p,r}(S)$. We define $\|f\|_{B_{p,r}(S)}$ as the infimum over all expressions $\sum_{n=1}^{\infty} \|\xi_n\|\|\phi_n\|$, where, for each $n \in \mathbb{N}$, there is $(\pi_n, E_n) \in \text{Cyc}_{q,r}(S)$ with $\xi_n \in E_n$ and $\phi_n \in E_n^*$ such that $\sum_{n=1}^{\infty} \|\xi_n\|\|\phi_n\| < \infty$ and

$$f(x) = \sum_{n=1}^{\infty} \langle \pi_n(x)\xi_n, \phi_n \rangle, \quad (x \in S).$$

The proof of the following theorem is similar to the group case.

Theorem 4.10. Let $S$ be an inverse semigroup, let $p \in (1, \infty)$, and let $f, g : S \rightarrow \mathbb{C}$ be coefficient functions of $(\pi, E)$ and $(\rho, F)$ in $\Sigma_{p,r}(S)$, respectively. Then the pointwise product of $f$ and $g$ is a coefficient function of $(\pi \otimes \rho, E \otimes F)$.

In the next theorem we give some result about our new constructed space and also the relation between semigroup restricted Herz algebra and our new space.

Theorem 4.11. Let $S$ be an inverse semigroup, let $p \in (1, \infty)$. Then:

(i) $B_{p,r}(S)$ is a commutative Banach algebra.

(ii) the inclusion $A_p(S) \subseteq B_{p,r}(S)$ is a contraction.

(iii) for $2 \leq p' \leq p$ or $p \leq p' \leq 2$, the inclusion $B_{p',r}(S) \subseteq B_{p,r}(S)$ is a contraction.

(iv) for $p = 2$, $B_{r,e}(S)$ is isometrically isomorphic to $B_{p,r}(S)$ as Banach algebras.

Proof. (i) Let $\frac{1}{p} + \frac{1}{q} = 1$. The space $B_{p,r}(S)$ is the quotient space of complete $q$-projective tensor product of $E \tilde{\otimes} E^*$, for the universal restricted representation $(\pi, E)$, on $\mathbb{Q}SL_q$-space $E$. Also Theorem 4.10 shows it is an algebra. The submultiplicative property for norm of $B_{p,r}(S)$ is similar to the group case in [20] and it is only based on characteristic property of infimum.

(ii) By the definition of semigroup Herz algebra in [17] for conjugate numbers $p, q$, each $f \in A_p(S)$ is a coefficient function of the restricted left regular representation on the $\ell^q$-space, $\ell^q(S)$. So $A_p(S) \subseteq B_{p,r}(S)$. By the definition of the norm of $f \in B_{p,r}(S)$, the infimum is taken on all expressions of $f$ as the coefficient function of some restricted representation on a $\mathbb{Q}SL_q$-space, and...
the norm on the $A_p(S)$ is the infimum only on expressions of $f$ as the coefficient function of restricted left regular representation, so the inclusion map is a contraction.

(iii) For $2 \leq p \leq p'$ or $p \leq p' \leq 2$ and $q, q'$ conjugate scalars to $p$ and $p'$ respectively. Then each restricted representation on a $QSL_{q'}$-space is a restricted representation on a $QSL_q$-space.

(iv) By the definition, each element of $B_{r,e}(S)$ is a coefficient function of a 2-restricted representation [3].

□

Remark 4.12. A very natural question is that when $A_p(S)$ is an ideal in $B_{p,r}(S)$. Even in $p = 2$ this question is not studied. If we want to go along the proof of the group case, a difficulty to prove this is that in general for $p \in (1, \infty)$, and $(\pi, E) \in \Sigma_{p,r}(S)$, the representations $(\lambda_p \otimes \pi, \ell_p(S, E))$ and $(\lambda_p \otimes \text{id}_E, \ell_p(S, E))$ are not equivalent. In fact we can not find a suitable substitution for representation $\text{id} : S \rightarrow B(E)$, $\text{id}(s) = \text{id}_E$ in the class of restricted representations.

But in a special case, such as Clifford semigroups, we can give a better result.

5. ORDER STRUCTURE OF THE $p$-ANALOG OF THE SEMIGROUP
FOURIER–STIELTJES ALGEBRAS $B_{p,r}(S)$

Studying the ordered spaces and order structures has a long history. The natural order structure of the Fourier-Stieltjes algebras was favorite in 80s. In [21] the authors studied the order structure of Figà–Talamanca– Herz algebra and generalized results on Fourier algebras. In this section, we consider the $p$-analog of the restricted Fourier–Stieltjes algebra, $B_{p,r}(S)$, introduced in Section 4, and study its order structure given by the $p$-analog of positive definite continuous functions.

A compatible couple of Banach spaces in the sense of interpolation theory (see [3]) is a pair $(E_0, E_1)$ of Banach spaces such that both $E_0$ and $E_1$ are embedded continuously in some (Hausdorff) topological vector space. In this case, the intersection $E_0 \cap E_1$ is again a Banach space under the norm $\| \cdot \|_{(E_0, E_1)} = \max\{\| \cdot \|_{E_0}, \| \cdot \|_{E_1}\}$. For example, for a locally compact group $G$, the pairs $(A_p(G), A_q(G))$ and $(L_p(G), L_q(G))$ are compatible couples.

**Definition 5.1.** Let $(\pi, E)$ be a restricted representation of $S$ on a Banach space $E$, such that $(E, E^*)$ is a compatible couple. We mean by a $\pi_r$-positive definite function on $S$, a function which has a representation as $f(x) = \langle \pi(x)\xi, \xi \rangle$, $(x \in S)$, where $\xi \in E \cap E^*$. For dual scalars $p, q \in (1, \infty)$, we call each element in the closure of the set of all $\pi_r$-positive definite functions on $S$ in $B_{p,r}(S)$, where $\pi$ is a restricted representation of $S$ on an $L_q$-space, a restricted $p$-positive definite function on $S$ and the set of all restricted $p$-positive definite functions on $S$, will be denoted by $P_{p,r}(S)$. 
It follows from [21] and the definition of \( P_{p,r}(S) \), that for each \( f \in P_{p,r}(S) \), associated to a representation \((\pi, E)\), for a \( QSL_p \)-space \( E \), there exists a sequence \((\pi_n, E_n)_{n=1}^{\infty}\) of cyclic restricted representations of \( S \) on closed subspaces \( E_n \) of \( E \cap E^* \), and \( \{\xi_n\} \) in \( E_n \), such that

\[
f(x) = \sum_{n=1}^{\infty} (\pi_n(x)\xi_n, \xi_n) \quad (x \in S).
\]

**Proposition 5.2.** The linear span of all finite support elements in \( P_{p,r}(S) \) is dense in \( A_p(S) \), and \( A_p(S) \) is an ordered space.

**Proof.** From [17, Proposition 3.2] \( A_p(S) \) is a norm closure of the set of elements of the form \( \sum_{i=1}^{n} f_i \cdot \hat{g}_i \) where \( f_i, g_i \) are finite support functions on \( S \), \( i = 1, \ldots, n \). Also \( f_i \cdot \hat{g}_i(x) = \langle \lambda_r(x^*) f_i, g_i \rangle \). Now by Polarization identity, we have the statement. \( \square \)

Since \( A_p(S) \) is the set of coefficient functions of the restricted left regular representation of \( S \) on \( \ell_p(S) \), we define the positive cone of \( A_p(S) \) as the closure in \( A_p(S) \), of the set of all function of the form \( f = \sum_{i=1}^{n} \xi_i \cdot \xi_i^* \), for a sequence \((\xi_i)\) in \( \ell_p(S) \cap \ell_q(S) \), and denote it by \( A_p(S)_+ \).

This order structure, in the case where \( p = 2 \), is the same as the order structure of \( A_{r,e}(S) \), induced by the set \( P_{r,e}(S) \cap A_{r,e}(S) \), as a positive cone. Because in the case \( p = 2 \), the extensible restricted positive definitive functions are exactly the closed linear span of \( h \cdot \hat{h} \), for \( h \in \ell^2(S) \).

6. \( p \)-Analog of the Fourier–Stieltjes Algebras on Clifford Semigroups

Let \( S \) be a semigroup. Then, by [13, Chapter 2], there is an equivalence relation \( D \) on \( S \) by \( sDt \) if and only if there exists \( x \in S \) such that

\[
Ss \cup \{s\} = Sx \cup \{x\} \quad \text{and} \quad ts \cup \{t\} = xS \cup \{x\}.
\]

If \( S \) is an inverse semigroup, then by [13, Proposition 5.1.2(4)], \( sDt \) if and only if there exists \( x \in S \) such that \( s^*s = xx^* \) and \( t^*t = x^*x \).

**Proposition 6.1.** [17, Proposition 4.1]. Let \( S \) be an inverse semigroup,

(i) and let \( D \) be a \( D \)-class of \( S \). Then \( \ell_p(D) \) is a closed \( \ell^1_p(S) \)-submodule of \( \ell_p(S) \).

(ii) and let \( \{D_\lambda; \lambda \in \Lambda\} \) be the family of \( D \)-classes of \( S \) indexed by some set \( \Lambda \). Then there is an isometric isomorphism of Banach \( \ell^1_p(S) \)-bimodules

\[
\ell^p(S) \cong \ell^p - \bigoplus_{\lambda \in \Lambda} \ell_p(D_\lambda). \tag{6.1}
\]

**Corollary 6.2.** Let \( S \) be an inverse semigroup, and let \( \{D_\lambda; \lambda \in \Lambda\} \) be the family of \( D \)-classes of \( S \) indexed by some set \( \Lambda \). Then for a \( QSL_{p} \)-space \( E \) of functions on \( S \), there is a family of \( QSL_p \)-spaces \( \{E_\lambda\}_{\lambda \in \Lambda} \), where for each \( \lambda \in \Lambda \), \( E_\lambda \) consists of functions on \( D_\lambda \), and \( E \cong \ell^p - \bigoplus_{\lambda \in \Lambda} E_\lambda \).
Proof. This is clear by the definition of a $QSL_p$-space, and the fact that the isomorphism 6.1 is compatible with taking quotients and subspaces of $\ell_p(D_\lambda)s$.

An inverse semigroup $S$ is called a Clifford semigroup if $s^*s = ss^*$ for all $s \in S$. For $e \in E(S)$ define $G_e := \{ s \in S | s^*s = ss^* = e \}$. Then $G_e$ is a group with identity $e$. Here each $D$-class $D$ contains a single idempotent (say $e$) and we have $D = G_e$.

We modified the isometrical isomorphism derived in [17, Section 5.3] in the following theorem.

**Theorem 6.3.** Let $S$ be a Clifford semigroup with the family of $D$-classes $\{G_e\}_{e \in E(S)}$, and let $p \in (1, \infty)$. Then

$$B_{p,r}(S) \cong \ell^1 - \bigoplus_{e \in E(S)} B_p(G_e)$$

Proof. Let $p, q$ are conjugate scalars. Fix $e \in E(S)$, assume that $G_e = \{ x \in S; x^*x = e \}$ Define $\pi : S \to B(\ell_q(G_e))$

$$\pi(s)(\delta_t) = \begin{cases} \delta_t & \text{if } s^*s = e, \\ 0 & \text{otherwise} \end{cases}$$

for $s \in S$. Then $\pi$ is a restricted representation and $\chi_{G_e}(s) = \langle \pi(s)\delta_t, \delta_t \rangle$. Hence $\chi_{G_e}$ is in $B_{p,r}(S)$, and indeed $\chi_{G_e}$ is a restricted positive definite function. Now for each $u \in B_{p,r}(S)$, $u \cdot \chi_{G_e}$ is in $B_{p,r}(S)$. In fact the set \{ $u \in B_{p,r}(S); u(s) = 0$ for all $s \in S \setminus G_e$ \} is a closed subspace of $B_{p,r}(S)$ and it is isometrically isomorphic to $B_p(G_e)$. This follows from the fact that, each coefficient function of a restricted representation of $S$ on a $QSL_q$-space that is zero on $G_e^-$, is a coefficient function of a representation on a $QSL_q$-space of $G_e$, using Corollary 6.2.

Let $u \in B_{p,r}(S)$, then we could decompose $u$ to $(u_e)_{e \in E(S)}$, for some $u_e \in B_p(G_e)$, by the above paragraph. Now for all $e \in E(S)$ and all explanations of $u_e$ as $u_e = \langle \pi_e(\cdot)\xi_e, \eta_e \rangle$, where $\pi_e \in \Sigma_q(G_e)$, $\xi_e$ in some $QSL_q$ and $\eta_e$ in some $QSL_p$-space for dual scalars $p, q$ we have $\|u_e\| \leq \|\xi_e\|\|\eta_e\|$ and also $u = (u_e)_{e \in E(S)} = \langle \oplus \pi_e(\cdot) \oplus \xi_e, \oplus \eta_e \rangle$ and then $\oplus \pi_e$ is a restricted representation of $S$ on a $QSL_q$-space and $\sum_{e \in E(S)} \|u_e\| \leq \sum_{e \in E(S)} \|\xi_e\|\|\eta_e\| \leq (\sum_{e \in E(S)} \|\xi_e\|^q)^{\frac{1}{q}}(\sum_{e \in E(S)} \|\eta_e\|^p)^{\frac{1}{p}} = \|(\xi_e)\|(\|\eta_e\|)$, the last equality comes from Proposition 6.1. Now we have:

$$\sum_{e \in E(S)} \|u_e\| \leq \|(u_e)_{e \in E(S)}\| = \|\sum_{e \in E(S)} u_e\|$$

by the definition of the norm in the Fourier–Stieltjes algebras. So we have an isometric isomorphism of Banach algebras.

The following corollary improves [22, Proposition 2.6].
Corollary 6.4. Let $S$ be a Clifford semigroup with the family of $D$-classes $\{G_e\}_{e \in E(S)}$, and let $p \in (1, \infty)$. Then $A_p(S)$ is an ideal of $B_{p,r}(S)$, and $B_{p,r}(S)$ is amenable if and only if $B_p(G_e)$ is amenable for all $e \in E(S)$.

Theorem 6.5. Let $S$ be a Clifford semigroup with the family of amenable $D$-classes $\{G_e\}_{e \in E(S)}$ and let $p \in (1, \infty)$. Then $A_p(S)$ is equal to the closure of $B_{p,r}(S) \cap F(S)$ in the norm of $A_p(S)$.

Proof. Since for each $e \in E(S)$ the group $G_e$ is amenable, the natural embedding $i_e : A_p(G_e) \to B_p(G_e)$ is an isometry by [20, Corollary 5.3]. Now let $f \in B_{p,r}(S) \cap F(S)$. Then by Theorem 6.3, $f = \sum_{e \in E(S)} f_e$, where $f_e \in B_p(G_e) \cap F(G_e)$, so $f_e$ belongs to $A_p(G_e)$ by [20]. Now since

$$A_p(S) \cong \ell_1 - \bigoplus_{e \in E(S)} A_p(G_e),$$

[17, Equation 5.1], we conclude that $f \in A_p(S)$. On the other hand let $f \in A_p(S) \cap F(S)$. Then for each $e \in E(S)$ the function $f_e$, which has been defined in Theorem 6.3, belongs to $A_p(G_e) \cap F(G_e)$ and so to $B_p(G_e) \cap F(G_e)$ with the same norm because of amenability of $G_e$. Now, by Theorem 6.3 we have $f \in B_{p,r}(S) \cap F(S)$. Since $F(S)$ is dense in $A_p(S)$ with norm of $A_p(S)$, [17, Proposition 3.2 vi], result follows.

\[\square\]

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