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Groups in polygroups

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Abstract. In this paper, we consider the fundamental relation β^* as the smallest equivalence relation on a polygroup P such that the quotient P/β^*, the set of all equivalence classes, is a group. The quotient P/β^* is called the fundamental group. We give some interesting results about the fundamental groups.

1 Introduction

This paper deals with certain algebraic system called a polygroup. Since Marty [14] had introduced the concept of hypergroups, several authors have studied about them. Application of hypergroups have mainly appeared in special subclasses. For example, polygroups which are certain subclasses of hypergroups are studied in [12] by Ioulidis and are used to study color algebra [2],[4]. Quasi-canonical hypergroups (called "polygroups" by Comer) were introduced in [1], as a generalization of canonical hypergroups, introduced in [15]. Some algebraic and combinatorial properties were developed in [3] by Comer. Davvaz and Poursalavati in [8] introduced matrix representations of polygroups over hyperrings and they introduced the notion of a polygroup hyperring generalizing the notion of a group ring. Davvaz in [10], using the concept of generalized permutation defined permutation polygroup and some concepts related to it. The reader will find in [2-6, 8-12] a deep disscussion of polygroup theory.

2 Basic definitions

A hypergroup is a non-empty set H equipped with an associative hyperoperation $\cdot$ from $H \times H$ into the family of non-empty subsets of H which satisfies

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the property $a \cdot H = H \cdot a = H$ for all a in H. If $A, B \subseteq H$ then $A \cdot B$ is given by $A \cdot B = \cup\{a \cdot b \mid a \in A,\ b \in B\}$. $x \cdot A$ is used for $\{x\} \cdot A$ and $A \cdot x$ for $A \cdot \{x\}$. A polygroup is a special case of a hypergroup. We recall the following definition from [2].

A polygroup is a system $\mathbf{P} = < P, ., e, ^{-1} >$, where $e \in P$, $\ ^{-1}$ is a unitary operation on P, $\cdot$ maps $P \times P$ into the non-empty subsets of P, and the following axioms hold for all x, y, z in P: i) $(x \cdot y) \cdot z = x \cdot (y \cdot z)$; ii) $e \cdot x = x \cdot e = x$; iii) $x \in y \cdot z$ implies $y \in x \cdot z^{-1}$ and $z \in y^{-1} \cdot x$. The following elementary facts about polygroups follow easily from the axioms: $e \in x \cdot x^{-1} \cap x^{-1} \cdot x$, $e^{-1} = e$, $(x^{-1})^{-1} = x$, and $(x \cdot y)^{-1} = y^{-1} \cdot x^{-1}$ where $A^{-1} = \{a^{-1} \mid a \in A\}$. We write ab instead of $a \cdot b$. Examples of polygroups are given in [2-6, 8, 16, 17] to indicate how these systems occur naturally in various context. There are examples to the following subjects: double coset algebra, Prenowitz algebra, regular graph, conjugate class polygroups, character polygroups, algebraic logic, relation algebra and etc.. Also an extension of polygroups by polygroups has been introduced in [4].

If N is a normal subpolygroup of P (i.e., $a^{-1}Na \subseteq N$ for all $a \in P$), then we define the relation $x \equiv y(modN)$ if and only if $xy^{-1} \cap N \neq \emptyset$. This relation is denoted by $xN_P y$. Clearly, N_P is an equivalence relation. Let $N_P(x)$ be the equivalence class of the element $x \in P$. On P/N, the set of all equivalence classes, we consider the hyperoperation $*$ defined as follows: $N_P(x) \odot N_P(y) = \{N_P(z) \mid z \in N_P(x)N_P(y)\}$. Then $< P/N, *, N_P(e), ^{-I} >$ is a polygroup, where $N_P(a)^{-I} = N_P(a^{-1})$.

Let $< P_1, \cdot, e_1, ^{-1} >$ and $< P_2, *, e_2, ^{-I} >$ be polygroups. A mapping φ from P_1 into P_2 is said to be a strong homomorphism if for all $a, b \in P_1$, $\varphi(ab) = \varphi(a) * \varphi(b)$ and $\varphi(e_1) = e_2$.

3 Fundamental groups

Let P be a polygroup. We define the relation β^* as the smallest equivalence relation on P such that the quotient P/β^*, the set of all equivalence classes, is a group. In this case β^* is called the fundamental equivalence relation on P and P/β^* is called the fundamental group. The equivalence relation β^* was introduced by Koskas [13] and studied mainly by Corsini [7] concerning hypergroups and Vougiouklis [18] concerning H_v-groups. The product $\odot$ in P/β^* is defined as follows: $\beta^*(x) \otimes \beta^*(y) = \beta^*(z)$ for all $z \in \beta^*(x)\beta^*(y)$. Let $\mathbf{U}_P$ be the set of all finite products of elements of P. We define the relation β as follows: $x\beta y$ if and only if $\{x, y\} \subseteq u$ for some $u \in \mathbf{U}_P$. Since polygroups are certain subclasses of hypergroups, we have $\beta^* = \beta$ (Theorem 81, [7]). The kernel of the canonical map $\varphi : P \longrightarrow P/\beta^*$ is called the core of P and is denoted by ω_P. Here we also denote by ω_P the unit of P/β^*. It is easy to prove that the following statements: $\omega_P = \beta^*(e)$ and $\beta^*(x)^{-1} = \beta^*(x^{-1})$ for all $x \in P$. Now, we give the main results about fundamental groups.

Theorem 3. 1. *Let β_1^*, β_2^* and β^* be fundamental equivalence relations on polygroups P_1, P_2 and $P_1 \times P_2$ respectively, then*

$$(P_1 \times P_2)/\beta^* \cong P_1/\beta_1^* \times P_2/\beta_2^*.$$

Proof. First we define the relation $\widetilde{\beta}$ on $P_1 \times P_2$ as follows:

$$(x_1, y_1)\widetilde{\beta}(x_2, y_2) \Longleftrightarrow x_1\beta_1^* x_2, y_1\beta_2^* y_2.$$

$\widetilde{\beta}$ is an equivalence relation. We define $\odot$ on $(P_1 \times P_2)/\widetilde{\beta}$ as follows:

$$\widetilde{\beta}(x_1, y_1) \odot \widetilde{\beta}(x_2, y_2) = \widetilde{\beta}(a, b),$$

for all $a \in \beta_1^*(x_1) \cdot \beta_1^*(x_2)$ and $b \in \beta_2^*(y_1) \cdot \beta_2^*(y_2)$.

Since P_1 and P_2 are associative, we see that is $\odot$ associative and consequently $(P_1 \times P_2)/\widetilde{\beta}$ is a group.

Now let θ be an equivalence relation on $P_1 \times P_2$ such that $(P_1 \times P_2)/\theta$ is a group. Let $\theta(x_1, y_1)$ be the class of (x_1, y_1). Then $\theta(x_1, y_1) \odot \theta(x_2, y_2)$ is singleton, i.e., $\theta(x_1, y_1) \odot \theta(x_2, y_2) = \theta(a, b), \forall (a, b) \in \theta(x_1, y_1).\theta(x_2, y_2)$. But also for every $(x_1, y_1), (x_2, y_2) \in P_1 \times P_2$ and $A \subseteq \theta(x_1, y_1), B \subseteq \theta(x_2, y_2)$ we have $\theta(x_1, y_1) \odot \theta(x_2, y_2) = \theta((x_1, y_1) \cdot \theta(x_2, y_2)) = \theta(AB)$, so this relation is valid for all finite products which means that the equality $\theta(x, y) = \theta(u, v)$ for every $(u, v) \in \mathbf{U}_{P_1 \times P_2}$ and $x \in u, y \in v$ holds.

Now, if $(x, y) \in \widetilde{\beta}(a, b)$, then $x\beta_1^* a$ and $y\beta_2^* b$. We have
$x\beta_1^* a$ if and only if $\exists x_1, \cdots, x_{m+1}$ with $x_1 = x,\ x_{m+1} = a$ and $u_1, \cdots, u_m \in \mathbf{U}_{P_1}$ such that

$$\{x_i, x_{i+1}\} \subseteq u_i, \quad i = 1, \cdots, m, \text{ and}$$

$y\beta_2^* b$ if and only if $\exists y_1, \cdots, y_{m+1}$ with $y_1 = y,\ y_{n+1} = b$ and $v_1, \cdots, v_n \in \mathbf{U}_{P_2}$ such that

$$\{y_j, y_{j+1}\} \subseteq v_j, \quad j = 1, \cdots, n.$$

Therefore

$$(x_i, y_j) \in (u_i, v_j), (x_{i+1}, y_{j+1}) \in (u_i, v_j), \quad i = 1, \cdots, m,\ j = 1, \cdots, n.$$

And so

$$\theta(x_i, y_j) = \theta(u_i, v_j), \theta(x_{i+1}, y_{j+1}) = \theta(u_i, v_j), \quad i = 1, \cdots, m,\ j = 1, \cdots, n$$

which implies that $\theta(x_i, y_j) = \theta(x_{i+1}, y_{j+1}), \quad i = 1, \cdots, m, \quad j = 1, \cdots, n$. Therefore $\theta(x, y) = \theta(a, b)$ or $(x, y) \in \theta(a, b)$. So we get

$$(x, y)\widetilde{\beta}(a, b) \Longrightarrow (x, y) \in \theta(a, b).$$

Thus, the relation $\widetilde{\beta}$ is the smallest equivalence relation on $P_1 \times P_2$ such that $(P_1 \times P_2)/\widetilde{\beta}$ is a group, i.e., $\widetilde{\beta} = \beta^*$. Now, we consider the map

$$f : P_1/\beta_1^* \times P_2/\beta_2^* \longrightarrow (P_1 \times P_2)/\beta^*$$

with $f(\beta_1^*(x), \beta_2^*(y)) = \beta^*(x, y)$. It is easy to see that f is an isomorphism. □

Corollary 3. 2. *If N_1, N_2 are normal subpolygroups of P_1, P_2 respectively, and β_1^*, β_2^* and β^* fundamental equivalence relations on P_1/N_1, P_2/N_2 and $(P_1 \times P_2)/(N_1 \times N_2)$ respectively, then*

$$((P_1 \times P_2)/(N_1 \times N_2))/\beta^* \cong (P_1/N_1)/\beta_1^* \times (P_2/N_2)/\beta_2^*.$$

Lemma 3. 3. *Let f be a strong homomorphism from P_1 into P_2 and let β_1^*, β_2^* be fundamental equivalence relations on P_1, P_2 respectively, we define $\overline{kerf} = \{\beta_1^*(x) \mid x \in P_1,\ \beta_2^*(f(x)) = \omega_{P_2}\}$. Then $\overline{kerf}$ is a normal subgroup of the fundamental group P_1/β_1^*.*

Proof. Assume that $\beta_1^*(x), \beta_1^*(y) \in \overline{kerf}$ then for every $z \in xy^{-1}$ we have $\beta_1^*(z) = \beta_1^*(x) \otimes \beta_1^*(y^{-1})$. On the other hand, we have

$$\beta_2^*(f(z)) = \beta_2^*(f(x)f(y^{-1})) = \beta_2^*(f(x)) \otimes \beta_2^*(f(y^{-1})) = \omega_{P_2} \otimes \omega_{P_2} = \omega_{P_2}.$$

Therefore $\beta_1^*(z) \in \overline{kerf}$. Now, let $\beta_1^*(a) \in P_1/\beta_1^*$ and $\beta_1^*(x) \in \overline{kerf}$ then for every $z \in axa^{-1}$ we have $\beta_1^*(z) = \beta_1^*(a) \otimes \beta_1^*(x) \otimes \beta_1^*(a^{-1})$. On the other hand, we have

$$\begin{aligned}\beta_2^*(f(z)) &= \beta_2^*(f(a)f(x)f(a^{-1})) \\ &= \beta_2^*(f(a)) \otimes \beta_2^*(f(x)) \otimes \beta_2^*(f(a^{-1})) \\ &= \beta_2^*(f(a)) \otimes \omega_{P_2} \otimes \beta_2^*(f(a^{-1})) \\ &= \beta_2^*(f(aa^{-1})) = \beta_2^*(f(e_1)) = \beta_2^*(e_2) = \omega_{P_2}.\end{aligned}$$

Hence we get $\beta_1^*(z) \in \overline{kerf}$. This complete the proof. □

Theorem 3. 4. *Let P be a polygroup, M, N two normal subpolygroups of P with $N \subseteq M$ and $\phi : P/N \longrightarrow P/M$ canonical map. Suppose β_M^*, β_N^* be the fundamental equivalence relations on P/M, P/N respectively then $((P/N)/\beta_N^*)/\overline{ker\phi} \cong (P/M)/\beta_M^*$.*

Proof. We define the map $\psi : (P/N)/\beta^*_N \longrightarrow (P/M)/\beta^*_M$ by $\psi : \beta^*_N(Nx) \longmapsto \beta^*_M(Mx)$ (for all $x \in P$). We must check that ψ is well-defined, that is, that if $x, y \in P$ and $\beta^*_N(Nx) = \beta^*_N(Ny)$ then $\beta^*_M(Mx) = \beta^*_M(My)$. Now $\beta^*_N(Nx) = \beta^*_N(Ny)$ if and only if $\{Nx, Ny\} \subseteq u$ for some $u \in \mathbf{U}_{P/N}$. We have $u = Nx_1 \odot Nx_2 \odot \ldots \odot Nx_n = \{Nz \mid z \in \prod_{i=1}^{n} x_i\}$. Therefore for some $z_1 \in \prod_{i=1}^{n} x_i$, $z_2 \in \prod_{i=1}^{n} x_i$ we have $Nx = Nz_1$ and $Ny = Nz_2$. So there exist $a \in xz_1^{-1} \cap N$ and $b \in yz_2^{-1} \cap N$, then $x \in az_1$ and $y \in bz_2$. Hence $Mx \in Ma \odot Mz_1$ and $My \in Mb \odot Mz_2$. Since $a, b \in N \subseteq M$, then $Ma = M$, $Mb = M$. Since $M \odot Mz_1 = Mz_1$ and $M \odot Mz_2 = Mz_2$, we have $Mx = Mz_1$ and $My = Mz_2$. From $\{Mz_1, Mz_2\} \subseteq \{Mz \mid z \in \prod_{i=1}^{n} x_i\}$, we get $\{Mx, My\} \subseteq \{Mz \mid z \in \prod_{i=1}^{n} x_i\} = Mx_1 \odot Mx_2 \odot \ldots \odot Mx_n$. Therefore $\beta^*_M(Mx) = \beta^*_M(My)$. This follows that ψ is well-defined. Moreover ψ is a strong homomorphism, for if $x, y \in P_1$ then

$$\begin{aligned}\psi(\beta^*_N(Nx) \otimes \beta^*_N(Ny)) &= \psi(\beta^*_N(Nxy)) = \beta^*_M(Mxy)\\ &= \beta^*_M(Mx) \otimes \beta^*_M(My)\\ &= \psi(\beta^*_N(Nx)) \otimes \psi(\beta^*_M(My)),\end{aligned}$$

and $\psi(\omega_{P/N}) = \psi(\beta^*_N(N)) = \beta^*_M(M) = \omega_{P/M}$. Clearly, ψ is onto. Now, we show that $ker\psi = \overline{ker\phi}$. We have

$$\begin{aligned}ker\psi &= \{\beta^*_N(Nx) \mid \psi(\beta^*_N(Nx)) = \omega_{P/N}\}\\ &= \{\beta^*_N(Nx) \mid \beta^*_M(Mx) = \omega_{P/N}\}\\ &= \{\beta^*_N(Nx) \mid \beta^*_M(\phi(Nx)) = \omega_{P/N}\}\\ &= \overline{ker\phi}.\end{aligned}$$

Theorem 3.5. *Let P_1 and P_2 be two polygroups and β^* be the fundamental equivalence relation on P_2. According to [19], we consider the group $AutP_1$ and the fundamental group P_2/β^*, let*

$$\widehat{\ }: P_2/\beta^* \longrightarrow AutP_1$$

$$\beta^*(b) = \widehat{\beta^*(b)} = \widehat{b}$$

be a homomorphism of groups. Then on $P_1 \times P_2$ we define a hyperproduct as follows: $(a_1, b_1) \circ (a_2, b_2) = \{(x, y) \mid x \in a_1 \cdot \widehat{b_1}(a_2),\ y \in b_1 \cdot b_2\}$ and we call this the semi-direct hyperproduct of polygroups P_1 and P_2. Then $P_1 \times P_2$ equipped with the semi-direct hyperproduct is a polygroup.

Proof. Similar to the theorem 2.4.1 of [10], associativity is valid. Since $a = a \cdot \widehat{b}(e_1), a = e_1 \cdot \widehat{e_2}(a)$ and $b = b * e_2 = e_2 * b$, we have $(a,b)o(e_1,e_2) = (a,b) = (e_1,e_2)o(a,b)$, i.e., (e_1,e_2) is the identity element in $P_1 \times P_2$, and we can check that $(\widehat{b^{-1}}(a^{-1}), b^{-1})$ is the inverse of (a,b) in $P_1 \times P_2$.

Now, we show that
$(z_1,z_2) \in (x_1,x_2)o(y_1,y_2) \Rightarrow (x_1,x_2) \in (z_1,z_2)o(y_1,y_2)^{-I}$ and $(y_1,y_2) \in (x_1,x_2)^{-I}o(z_1,z_2)$.
We have $(z_1,z_2) \in (x_1,x_2)o(y_1,y_2) = \{(a,b) | a \in x_1 \cdot \widehat{x_2}(y_1), b \in x_2 * y_2\}$ which implies $z_1 \in x_1 \cdot \widehat{x_2}(y_1)$ and $z_2 \in x_2 * y_2$. Since $z_1 \in x_1.\widehat{x_2}(y_1)$ we get $x_1 \in z_1 \cdot \widehat{x_2}(y_1)^{-1}$ or $x_1 \in z_1 \cdot \widehat{x_2}(y_1^{-1})$. Since $z_2 \in x_2 * y_2$ then $x_2 \in z_2 * y_2^{-1}$. Therefore $\beta^*(x_2) = \beta^*(z_2) \odot \beta^*(y_2^{-1})$ and so $\widehat{\beta^*(x_2)} = \widehat{\beta^*(z_2) \odot \beta^*(y_2^{-1})} = \widehat{\beta^*(z_2)} \cdot \widehat{\beta^*(y_2^{-1})}$ or $\widehat{x_2} = \widehat{z_2}\widehat{y_2^{-1}}$. Therefore we get $x_1 \in z_1 \cdot \widehat{z_2}\widehat{y_2^{-1}}(y_1^{-1})$. Now, we have $(x_1,x_2) \in \{(a,b) | a \in z_1 \cdot \widehat{z_2}\widehat{y_2^{-1}}(y_1^{-1}), b \in z_2 * y_2^{-1}\}$ or $(x_1,x_2) \in (z_1,z_2)o(y_1,y_2)^{-I}$.

On the other hand, we have

$$\begin{aligned}(x_1,x_2)^{-I}o(z_1,z_2) &= (\widehat{x_2^{-1}}(x_1^{-1}),\ x_2^{-1})o(z_1,z_2)\\ &= \{(a,b) | a \in \widehat{x_2^{-1}}(x_1^{-1}) \cdot \widehat{x_2^{-1}}(z_1),\ b \in x_2^{-1} * z_2\}\\ &= \{(a,b) | a \in \widehat{x_2^{-1}}(x_1^{-1} \cdot z_1),\ b \in x_2^{-1} * z_2\}.\end{aligned}$$

Since $z_1 \in x_1 \cdot \widehat{x_2}(y_1)$ implies $\widehat{x_2}(y_1) \in x_1^{-1} \cdot z_1$, hence $y_1 \in \widehat{x_2^{-1}}(x_1^{-1} \cdot z_1)$. Therefore $(y_1,y_2) \in (x_1,x_2)^{-I}o(z_1,z_2)$. □

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