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Characterizations of the simple group $D_n(3)$ by prime graph and spectrum

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Abstract. We prove that $D_n(3)$, where $n \geq 6$ is even, is uniquely determined by its prime graph. Also, if G is a finite group with the same prime graph as $D_4(3)$, then $G \cong D_4(3), B_3(3), C_3(3)$ or $G/O_2(G) \cong \mathrm{Aut}({}^2B_2(8))$.

Keywords: Characterization, prime graph, the classification theorem of finite simple group, spectrum.

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1. Introduction

For a finite group G, we denote by $\pi(G)$ the set of all prime divisors of $|G|$ and the *spectrum* $\omega(G)$ of G is the set of element orders of G. The *prime graph* (or *Gruenberg-Kegel graph*) $GK(G)$ of G is an undirected and simple graph with vertex set $\pi(G)$ where two distinct vertices p and q are adjacent by an edge (briefly, adjacent) if $pq \in \omega(G)$, in which case, we write $(p, q) \in GK(G)$. The concept of prime graph arose during the investigation of certain cohomological questions associated with integral representations of finite groups. It has been proved that for every finite group G the number of connected components of

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$GK(G)$ is at most 6 [15, 20].

The group G is said to be *characterizable by spectrum*, if for every finite group H, the equality $\omega(G) = \omega(H)$ implies the group isomorphism $G \cong H$. Since the knowledge of $\omega(G)$ determines $GK(G)$, alongside the above definition, in [10] the same concept related to the prime graph has been introduced. A finite group G is called *characterizable by prime graph* if $H \cong G$ for every finite group H with $GK(H) = GK(G)$. In these both characterizations, when the number of connected components of prime graph increases, dealing with the groups gets much easier. So, the recognition problem by prime graph has been considered for many simple groups with more than two prime graph components (for a survey see the references of [5]). In [21], as the first example of finite groups with two prime graph components, Zavarnitsine proved that $G_2(7)$ is characterizable by its prime graph. Moreover, by mentioning that it is not easy to find examples of groups which are characterizable by prime graph and their prime graphs are connected, he put forward the following open problem:

Open problem. Is there a finite group characterizable by its prime graph whose prime graph is connected?

The simple group $L_{16}(2)$ is the first positive answer to this problem [12, 22]. Except $G_2(7)$, the groups $B_p(3)$ and $C_p(3)$, where $p > 3$ is an odd prime [17], and ${}^2D_n(3)$, where $n \geq 5$ is an odd number and $n = 2^m + 1$ is a non-prime or $n \neq 2^m + 1$ is a prime number [6], are the only groups with two prime graph components, which their recognizability by prime graph have been solved thoroughly. Also, except $L_{16}(2)$, the group ${}^2D_n(3)$, where $n \geq 5$ and $n \neq 2^m + 1$ is an odd non-prime [6], is the only positive answer to Zavarnitsine's problem, thus far. In this paper, we will consider the characterizability of finite simple group $D_n(3)$ by prime graph. Since $GK(D_n(3))$, where n is even, has at most two connected components, we actually find another infinite series of characterizable finite simple groups by prime graph which either are a positive answer to Zavarnitsine's problem or have two prime graph components. In fact, we prove the following main theorem:

Main theorem. Let $n \geq 6$ be an even number. The simple group $D_n(3)$ is characterizable by prime graph. Moreoever, if G is a finite group with $GK(G) = GK(D_4(3))$, then $G \cong D_4(3), B_3(3), C_3(3)$ or $G/O_2(G) \cong \mathrm{Aut}({}^2B_2(8))$.

In [9], the recognition problem of the group $D_n(3)$ by its spectrum, where its prime graph is disconnected, has been considered. As the first consequence of the main theorem, the characterizability of the group $D_n(3)$ by spectrum, where $n \geq 6$ is an even number, is obtained. It is worth to mention that our proof for the special case $n-1$ is a prime number, is different from [9].

The non-commuting graph of a nonabelian group G, denoted by Γ_G is the graph with vertex set $G \setminus Z(G)$, where two distinct vertices x and y are adjacent by an edge if $xy \neq yx$. Problem 16.1 in the Kourovka notebook [13] is AAM's conjecture, which says simple groups are determined uniquely by the non-commuting graph. This conjecture is valid for all non-abelian finite simple groups with disconnected prime graph (for examples see paper by Darafsheh [4] and references quoted in that paper), while this conjecture is open yet for finite simple groups with connected prime graph. As a prominent corollary of the main theorem, the validity of the AAM's conjecture can be obtained for the groups under study, as well.

2. **Notation and preliminary results**

Throughout this paper, we use the following notation: By $[x]$ we denote the integer part of x and by $\gcd(m,n)$ we denote the greatest common divisor of m and n. The notation for groups of Lie type is according to [3] and sometimes for abbreviation, we write $A_n^{\varepsilon}(q)$ and $D_n^{\varepsilon}(q)$, where $\varepsilon \in \{+,-\}$, and $A_n^+(q) = A_n(q)$, $A_n^-(q) = {}^2A_n(q)$, $D_n^+(q) = D_n(q)$, $D_n^-(q) = {}^2D_n(q)$.

Also, we use the following definitions and notation related to $GK(G)$: A set of vertices of a graph is called a coclique (or independent), if its elements are pairwise nonadjacent. We denote by $\rho(G)$ and $\rho(r,G)$ a coclique of maximal size in $GK(G)$ and a coclique of maximal size, containing r, in $GK(G)$, respectively. We put $t(G) = |\rho(G)|$ and $\mathrm{t}(r,G) = |\rho(r,G)|$. Also, for an integer n, by $\nu(n)$ and $\eta(n)$ we denote the following functions:

$$\nu(n) = \begin{cases} n \text{ if } n \equiv 0 \pmod 4; \\ \frac{n}{2} \text{ if } n \equiv 2 \pmod 4; \\ 2n \text{ if } n \equiv 1 \pmod 2. \end{cases}, \quad \eta(n) = \begin{cases} n \text{ if } n \text{ is odd}; \\ \frac{n}{2} \text{ otherwise}. \end{cases}$$

All further unexplained notation are standard and refer to [3], for example. For a finite group G, with $t(G) \geq 3$ and $t(2,G) \geq 2$, Vasil'ev and Gorshkov have proved the following theorem:

Lemma 2.1. [18, Theorem 1] *Let G be a finite group with $t(G) \geq 3$ and $t(2,G) \geq 2$. Then the following hold:*

(1) *There exists a finite nonabelian simple group S such that $S \leq \bar{G} = G/K \leq \mathrm{Aut}(S)$ for the maximal normal soluble subgroup K of G.*
(2) *For every independent subset ρ of $\pi(G)$ with $|\rho| \geq 3$ at most one prime in ρ divides the product $|K|.|\bar{G}/S|$. In particular, $t(S) \geq t(G) - 1$.*
(3) *One of the following holds:*

(a) *every prime* $r \in \pi(G)$ *nonadjacent to* 2 *in* $GK(G)$ *does not divide the product* $|K|.|\bar{G}/S|$; *in particular,* $t(2,S) \geq t(2,G)$;

(b) *there exists a prime* $r \in \pi(K)$ *nonadjacent to* 2 *in* $GK(G)$; *in which case* $t(G) = 3$, $t(2,G) = 2$, *and* $S \cong A_7$ *or* $A_1(q)$ *for some odd* q.

Let s be a prime and let m be a natural number. The s-part of m is denoted by m_s, i.e., $m_s = s^t$ if $s^t \mid m$ and $s^{t+1} \nmid m$. If q is a natural number, r is an odd prime and $\gcd(r,q) = 1$, then by $e(r,q)$ we denote the smallest natural number m such that $q^m \equiv 1 \pmod r$. Obviously if $q^n \equiv 1 \pmod r$, then $e(r,q) \mid n$. Also, by Fermat's little theorem it follows that $e(r,q) \mid (r-1)$. If q is odd, we put $e(2,q) = 1$ if $q \equiv 1 \pmod 4$, and $e(2,q) = 2$ otherwise. The prime r with $e(r,q) = m$ is called a *primitive prime divisor* of $q^m - 1$. It is obvious that $q^m - 1$ can have more than one primitive prime divisor. We denote by $r_m(q)$ some primitive prime divisor of $q^m - 1$. If there is no ambiguous, we write r_m instead of $r_m(q)$.
The following easy lemma will be used in the proof of the main theorem:

Lemma 2.2. *Let* G *be a finite group. If* H *is a subgroup of* G *and* N *is a normal subgroup of* G, *then:*

(1) *If* $(p,q) \in GK(H)$, *then* $(p,q) \in GK(G)$.
(2) *If* $(p,q) \in GK(G/N)$, *then* $(p,q) \in GK(G)$.
(3) *If* $(p,q) \in GK(G)$ *and* $\{p,q\} \cap \pi(N) = \emptyset$, *then* $(p,q) \in GK(G/N)$.
(4) *If* G/N *is a* p-*group and* $x \in G - N$, *then* $p \mid O(x)$.

3. Proof of the main theorem

If $n \geq 8$ is an even number, then by Tables 6 and 8 in [19], we have $t(2, D_n(3)) = 2$, $\rho(2, D_n(3)) = \{2, r_{n-1}(3)\}$ and $t(D_n(3)) = [\frac{3n+1}{4}]$. Hence, if G is a finite group with $GK(G) = GK(D_n(3))$, then Lemma 2.1 implies that G has a unique nonabelian composition factor S, in which case $S \leq \bar{G} = G/K \leq \mathrm{Aut}(S)$, $t(S) \geq t(D_n(3)) - 1$, $t(2,S) \geq 2$ and $r_{n-1}(3) \in \pi(S)$. One should remark that $r_m(q)$ stands for any primitive prime divisor of $q^m - 1$.

3.1. **The group S is isomorphic to the group $D_n(3)$.** Since S is a finite nonabelian simple group, it follows by the classification of the finite simple groups that S is a sporadic simple group, an alternating group or a simple group of Lie type. We will prove that $S \cong D_n(3)$, by a sequence of lemmas.

Lemma 3.1. *S cannot be isomorphic to a sporadic simple group.*

Proof. If $n \geq 18$, then by Lemma 2.1(2), $t(S) \geq t(D_n(3)) - 1 \geq 12$ and hence, the conclusion immediately holds by Table 2 in [19]. Otherwise, we have $n \in \{8, 10, 12, 14, 16\}$ and since $r_{n-1}(3) \in \pi(S)$, thus one of the numbers

$r_7(3) = 1093$, $r_9(3) = 757$, $r_{11}(3) = 3851$, $r_{13}(3) = 797161$ or $r_{15}(3) = 4561$ divides $|S|$. But, according to the orders of sporadic simple groups, we can easily get a contradiction.□

Lemma 3.2. *S cannot be isomorphic to an alternating group.*

Proof. If $S \cong A_m$, where $m \geq 5$, then by considering the cases $n \geq 18$ and $8 \leq n \leq 16$ separately, we get a contradiction.
Case 1. $n \geq 18$. Then $t(S) \geq 12$ and hence, $|\pi(A_m)| \geq 12$. Thus according to the set $\pi(A_m)$, we can assume that $m \geq 37$ and it implies that $19 \in \pi(S)$. First, we find an upper bound for $t(19, A_m)$. If $x \in \rho(19, A_m) \setminus \{19\}$, then by Proposition 1.1 in [19], $x \neq 2$ and $x + 19 > m$. Also, since $x \in \pi(A_m)$, we conclude that $x \in \{s \mid s \text{ is a prime}, m - 18 \leq s \leq m\}$. By an easy computation, we can see that there are at most six integers coprime to 30 between $m - 18$ and m, where $m \geq 37$ and hence, there exist at most six choices for x. Thus $t(19, A_m) \leq 7$. Since $S \leq G/K$ and $\pi(G) = \pi(D_n(3))$, we have $\pi(S) \subseteq \pi(D_n(3))$ and hence, $19 \in \pi(D_n(3))$. Now we find a lower bound for $t(19, D_n(3))$. Since $e(19, 3) = 18$, n is an even number and $n \geq 18$, we can use the set

$$\tau = \{19, r_{n-1}, r_{2(n-2)}, r_{n-3}, r_{2(n-4)}, r_{n-5}, r_{2(n-6)}, r_{n-7}, r_{2(n-8)}\}$$

which is a coclique of $GK(D_n(3))$, according to Proposition 2.4 in [19] and hence, $t(19, D_n(3)) \geq 9$. By Lemma 2.1(2) for the set $\rho(19, D_n(3))$, we have $|\rho(19, D_n(3)) \bigcap \pi(A_m)| \geq t(19, D_n(3)) - 1 \geq 8$. On the other hand, since $S \leq G/K$, it follows by Lemma 2.2(1,2) that $|\rho(19, D_n(3)) \bigcap \pi(A_m)| \leq t(19, A_m)$ and hence, $8 \leq t(19, A_m) \leq 7$, which is impossible.
Case 2. $8 \leq n \leq 16$. We know that $r_{n-1}(3) \in \pi(A_m)$. Thus according to the numbers $r_7(3)$, $r_9(3)$, $r_{11}(3)$, $r_{13}(3)$ and $r_{15}(3)$ obtained in Lemma 3.1, we conclude that $53 \in \pi(A_m)$ and hence, since $\pi(A_m) \subseteq \pi(D_n(3))$, we have $53 \in \pi(D_n(3))$. But, $e(53, 3) = 52$. This is a contradiction, considering the prime divisors of $|D_n(3)|$, where $n \leq 16$.□

Lemma 3.3. *S cannot be isomorphic to a finite simple group of Lie type in characteristic different from* 3.

Proof. Assume that S is isomorphic to a finite simple group of Lie type in characteristic s, where $s \neq 3$. We get a contradiction considering three parts A, B and C.
Part A. $n \geq 20$. In this part, since $t(S) \geq 14$, by Table 9 in [19], we conclude that S cannot be an exceptional group of Lie type. Thus S is one of the classical groups $A^{\varepsilon}_{m-1}(q)$, $D^{\varepsilon}_m(q)$, $C_m(q)$ or $B_m(q)$, where $q = s^{\alpha}$. We get a contradiction case by case:

Case 1. $S \cong A_{m-1}(q)$. Set $t = e(s,3)$. If t is an odd number except 1,3, set

$$\rho = \{s, r_{2(n-1)}, r_{2(n-2)}, r_{2(n-3)}, r_{2(n-4)}\}.$$

Since $GK(G) = GK(D_n(3))$, by Proposition 2.4 in [19], we can see that ρ is a coclique of $GK(G)$, containing s and hence, by Lemmas 2.1(2) and 2.2(1,2), we conclude that $t(s,S) \geq |\rho| - 1$. On the other hand, since $t(S) \geq 14$, by Table 8 in [19], we can assume that $m \geq 27$ and hence, Table 4 in [19] implies that $t(s,S) = 3$. Thus $3 \geq |\rho| - 1 = 4$, which is impossible. Also, if t is an even number except 2,6, where $\frac{t}{2}$ is odd, then it is enough to replace ρ with the coclique $\{s, r_{n-1}, r_{2(n-2)}, r_{n-3}, r_{2(n-4)}\}$ of $GK(D_n(3))$ and get a contradiction. If t and $\frac{t}{2}$ are even numbers and $t \neq 4$, then by replacing ρ with the coclique $\{s, r_{n-1}, r_{2(n-1)}, r_{n-3}, r_{2(n-3)}\}$ of $GK(D_n(3))$ in the previous argument, we can get a contradiction. Therefore, we should only consider different cases for t, where $t \in \{1,2,3,4,6\}$. Since $t = e(s,3)$, by Lemma 1.4 in [19], we can see that $s \in \{2,5,7,13\}$. Since $m \geq 27$, according to $|A_{m-1}(q)|$, we have $r_7(q) \in \pi(S)$ and if $q \neq 2$, then $r_1(q) \in \pi(S)$. For considering the remaining cases, first we find an upper bound for $t(r_1(q), S)$ and $t(r_7(q), S)$. If $r_1(q) \in \pi(S)$, then Proposition 4.1 in [19] implies that $t(r_1(q), S) \leq 3$. We claim that $t(r_7(q), S) = 7$:
By Propositions 3.1(1) and 4.1 in [19], we can see that

$$(2, r_7(q)), (r_1(q), r_7(q)), (s, r_7(q)) \in GK(S).$$

Thus if $x \in \rho(r_7(q), S) \setminus \{r_7(q)\}$, then $x \notin \{2, s, r_1\}$ and if $e(x, s^{\alpha}) = l$, then by Proposition 2.1 in [19] we conclude that $l + 7 > m$ and $7 \nmid l$. Also, according to $|S|$, we have $l \leq m$ and hence, $l \in \{m-6, m-5, \cdots, m\}$ and $7 \nmid l$. Since $m-6, m-5, \cdots, m$ are seven consecutive numbers, so 7 divides exactly one of them and we have exactly six choices for l and hence, $t(r_7(q), S) = 7$. If $s = 2$, then since $r_7(2) = 127$, we have $127 \in \{r_1(2^{\alpha}), r_7(2^{\alpha})\} \subseteq \pi(S)$ and by the above statements we conclude that $t(127, S) \leq 7$. On the other hand, since $\pi(S) \subseteq \pi(D_n(3))$, we have $127 \in \pi(D_n(3))$. Also, we know that $e(127,3) = 126$ and hence, according to $|D_n(3)|$, we conclude that $n \geq 64$. Moreover, since n is an even number, it follows by Proposition 2.4 in [19] that the set $\tau \bigcup \{127\}$ is a coclique of $GK(D_n(3))$, where $\tau = \{r_i(3) \mid n-15 \leq i \leq n-1,\ i \equiv 1 \pmod 2\} \bigcup \{r_{2i}(3) \mid n-16 \leq i \leq n-2,\ i \equiv 0 \pmod 2\}$. Since $e(127,3) = 126 = 2 \times 63$, according to the choice of τ, we have $127 \notin \tau$ and hence, $t(127, D_n(3)) \geq 17$. Also, since $S \leq G/K$, it follows by Lemma 2.2(1,2) that $|\rho(127, D_n(3)) \bigcap \pi(S)| \leq t(127, S)$. By Lemma 2.1(2) for $\rho(127, D_n(3))$, we have $7 \geq t(127, S) \geq t(127, D_n(3)) - 1 \geq 16$, which is impossible. It easy to check that $r_7(5) = 19531$ and $r_7(13) = 5229043$ and also, $e(19531, 3) = 6510$ and $e(5229043, 3) = 249002$. Thus Proposition 2.4 in [19] implies that the set $\tau \bigcup \{r_7(s)\}$ is a coclique of $GK(D_n(3))$ and hence, $t(r_7(s), D_n(3)) \geq 17$, where $s \in \{5, 13\}$ and as in the previous argument we can get a contradiction.

If $s = 7$, then $r_7(7) = 4733$ and as in the case $s = 2$, we conclude that $t(4733, S) \leq 7$ and $4733 \in \pi(D_n(3))$. Also, since $e(4733, 3) = 676$, Proposition 2.4 in [19] implies that the set $\tau' \bigcup\{4733\}$, where $\tau' = \{r_i(3), r_{2i}(3) \mid n-15 \leq i \leq n-1,\ i \equiv 1 \pmod 2\}$ is a coclique of $GK(D_n(3))$. Moreover, since $e(4733, 3) = 676 = 2 \times 338$, according to the choice of τ', it is obvious that $4733 \notin \tau'$ and hence, $t(4733, D_n(3)) \geq 17$. Now as in the previous arguments, we can get a contradiction.

Case 2. $S \cong {}^2A_{m-1}(q)$. Since $t(S) \geq 14$, by Table 8 in [19] we can see that $m \geq 27$ and hence, $t(s, S) = 3$, by Table 4 in [19]. Thus as in Case 1, it is enough to consider $s \in \{2, 5, 7, 13\}$. Since $m \geq 27$, according to $|{}^2A_{m-1}(q)|$, we have $r_7(q) \in \pi(S)$ and if $q \neq 2$, then $r_1(q) \in \pi(S)$. We want to find an upper bound for $t(r_7(q), S)$. Since $m \geq 27$, by Propositions 3.1(2) and 4.2 in [19], we have

$$(2, r_7(q)), (r_2(q), r_7(q)), (s, r_7(q)) \in GK(S).$$

Thus if $x \in \rho(r_7(q), S) \setminus \{r_7(q)\}$, then $x \notin \{2, s, r_2(q)\}$ and if $e(x, s^\alpha) = l$, then by Proposition 2.2 in [19], we have $\nu(l) + 14 > m$ and $14 \nmid \nu(l)$. Furthermore, by $|{}^2A_{m-1}(q)|$, we can see that $\nu(l) \leq m$. Thus $\nu(l) \in \{m-13, m-12, \cdots, m\}$ and $14 \nmid \nu(l)$. Since $m-13, m-12, \cdots, m$ are fourteen consecutive numbers, so 14 divides exactly one of them and hence, we have thirteen choices for l. Also, $r_7(q) \neq r_i(q)$, where $m - 13 \leq i \leq m$, because $m \geq 27$. Therefore, $t(r_7(q), S) = 14$. If $r_1(q) \in \pi(S)$, by the same procedure, we can show that $t(r_1(q), S) \leq 2$. Hence, since $r_7(s) \in \{r_1(s^\alpha), r_7(s^\alpha)\}$, we have $t(r_7(s), S) \leq 14$. Now we can apply all the statements in the Case 1 to get a contradiction.

Case 3. S is isomorphic to one of the groups $B_m(q), C_m(q)$ or $D_m(q)$. Since $t(S) \geq 14$, by Table 8 in [19], we can see that $m \geq 17$ and hence, $t(s, S) \leq 3$, by Table 4 in [19]. Thus as in Case 1, it is enough to consider $s \in \{2, 5, 7, 13\}$. If $S \cong B_m(q)$ or $C_m(q)$, then since $m \geq 17$, according to $|S|$, we can see that $r_7(q) \in \pi(S)$ and if $q \neq 2$, then $r_1(q) \in \pi(S)$. By Propositions 3.1(3,4) and 4.3 in [19], we have $(2, r_7(q)), (s, r_7(q)) \in GK(S)$. Thus if $x \in \rho(r_7(q), S) \setminus \{r_7(q)\}$, then $x \notin \{2, s\}$ and if $e(x, q) = l$, then by Proposition 2.3 in [19], we have $\eta(l) + 7 > m$. Furthermore, according to the order of $B_m(q)$ and $C_m(q)$, we can see that $\eta(l) \leq m$. Thus $\eta(l) \in \{m-6, m-5, \cdots, m\}$ and by the definition of $\eta(l)$, there are at most eleven choices for l. Therefore, $t(r_7(q), S) \leq 12$. Also, if $r_1(q) \in \pi(S)$, then by the same argument, we can show that $t(r_1(q), S) \leq 3$. If $S \cong D_m(q)$, then by Propositions 2.4, 3.1(5), 4.4 in [19] and the previous argument, we conclude that $t(r_7(q), S) \leq 13$ and if $r_1(q) \in \pi(S)$, then $t(r_1(q), S) \leq 4$. Now we can use all the statements in Case 1, part A to get a contradiction.

Case 4. $S \cong {}^2D_m(q)$. Since $t(S) \geq 14$, by Table 8 in [19] we can see that $m \geq 18$ and hence, $t(s, S) \leq 4$, according to Table 4 in [19]. By similar argument in Case 1, if $t = e(s, 3)$ and t is an odd number except 1,3, it is enough to replace ρ with the coclique $\rho \bigcup\{r_{2(n-5)}\}$ of $GK(D_n(3))$, and if t is an even number except 2,6 where $\frac{t}{2}$ is odd, we replace ρ with the coclique

$$\{s, r_{n-1}, r_{2(n-2)}, r_{n-3}, r_{2(n-4)}, r_{n-5}\}$$

of $GK(D_n(3))$ and if $t \notin \{4,8\}$ and t and $\frac{t}{2}$ are even numbers, we replace ρ with the coclique $\{s, r_{n-1}, r_{2(n-1)}, r_{n-3}, r_{2(n-3)}, r_{n-5}\}$ of $GK(D_n(3))$. Thus in this case, $s \in \{2,5,7,13,41\}$. By the same procedure in Case 3 for $S \cong D_m(q)$, we can prove that $t(r_7(q), S) \leq 12$ and if $r_1(q) \in \pi(S)$, then $t(r_1(q), S) \leq 4$. If $s \in \{2,3,7,13\}$, then using all the statements in Case 1, part A leads us to a contradiction. Since $r_7(41) = 113229229$ and $e(113229229, 3) = 56614614$, we use the coclique $\tau \bigcup \{113229229\}$ of $GK(D_n(3))$. Hence, as in Case 1, part A, we can get a contradiction.

Part B. $10 \leq n \leq 18$. In this part, since $t(S) \geq t(D_n(3)) - 1$ and

$$(n, t(D_n(3))) \in \{(10,7), (12,9), (14,10), (16,12), (18,13)\},$$

Tables 8 and 9 in [19], imply that in addition to classical groups of Lie type, S can be isomorphic to exceptional groups of Lie type $E_7(q)$ and $E_8(q)$. Moreover, by Tables 4 and 5 in [19], we conclude that $t(s,S) \leq 5$. If $t = e(s,3)$, then since $\pi(S) \subseteq \pi(D_n(3))$ and according to $|D_n(3)|$, we conclude that $t \leq 34$. Thus by considering the cases "t is odd" and "t is even" separately and according to the coclique $\rho(D_n(3))$, it is easy to check that if $t \notin \{1,2,3,4,6,8\}$, then we can find some coclique, containing s with seven elements in $GK(D_n(3))$ and conclude that $t(s, D_n(3)) \geq 7$. We omit the details for convenience. Hence, as in Case 1, Part A, by Lemmas 2.1(2) and 2.2(1,2), we can get a contradiction. If $t \in \{1,2,3,4,6,8\}$, then since $t = e(s,3)$, by Lemma 1.4 in [19], we can see that $s \in \{2,5,7,13,41\}$. For $s \in \{5,7,13,41\}$, according to $t(S) \geq t(D_n(3)) - 1 = 6$, by checking $|S|$ in different cases, we conclude that $\{r_1(s^\alpha), r_5(s^\alpha)\} \subseteq \pi(S)$ and hence, $r_5(s) \in \pi(S) \subseteq \pi(D_n(3))$. Also, it is easy to check that $e(r_5(s), 3) > 34$. On the other hand, according to $|D_n(3)|$, if $x \in \pi(D_n(3)) \setminus \{3\}$, then $e(x,3) \leq 2(n-1)$. Thus we get a contradiction, because $n \leq 18$. If $s = 2$, then by checking $|S|$ in different cases, we can see that if $S \ncong {}^2A_{m-1}(2^\alpha)$ and $S \ncong {}^2D_m(2^\alpha)$, then $r_7(2^\alpha) \in \pi(S)$, and if $\alpha \neq 1$, then $r_1(2^\alpha) \in \pi(S)$. Since $r_7(2) = 127$, thus $127 \in \{r_1(2^\alpha), r_7(2^\alpha)\} \subseteq \pi(S) \subseteq \pi(D_n(3))$, but $e(127,3) = 126 > 34$ and as in the previous argument, we can get a contradiction. If $S \cong {}^2A_{m-1}(2^\alpha)$ or $S \cong {}^2D_m(2^\alpha)$, then since $t(S) \geq 6$, by Table 8 in [19], we can see that $m \geq 11$ or $m \geq 7$ respectively and hence, $\{r_2(2^\alpha), r_{14}(2^\alpha)\} \subseteq \pi(S)$. If α is an odd number, then $43 = r_{14}(2) \in \{r_2(2^\alpha), r_{14}(2^\alpha)\} \subseteq \pi(S) \subseteq \pi(D_n(3))$, but $e(43,3) = 42 > 34$, which is impossible. Otherwise, there exists a natural number β such that $q = 4^\beta$ and since $r_8(4) = 257$ and according to $|{}^2A_{m-1}(4^\beta)|$ and ${}^2D_m(2^\alpha)$, we have $257 \in \{r_1(q), r_2(q), r_4(q), r_8(q)\} \subseteq \pi(S) \subseteq \pi(D_n(3))$, but $e(257,3) = 256 > 34$, which is impossible.

Part C. $n = 8$. Since $\pi(D_8(3)) = \{2,3,5,7,11,13,41,61,73,547,1093\}$, and $5 = t(D_8(3)) - 1 \leq t(S) \leq |\pi(D_8(3))| = 11$, according to Tables 8 and 9 in

[19], and $|S|$ in different cases, we can see that, if $S \not\cong {}^2F_4(2^{2m+1})$ then we can conclude that $s^8 - 1 \mid |S|$ and hence,

$$\pi(S) \bigcap \{17, 113, 137, 313, 1201, 7321, 14281,\\ 6922921, 14199121, 17404710161\} \neq \emptyset$$

which is a contradiction. If $S \cong {}^2F_4(2^{2m+1})$, then since $r_7(3) = 1093 \in \pi(S)$ and $(2, 1093) \notin GK(S)$, by Proposition 3.3(3) in [19], we conclude that $1093 \mid 2^{12(2m+1)} - 1$ and hence, $364 = e(1093, 2) \mid 12(2m + 1)$, which implies that $91 \mid 2m + 1$. On the other hand, since $2^{2m+1} - 1 \mid |S|$, we can conclude that $r_{91}(2) = 911 \in \pi(S) \subseteq \pi(D_8(3))$, which is a contradiction. Thus S cannot be a group of Lie type in characteristic different from 3.□

Lemma 3.4. *Let $n \geq 8$ be an even number. If S is isomorphic to a finite simple group of Lie type in characteristic 3, then $S \cong D_n(3)$.*

Proof. If S is isomorphic to a finite simple group of Lie type over a field of order 3^α, then by Lemma 2.1(3-a), we have $r_{n-1}(3) \in \pi(S)$. Put $e_n = e(r_{n-1}(3), 3^\alpha)$. Since $r_{n-1}(3)$ divides $3^{\alpha e_n} - 1$, we have $n - 1$ divides αe_n. Suppose that $\alpha e_n > n - 1$. Then a prime r with $e(r, 3) = \alpha e_n$ divides the order of S and hence, r divides the order of $D_n(3)$ and by $|D_n(3)|$, we conclude that $\alpha e_n \leq 2(n - 1)$. Consequently, $\alpha e_n \in \{n - 1, 2(n - 1)\}$. Now for proving the lemma, we consider classical and exceptional groups of Lie type separately:

Part A. If S is a classical group of Lie type in characteristic 3, then S is isomorphic to one of the groups $A^\varepsilon_{m-1}(3^\alpha)$, $D^\varepsilon_m(3^\alpha)$, $C_m(3^\alpha)$ or $B_m(3^\alpha)$. Now with a case by case analysis, we prove that $S \cong D_n(3)$:

Case 1. $S \cong A_{m-1}(3^\alpha)$. Since $(r_{n-1}(3), 2) \notin GK(S)$ and $e_n = e(r_{n-1}(3), 3^\alpha)$, it follows by Proposition 4.1 in [19] that $e_n \in \{m, m - 1\}$. Moreover, since $\alpha e_n \in \{n - 1, 2(n - 1)\}$, thus we are supposed to consider the cases $\alpha m = i(n - 1), \alpha(m - 1) = i(n - 1)$, where $i \in \{1, 2\}$.

If $\alpha \geq i$, then we easily conclude that $n \geq m$. On the other hand, since $t(S) \geq t(G) - 1 = t(D_n(3)) - 1$ and $n \geq 8$, by Table 8 in [19], we have $[\frac{m+1}{2}] \geq [\frac{3(n-1)}{4}]$ and $m \geq 9$ which imply that $3(n - 1) < 2(m + 1)$ or $3(n - 1) - 2(m + 1) < 4$. Now, by a simple computation, we can easily get a contradiction. Thus $\alpha < i$. Since $i \in \{1, 2\}$ and $\alpha \in \mathbb{N}$, we conclude that $\alpha = 1$ and $i = 2$. So, we should only consider the cases $(m - 1) = 2(n - 1)$ and $m = 2(n - 1)$. If $m = 2(n - 1)$, then by considering $|S|$, we have $r_{m-1}(3) = r_{2(n-1)-1}(3) \in \pi(S) \subseteq \pi(D_n(3))$. But $2(n - 1) - 1$ is odd and we thus get $r_{2(2(n-1)-1)}(3) \in \pi(D_n(3))$, which is impossible according to $|D_n(3)|$.

Also, if $m-1=2(n-1)$, then $r_{m-2}(3) \in \pi(S) \subseteq \pi(D_n(3))$ and similar argument leads us to a contradiction.

Case 2. $S \cong {}^2A_{m-1}(3^\alpha)$. Since $(r_{n-1}(3), 2) \notin GK(S)$ and $e_n = e(r_{n-1}(3), 3^\alpha)$, by Proposition 4.2 in [19] and the definition of $\nu(m)$, we have $e_n \in \{m, 2m, 2(m-1), \frac{m}{2}\}$ and since $\alpha e_n \in \{n-1, 2(n-1)\}$ and n is even, we have four cases $\alpha m = n-1$, $\alpha(m-1) = n-1$, $\alpha m = 2(n-1)$ and $\alpha m = 4(n-1)$. Also, since $t(S) \geq t(G)-1$ and $n \geq 8$, by Table 8 in [19], we have $[\frac{m+1}{2}] \geq [\frac{3(n-1)}{4}]$ and hence, we can get a contradiction using the same argument in the Case 1 for all these cases except $\alpha m = 4(n-1)$. If $\alpha m = 4(n-1)$, then according to $|{}^2A_{m-1}(3^\alpha)|$, we conclude that $\{r_{4(n-1)}(3), r_{8(n-1)}(3)\} \bigcap \pi(S) \neq \emptyset$ and hence, $\{r_{4(n-1)}(3), r_{8(n-1)}(3)\} \bigcap \pi(D_n(3)) \neq \emptyset$, which is impossible according to the prime divisors of $|D_n(3)|$.

Case 3. $S \cong B_m(3^\alpha)$ or $S \cong C_m(3^\alpha)$. Since $(r_{n-1}(3), 2) \notin GK(S)$ and $e_n = e(r_{n-1}(3), 3^\alpha)$, by Proposition 4.3 in [19], we have $e_n \in \{m, 2m\}$. Moreover, we know that $\alpha e_n \in \{n-1, 2(n-1)\}$ and n is even and hence, there are the following two subcases:

Subcase a. $\alpha m = 2(n-1)$. In this case, since $n \geq 8$ is even and $t(S) \geq t(G)-1$, according to Table 8 in [19], we have $[\frac{3m+5}{4}] \geq [\frac{3(n-1)}{4}]$ and $m \geq 5$. Thus by the same method in Case 1, we can assume that $\alpha \in \{1, 2\}$. Also, since $r_{2m}(3^\alpha) \in \pi(S)$, we have $r_{4(n-1)}(3) \in \pi(S) \subseteq \pi(D_n(3))$, which is a contradiction, according to $|D_n(3)|$.

Subcase b. $\alpha m = n-1$. As in the previous case, according to Table 8 in [19], we have $[\frac{3m+5}{4}] \geq [\frac{3(n-1)}{4}]$ and $m \geq 5$. Thus by the same method in Case 1, we can assume that $\alpha = 1$ and hence, $S \cong B_{n-1}(3)$ or $S \cong C_{n-1}(3)$. Since $S \leq G/K$, we first claim that $K = 1$:

• Suppose that $K \neq 1$. We are going to reach a contradiction under this assumption. Let K_1 be a maximal element of the following set:

$$\Sigma = \{M \leq G \mid M \trianglelefteq G,\ M < K\}.$$

Replacing K by K/K_1 allows us to assume that K is an elementary abelian p-group and S acts on K faithfully and irreducibly. If $p = 3$, then by Theorem 1.3 in [8] each element of S centralizes some nontrivial element of K and hence, $(3, r_{2(n-1)}(3)) \in GK(G) = GK(D_n(3))$, which is impossible according to Proposition 3.1(5) in [19]. Thus we can assume that $p \neq 3$ and $l = e(p, 3)$. According to Lemma 3.1 in [7], S contains Frobenius subgroups of the forms $U : Z_{r_{n-1}(3)}$ and $T : Z_{r_{n-3}(3)}$, where U and T are non-trivial 3-groups. Since $(|T|, p) = (|U|, p) = 1$, we can apply Lemma 1 in [16] to conclude that $(p, r_{n-1}(3))$ and $(p, r_{n-3}(3)) \in GK(D_n(3))$. Thus Proposition 4.4 in [19] shows that $p \neq 2$ and hence, $l \geq 3$, which implies that $2(n-1) + 2\eta(l) > 2n - (1-(-1)^{n-1+l})$. Thus Proposition 2.4 in [19] guarantees that $\frac{n-1}{l}$ is an odd number and hence, l is odd, $l \mid n-1$ and also, $l \geq 3$ which forces l not to divide $n-3$. Thus since $(p, r_{n-3}(3)) \in GK(D_n(3))$, by Proposition 2.4 in [19], we conclude that $2(n-3)+2\eta(l) = 2(n-3)+2l \leq 2n-(1-(-1)^{n-1+l}) = 2n$. This shows that $l \leq 3$ and hence, $l = 3$. This allows us to conclude that $p = 13$ and $l = 3 \mid n-1$. Thus K is a 13-group. Also, Proposition 2.4 in [19] forces $(r_{n-2}(3), r_{n+2}(3)) \in GK(D_n(3)) = GK(G)$, so G contains an element g of order $r_{n-2}(3)r_{n+2}(3)$. Since K is a 13-group, $S \leq \bar{G} = G/K \leq \mathrm{Aut}(S)$ and $[\mathrm{Aut}(S) : S] = 2$, we can deduce that $gK \in S$ and $O(gK) = r_{n-2}(3)r_{n+2}(3)$. Thus $(r_{n-2}(3), r_{n+2}(3)) \in GK(S) = GK(B_{n-1}(3)) = GK(C_{n-1}(3))$, which is a contradiction with Proposition 2.4 in [19].

Therefore, $K = 1$ and $S \leq G \leq \mathrm{Aut}(S)$. If $G/S \neq 1$, then the proof of the main theorem in [7] implies that $(2, r_{n-1}) \in GK(G/S)$, so $(2, r_{n-1}) \in GK(G)$, which is a contradiction. Thus $G = S \cong B_{n-1}(3)$ or $C_{n-1}(3)$, which is a contradiction with the main theorem in [7].

Case 4. $S \cong {}^2D_m(3^\alpha)$. Since

$$(r_{n-1}(3), 2) \notin GK(S) \text{ and } e_n = e(r_{n-1}(3), 3^\alpha),$$

by Proposition 4.4 in [19], we have $e_n \in \{2m, 2(m-1)\}$. Moreover, we know that $\alpha e_n \in \{n-1, 2(n-1)\}$ and n is even and hence, there are the following two subcases:

Subcase a. $\alpha m = n-1$. Since $n \geq 8$ and $t(S) \geq t(G)-1 = t(D_n(3))-1$, by Table 8 in [19], we have $[\frac{3m+4}{4}] \geq [\frac{3(n-1)}{4}]$ and $m \geq 6$ and hence, by the same argument in Case 1, we can assume that $\alpha = 1$, which implies that $S \cong {}^2D_{n-1}(3)$. Since n is even, according to $|D_n(3)|$ and $|{}^2D_{n-1}(3)|$, we can see that $r_{n-1} \in \pi(D_n(3)) \setminus \pi({}^2D_{n-1}(3))$. Also, since $S \leq \bar{G} = G/K \leq \mathrm{Aut}(S)$ and $\mathrm{Out}({}^2D_{n-1}(3))$ is a 2-group, we conclude that $r_{n-1} \in \pi(K)$. Hence, using the cocliques $\rho = \{r_{n-1}, r_{2(n-5)}, r_{12}\}$ and $\tau = \{r_{n-1}, r_{2(n-3)}, r_8\}$ of $GK(D_n(3))$ in Lemma 2.1(2), implies that $\{r_8, r_{2(n-5)}\} \bigcap \pi(K) = \emptyset$. By Proposition 2.4 in [19], we can see that $(r_8, r_{2(n-5)}) \in GK(D_n(3))$. Therefore, by Lemma 2.2(3), we conclude that $\bar{G}$ has an element g of order $r_8.r_{2(n-5)}$. On the other hand, since $\bar{G}/S \leq \mathrm{Out}(S)$ and $\mathrm{Out}({}^2D_{n-1}(3))$ is a 2-group, we can assume that $g \in S$ and hence, $(r_8, r_{2(n-5)}) \in GK({}^2D_{n-1}(3))$, which is impossible according to Proposition 2.4 in [19].

Subcase b. $\alpha(m-1) = n-1$. Since n is even, we can assume that α is odd. If $\alpha \geq 3$, then by the relation $t(S) \geq t(G)-1$ and the same argument in the previous subcase, we can easily get a contradiction. Thus $\alpha = 1$ and $S \cong {}^2D_n(3)$. But $r_{2n}(3) \in \pi({}^2D_n(3)) \setminus \pi(D_n(3))$ and this is a contradiction.

Case 5. $S \cong D_m(3^\alpha)$. In this case, we are supposed to show that $m = n$ and $\alpha = 1$. As in the previous case, Proposition 4.4 in [19] imposes some restrictions on e_n and we have $e_n \in \{2(m-1), m-1, m\}$. Also, since n is even and $\alpha e_n \in \{n-1, 2(n-1)\}$, we have four cases $\alpha m = n-1$, $\alpha m = 2(n-1)$, $\alpha(m-1) = 2(n-1)$ and $\alpha(m-1) = n-1$. Moreover, since $n \geq 8$ and $t(S) \geq t(G)-1$, by Table 8 in [19], we have $t(D_m(3^\alpha)) \in \{\frac{3m+3}{4}, [\frac{3m+1}{4}]\}$ and

$m \geq 6$. Since the method for considering the case $t(S) = \frac{3m+3}{4}$ is similar to the case $t(S) = [\frac{3m+1}{4}]$, we just deal with the case $t(S) = [\frac{3m+1}{4}]$ in the following four subcases:

Subcase a. $\alpha m = n - 1$. Since $t(S) \geq t(G) - 1$ and n is even, similar argument in Case 1 concludes that $\alpha = 1$ and hence, $m = n - 1$ and $S \cong D_{n-1}(3)$. According to $|D_n(3)|$ and $|D_{n-1}(3)|$, we can see that $r_{2(n-1)}(3) \in \pi(D_n(3)) \backslash \pi(D_{n-1}(3))$. Also, since $S \leq \bar{G} = G/K \leq \mathrm{Aut}(S)$ and $\mathrm{Out}(D_{n-1}(3))$ is a 2-group, we conclude that $r_{2(n-1)} \in \pi(K)$. Hence, using the cocliques $\rho = \{r_{2(n-1)}, r_{n-3}, r_8\}$ and $\tau = \{r_{2(n-1)}, r_{n-1}, r_4\}$ of $GK(D_n(3))$ in Lemma 2.1(2), implies that $\{r_4, r_{n-3}\} \bigcap \pi(K) = \emptyset$. By Proposition 2.4 in [19], we can see that $(r_4, r_{n-3}) \in GK(D_n(3))$ and $(r_4, r_{n-3}) \notin GK(D_{n-1}(3))$, which is impossible according to Lemma 2.2(3).

Subcase b. $\alpha m = 2(n - 1)$. According to $t(S) \geq t(G) - 1$ and by a similar argument in Case 1, we can assume that $\alpha \in \{1, 2\}$ and hence, $S \cong D_{2(n-1)}(3)$ or $D_{n-1}(3^2)$. But since $q^{2(m-1)} - 1 \mid |D_m(q)|$, we conclude that $\{r_{2(2(n-1)-1)}(3), r_{4(n-2)}(3)\} \bigcap \pi(D_n(3)) \neq \emptyset$, which is impossible.

Subcase c. $\alpha(m-1) = 2(n-1)$. Since $t(S) \geq t(G) - 1$ and by a similar argument in Case 1, we can assume that $\alpha \in \{1, 2\}$ and hence, $S \cong D_{2(n-1)+1}(3)$ or $D_n(3^2)$. But according to the order S and $D_n(3)$, we can see that $r_{4(n-1)}(3) \in \pi(S) \setminus \pi(D_n(3))$, which is a contradiction.

Subcase d. $\alpha(m - 1) = n - 1$. According to $t(S) \geq t(G) - 1 = [\frac{3(n-1)}{4}]$, as in the previous subcase, we can see that $\alpha = 1$ and hence, $S \cong D_n(3)$.

Thus if S is a classical group of Lie type in characteristic 3, then $S \cong D_n(3)$.

Part B. If S is isomorphic to a finite simple exceptional group of Lie type in characteristic 3, then by Table 9 in [19], we can see that $t(S) \leq 12$. Since $t(S) \geq t(D_n(3)) - 1 = [\frac{3(n-1)}{4}]$, $n \geq 8$ and n is even, we conclude that $n \in \{8, 10, 12, 14, 16, 18\}$ and $t(S) \geq 5$. In the following, we will get a contradiction for the cases $n \geq 10$ and $n = 8$, separately:

Case 1. If $n \geq 10$, then according to Table 9 in [19], $S \cong E_7(3^\alpha)$ or $S \cong E_8(3^\alpha)$.

- $S \cong E_7(3^\alpha)$. Since $(r_{n-1}(3), 2) \notin GK(S)$ and $e_n = e(r_{n-1}(3), 3^\alpha)$, by Proposition 4.5(5) in [19], we have $e_n \in \{7, 9\}$ or $e_n \in \{14, 18\}$. Also, since $n \in \{10, 12, 14, 16, 18\}$ and $\alpha e_n \in \{n - 1, 2(n - 1)\}$, by checking all different cases, we conclude that $n = 10$ and $\alpha \in \{1, 2\}$. Thus $S \cong E_7(3)$ or $S \cong E_7(3^2)$, when $GK(G) = GK(D_{10}(3))$. If $S \cong E_7(3^2)$, then by checking $|E_7(3^2)|$, we can see that $r_{18}(3^2) = r_{36}(3) \in \pi(S) \subseteq \pi(D_{10}(3))$, which is impossible. If $S \cong E_7(3)$, then according to $|D_{10}(3)|$ and $|E_7(3)|$, we can see that $r_{16}(3) \in \pi(D_{10}(3)) \setminus \pi(E_7(3))$. Also, since $S \leq \bar{G} \leq \mathrm{Aut}(S)$ and $\mathrm{Out}(E_7(3))$ is a 2-group, we conclude that $r_{16}(3) \in \pi(K)$. Hence, using the cocliques $\rho = \{r_3, r_{16}, r_{18}\}$ and $\tau = \{r_7, r_{10}, r_{16}\}$ of $GK(D_{10}(3))$ in Lemma 2.1(2), implies that $\{r_3, r_7\} \bigcap \pi(K) = \emptyset$. Also, by Propositions 2.4 and 2.5(5) in [19], we can see that $(r_3, r_7) \in GK(D_{10}(3))$ and $(r_3, r_7) \notin GK(S)$, which is a contradiction according to Lemma 2.2(3).

• If $S \cong E_8(3^\alpha)$, then since $(r_{n-1}(3), 2) \notin GK(S)$ and $e_n = e(r_{n-1}(3), 3^\alpha)$, by Proposition 4.5(6) in [19], we have $e_n \in \{15, 20, 24, 30\}$ and by checking all different cases, we conclude that $n = 16$ and $\alpha \in \{1, 2\}$. Thus $S \cong E_8(3)$ or $S \cong E_8(3^2)$, when $GK(G) = GK(D_{16}(3))$. If $S \cong E_8(3^2)$, then by checking $|E_8(3^2)|$, we can see that $r_{18}(3^2) = r_{36}(3) \in \pi(S) \subseteq \pi(D_{16}(3))$, which is impossible. If $S \cong E_8(3)$, then according to $|D_{16}(3)|$ and $|E_8(3)|$, we can see that $r_{22}(3) \in \pi(D_{16}(3)) \setminus \pi(E_8(3))$. Also, since $S \leq \bar{G} \leq \mathrm{Aut}(S)$ and $|\mathrm{Out}(E_8(3))| = 1$, we conclude that $r_{22}(3) \in \pi(K)$. Hence, using the cocliques $\rho = \{r_{12}, r_{15}, r_{22}\}$ and $\tau = \{r_7, r_{22}, r_{28}\}$ of $GK(D_{16}(3))$ in Lemma 2.1(2), implies that $\{r_7, r_{12}\} \bigcap \pi(K) = \emptyset$. Also, by Propositions 2.4 and 2.5(6) in [19], we can see that $(r_7, r_{12}) \in GK(D_{16}(3))$ and $(r_7, r_{12}) \notin GK(S)$, which is a contradiction according to Lemma 2.2(3).

Case 2. If $n = 8$, then according to Table 9 in [19], S can be isomorphic to one of the simple group

$$E_8(3^\alpha),\ E_7(3^\alpha),\ F_4(3^\alpha),\ E_6(3^\alpha),\ {}^2E_6(3^\alpha) \text{ or } {}^2G_2(3^{2m+1}).$$

Since $(r_7(3), 2) \notin GK(S)$, $e_n = e(r_7(3), 3^\alpha)$ and $\alpha e_n \in \{7, 14\}$, by Proposition 4.5(2-6,8) in [19], we can conclude that $S \cong E_7(3^\alpha)$ and $\alpha \in \{1, 2\}$ or $S \cong {}^2G_2(3^{2m+1})$ and $r_7(3) \mid 3^{6(2m+1)} - 1$. If the first case occurs, then by $|E_7(q)|$, we can conclude that $r_{18}(3) \in \pi(S) \setminus \pi(D_8(3))$, which is a contradiction. Otherwise, since $r_7(3) \mid 3^{6(2m+1)} - 1$, there exists an integer k such that $2m + 1 = 7k$. On the other hand, since $3^{3(2m+1)} + 1 \mid |{}^2G_2(3^{2m+1})|$ and k is odd, we have $r_{42}(3) \in \pi(S) \subseteq \pi(D_8(3))$ and this is impossible, according to $|D_8(3)|$.$\square$

Now, Lemmas 3.1, 3.2, 3.3 and 3.4 show that $S \cong D_n(3)$.

3.2. **The group G is isomorphic to $D_n(3)$.** According to the previous section, we have proved the following statement:

If $n \geq 8$ is an even number and G is a finite group with the same prime graph as the simple group $D_n(3)$, then

$$D_n(3) \leq G/K \leq \mathrm{Aut}(D_n(3)),$$

where K is the maximal normal solvable subgroup of G.

So, in this section, we are supposed to complete the proof of characterizability of the group $D_n(3)$, by showing $K = 1$ and $G = D_n(3)$. Since $B_{n-1}(3) \hookrightarrow D_n(3)$, $\Omega^-_{2(n-1)}(3) \hookrightarrow D_n(3)$ and $n \geq 8$ is an even number, by Lemmas 3.1 in [7] and 3.5 in [6] and the same procedure which has been used in Lemma 3.4, part A, Case 3, subcase b, we can see that $K = 1$, we omit the details for the sake of convenience. Thus we just need to prove the following lemma:

Lemma 3.5. *Let $n \geq 8$ be an even number. If $D_n(3) \leq G \leq \mathrm{Aut}(D_n(3))$ and $GK(D_n(3)) = GK(G)$, then $G = D_n(3)$.*

Proof. Let $J_n = \begin{pmatrix} J_1 & \mathbf{0} \\ \mathbf{0} & J^{n-1} \end{pmatrix}$, where $J^{n-1} = \begin{pmatrix} \mathbf{0} & I_{n-1} \\ I_{n-1} & \mathbf{0} \end{pmatrix}$ and $J_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. We have

$$GO^+_{2n}(q) = \{X \in GL_{2n}(q) \mid X^t J_n X = J_n\}$$

and

$$GO^+_{2(n-1)}(q) = \{X \in GL_{2(n-1)}(q) \mid X^t J^{n-1} X = J^{n-1}\}.$$

Fix $N = D_n(3)$. Let $\mathrm{bd}(C_1, C_2, ..., C_m)$ denote a block-diagonal matrix with square blocks $C_1, C_2, ..., C_m$. By Propositions 2.5.13 and 2.7.3 in [14], $\mathrm{Out}(N) \cong D_8$ and $\mathrm{Aut}(N) = \langle N, \bar{\alpha}, \bar{\beta}, \bar{\gamma} \rangle$, where

$$\alpha = \mathrm{bd}(-J_1, I_{2(n-1)}),\ \beta = \mathrm{bd}(J_1, I_{2(n-1)}),\ \gamma = \mathrm{bd}(-1, 1, -I_{n-1}, I_{n-1})$$

and $\bar{x}$ is the image of an element x of $GL_{2n}(3)$ in $GL_{2n}(3)/\{\pm I_{2n}\}$. We claim that $G = N$. If not, then there exists an element $\bar{\delta} \in G - N$, so without loss of generality we can assume that $\bar{\delta} \in \langle \bar{\alpha}, \bar{\beta}, \bar{\gamma} \rangle$. Thus $\bar{\delta} = \bar{\alpha}^i \bar{\beta}^j \bar{\gamma}^k$, so an easy computation shows that

$$\{\mathrm{bd}(I_2, w, (w^{-1})^t) \mid w \in GL_{n-1}(3)\} \leq C_{SO^+_{2n}(3)}(\delta)$$

and hence, $\mathrm{bd}(I_2, y, (y^{-1})^t) \in C_{\Omega^+_{2n}(3)}(\delta)$, where y is an element of order $r_{n-1}(3)$ in $SL_{n-1}(3)$. On the other hand, since $\bar{\delta}N \in D_8$, we have $2 \mid O(\bar{\delta})$, by Lemma 2.2(4). Thus $(2, r_{n-1}(3)) \in GK(G) = GK(D_n(3))$, which is a contradiction, as required. This shows that $G = D_n(3)$.□

Thus according to the subsections 3.1 and 3.2, the main theorem is obtained for the case $n \geq 8$, where n is even. Also, if $n = 6$, then Theorem 3.2 in [11] implies that $D_6(3)$ is characterizable by prime graph. Moreover, if $n = 4$, then the main theorem in [17] completes the proof of the main theorem.

We have the following remark, when n is odd:

Remark 3.6. *If $n \geq 5$ is an odd number, then by Tables 6 and 8 in [19], we have $t(2, D_n(3)) = 2$, $\rho(2, D_n(3)) = \{2, r_n(3)\}$ and $t(D_n(3)) \in \{[\frac{3n+1}{4}], \frac{3n+4}{4}\}$. Hence, if G is a finite group with $GK(G) = GK(D_n(3))$, then Lemma 2.1 implies that G has a unique nonabelian composition factor S, in which case $S \leq \bar{G} = G/K \leq \mathrm{Aut}(S)$, $t(S) \geq t(D_n(3)) - 1$, $t(2, S) \geq 2$ and $r_n(3) \in \pi(S)$. Now, by the same method which is used in subsection 3.1 and considering appropriate cocliques in $GK(D_n(3))$, we can conclude that $S \cong D_n(3)$. But it is worth to* *mention that the method which is used in subsection 3.2 cannot complete the characterization of $D_n(3)$.*

4. Corollaries

In this section, we will consider some significant corollaries of the main theorem. If G is a finite group, by $M(G)$ we denote the set of orders of maximal abelian subgroups of G and a group G is said to be characterizable by the set of orders of its maximal abelian subgroups, if G is uniquely determined by $M(G)$.

Lemma 4.1. [2, Lemma 2] *Let G and H be finite groups. If $M(G) = M(H)$, then $GK(G) = GK(H)$.*

Lemma 4.2. [1, Corpllary 2.6.] *Let P be a finite nonabelian simple group and G is a group such that $\Gamma_G \cong \Gamma_P$, then $GK(G) = GK(P)$ and $M(G) = M(P)$.*

Corollary 4.3. *If $n \geq 6$ is an even number, then*

(i) *the simple group $D_n(3)$ is characterizable by spectrum;*
(ii) *the simple group $D_n(3)$ is characterizable by its set of orders of maximal abelian subgroups;*
(iii) *the AAM's Conjecture is true for the groups under study.*

Proof. Let G be a finite group with $\omega(G) = \omega(D_n(3))$. Therefore, $GK(G) = GK(D_n(3))$ and hence, the main theorem completes the proof of (i). From Lemmas 4.1, 4.2 and the main theorem, we get (ii) and (iii).□ **Acknowledgments.** The authors wish to thank the referees for their invaluable comments. The second author likes to thank Shahrekord University for the financial support .

References

1. N. Ahanjideh and A. Iranmanesh, On the relation between the non-commuting graph and the prime graph, *International Journal of Group Theory*, **1** (1) (2012), 25–28.
2. G. Chen, A characterization of alternating groups by the set of orders of maximal abelian subgroups, *Siberian Math. J.*, **47** (3) (2006), 594–596.
3. J. Conway, R. Curtis, S. Norton, P. Parker and R. Wilson, *Atlas of finite groups*, Clarendon Press, Oxford, 1985.
4. M. R. Darafsheh, Groups with the same non-commuting graph, *Discrete Applied Math.*, **157** (2009), 833–837.
5. M. Foroudi Ghasemabadi and A. Iranmanesh, 2-quasirecognizability of the simple groups $B_n(p)$ and $C_n(p)$ by prime graph, *Bulletin of the Iranian Mathematical Society*, in press.
6. M. Foroudi Ghasemabadi, A. Iranmanesh and N. Ahanjideh, Characterizations of the simple group ${}^2D_n(3)$ by prime graph and spectrum, *Monatsh. Math.*, DOI: 10.1007/s00605-011-0336-y.
7. M. Foroudi Ghasemabadi, A. Iranmanesh and N. Ahanjideh, 2-recognizability of the simple groups $B_n(3)$ and $C_n(3)$ by prime graph, *Bulletin of the Iranian Mathematical Society*, in press.

8. R. M. Guralnick and P. H. Tiep, Finite simple unisingular groups of Lie type, *J. Group theory*, **6** (2003), 271-310.
9. H. Y. He and W. J. Shi, Recognition of some finite simple groups of type $D_n(q)$ by spectrum, *International J. Algebra and Computation*, **19** (5) (2009), 681–698.
10. A. Khosravi and B. Khosravi, Quasirecogniton of the simple group ${}^2G_2(q)$ by the prime graph, *Siberian Math. J.*, **48** (3) (2007), 570–577.
11. B. Khosravi, Z. Akhlaghi and M. Khatami, Quasirecognition by prime graph of the simple group $D_n(3)$, *Publ. Math. Debrecen*, **78** (2) (2011), 469–484.
12. B. Khosravi, B. Khosravi and B. Khosravi, A characterization of the finite simple group $L_{16}(2)$ by its prime graph, *Manuscripta Math.*, **126** (2008), 49–58.
13. E. I. Khukhro and V. D. Mazurov, *Unsolved problems in group theory: The Kourovka Notebook*, Sobolev Institute of Mathematics, Novosibirsk, 17th edition, 2010.
14. P. B. Kleidman and M. Liebeck, *The subgroup structure of finite classical groups*, Cambridge Univ. Press, Cambridge, 1990.
15. A. S. Kondrat'ev, Prime graph components of finite simple groups, *Mat. Sb.*, **180** (6) (1989), 787-797 (translated in *Math. USSR-Sb.*, **67** (1990), 235–247).
16. V. D. Mazurov, Characterization of finite groups by the sets of element orders, *Algebra and Logic*, **36** (1) (1997), 23–32.
17. Z. Momen and B. Khosravi, On r-recognition by prime graph of $B_p(3)$ where p is an odd prime, *Monatsh. Math.*, (2010), DOI 10.1007/s00605-010-0276-y.
18. A. V. Vasil'ev and I. B. Gorshkov, On recognition of finite simple groups with connected prime graph. *Siberian Math. J.*, **50** (2) (2009), 233–238.
19. A.V. Vasil'ev and E.P. Vdovin, An adjacency criterion for the prime graph of a finite simple group, (2010), http://arxiv.org/abs/math/0506294.
20. J. S. Williams, Prime graph components of finite groups, *Journal of Algebra*, **69** (1981), 487–513.
21. A. V. Zavarnitsine, Recognition of finite groups by the prime graph, *Algebra and Logic*, **45** (4) (2006), 220–231.
22. A. V. Zavarnitsine, Uniqueness of the prime graph of $L_{16}(2)$, *Sib. Èlektron. Mat. Izv.*, **7** (2010), 119–121.