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Some Further Refinements of Hermite-Hadamard Type Inequalities for Harmonically Convex and P-Convex Functions via Fractional Integrals

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Abstract. It is well known that Hermite-Hadamard inequality generates an estimate of the mean value of the convex function over a bounded interval, in this work we investigate some Hermite-Hadamard type integral inequalities for p-convex functions and harmonically convex functions in fractional integral forms. Precisely, we provide extensions better than those existing in earlier works.

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1. Introduction

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function defined on a real interval I and let a, b be two real constants in I such that $a < b$. The following inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x \leq \frac{f(a)+f(b)}{2}, \tag{1.1}$$

is well known in the literature as the inequality of Hermite-Hadamard (proved by Charles Hermite in 1883 and rediscovered ten years later by Jack Hadamard) [12, 13], it generates an estimate of the mean value of the convex function over a bounded interval. In this context, the Hermite-Hadamard inequality, which is, we can say, the first fundamental result for convex functions with a natural geometric interpretation and many applications, has attracted and continues to attract a great deal of interest in elementary mathematics. Many mathematicians have devoted their efforts to generalize, refine, counterbalance and extend to different classes of functions: quasi-convex functions, log-convex functions, p-convex functions, etc. or apply it to special means (p-logarithmic means, same average, etc.). In recent years there have been many extensions, generalizations and results similar to the Hermite Hadamard inequality for convex functions [1, 2, 3, 4, 5, 7, 9, 10, 14, 15, 17, 18, 19, 20, 22, 23, 24, 26, 28, 29, 30, 31, 32, 33, 34, 35, 37].

In [11], Fejér established the following Hermite-Hadamard-Fejér inequality which is the weighted generalization of the Hermite–Hadamard inequality (1.1):

Theorem 1.1. *Let $f : [a, b] \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function. Then the inequality*

$$f\left(\frac{a+b}{2}\right)\int_a^b w(x)\mathrm{d}x \leq \frac{1}{b-a}\int_a^b f(x)w(x)\,\mathrm{d}x \leq \frac{f(a)+f(b)}{2}\int_a^b w(x)\mathrm{d}x, \tag{1.2}$$

holds where $w : [a, b] \to \mathbb{R}$ is nonnegative, integrable and symmetric with respect to $(a + b)/2$.

Definition 1.2. Let $f \in L^1[a, b]$. The right-hand side and left-hand side Riemann-Liouville fractional integrals $J^{\alpha}_{a+}f$ and $J^{\alpha}_{b-}f$ of order $\alpha \in \mathbb{R}_+$, with $b > a \geq 0$ are defined by :

$$\begin{aligned} J^{\alpha}_{a+}f(x) &= \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}f(t)\mathrm{d}t \qquad x > a, \\ J^{\alpha}_{b-}f(x) &= \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}f(t)\mathrm{d}t \qquad b > x, \end{aligned}$$

respectively, where $\Gamma(\alpha)$ is the well known Gamma function.

In [31], Sarıkaya et. al. gave Hermite–Hadamard's inequalities in fractional integral forms as follows.

Theorem 1.3 ([31]). *Let $f : [a, b] \to \mathbb{R}$ be a positive function with $0 \le a < b$ and $f \in L^1[a, b]$. If f is a convex function on $[a, b]$, then the following inequalities for fractional integrals hold:*

$$f\left(\frac{a+b}{2}\right) \le \frac{\Gamma(\alpha+1)}{2\,(b-a)^{\alpha}}\left[J^{\alpha}_{a+}f(b) + J^{\alpha}_{b-}f(a)\right] \le \frac{f(a)+f(b)}{2} \tag{1.3}$$

with $\alpha > 0$.

In [9], Farissi obtained the following refinement of the inequality (1.1) with a simple proof method.

Theorem 1.4. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function defined on a real interval I and let a, b be two real constants in I such that $a < b$. The inequalities*

$$f\left(\frac{a+b}{2}\right) \le s \le \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x \le S \ \le \frac{f(a)+f(b)}{2}, \tag{1.4}$$

hold. Where

$$s = \frac{1}{2}f\left(\frac{b+3a}{4}\right) + \frac{1}{2}f\left(\frac{3b+a}{4}\right), \tag{1.5}$$

$$S = \frac{1}{2}\left[\frac{f(b)+f\,(b)}{2} + f\left(\frac{a+b}{2}\right)\right]. \tag{1.6}$$

In [9], the author also proved the result:

Theorem 1.5. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function defined on a real interval I and let a, b be two real constants in I such that $a < b$. The inequalities*

$$f\left(\frac{a+b}{2}\right) \le s(t) \le \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x \le S(t) \ \le \frac{f(a)+f(b)}{2}, \tag{1.7}$$

hold. Where

$$s(t) = tf\left(\frac{(2-t)a+tb}{2}\right) + (1-t)f\left(\frac{(1-t)a+(1+t)b}{2}\right),$$

$$S(t) = \frac{1}{2}\left[f\left(tb+(1-t)a\right) + tf(a) + (1-t)f(b)\right],\ t \in [0,1].$$

Definition 1.6 ([18]). A function $f : I \subset \mathbb{R}\setminus\{0\} \to \mathbb{R}$ is said to be harmonically convex, if

$$f\left(\frac{ab}{ta+(1-t)b}\right) \le tf(a) + (1-t)f(b), \tag{1.8}$$

for all $a, b \in I$ and $t \in [0, 1]$. If the inequality in (1.8) is reversed, then f is said to be harmonically concave.

Theorem 1.7. [22] *Let $f : I \subset \mathbb{R}\setminus\{0\} \to \mathbb{R}$ be a harmonically convex function and $a, b \in I$ with $a < b$, if $f \in L^1([a, b])$. Then, we have:*

$$f\left(\frac{2ab}{a+b}\right) \le \left(\frac{ab}{b-a}\right)\int_a^b \frac{f(x)}{x^2}\ \mathrm{d}x \ \le \frac{f(a)+f(b)}{2}. \tag{1.9}$$

Theorem 1.8. [24] *Let $f : I \subset (0,\infty) \to \mathbb{R}$ be a function such that $f \in L^1[a,b]$, where $a,b \in I$ with $a < b$. If f is a harmonically convex function on $[a;b]$, then the following inequalities for fractional integrals hold:*

$$f\left(\frac{2ab}{a+b}\right) \le \left(\frac{ab}{b-a}\right)^{\alpha} \frac{\Gamma(\alpha+1)}{2^{1-\alpha}} \left[J^{\alpha}_{\frac{a+b}{2ab}+}(f\circ g)(\frac{1}{a}) + J^{\alpha}_{\frac{a+b}{2ab}-}(f\circ g)(\frac{1}{b})\right] \le \frac{f(a)+f(b)}{2}, \tag{1.10}$$

with $\alpha > 0$ and $g(x) = \frac{1}{x},\ x \in [\frac{1}{a}, \frac{1}{b}]$.

Definition 1.9 ([27])**.** A function $w : [a,b] \subset \mathbb{R}\setminus\{0\} \to \mathbb{R}$ is said to be harmonically symmetric with respect to $\frac{2ab}{a+b}$ if $w(x) = w\left(\left[a^{-1}+b^{-1}-x^{-1}\right]^{-1}\right)$ holds for all $x \in [a,b]$.

In [6], Chen and Wu presented a Hermite–Hadamard–Fejer type inequality for harmonically convex functions as follows:

Theorem 1.10. *Let $f : I \subset \mathbb{R}\setminus\{0\} \to \mathbb{R}$ be a harmonically convex function and $a,b \in I$ with $a < b$, if $f \in L^1([a,b])$ and $w : [a,b] \subset \mathbb{R}\setminus\{0\} \to \mathbb{R}$ is nonnegative, integrable and harmonically symmetric with respect to $\frac{2ab}{a+b}$, then*

$$f\left(\frac{2ab}{a+b}\right)\int_a^b \frac{w(x)}{x^2}\mathrm{d}x \leqslant \int_a^b \frac{f(x)w(x)}{x^2}\mathrm{d}x \leqslant \frac{f(a)+f(b)}{2}\int_a^b \frac{w(x)}{x^2}\mathrm{d}x. \tag{1.11}$$

In [36], Zhang and Wan gave the definition of a p-convex function on $I \subset \mathbb{R}$, in [16], İşcan gave a different definition of a p-convex function on $I \subset (0,\infty)$:

Definition 1.11. Let $I \subset (0,\infty)$ be a real interval and $p \in \mathbb{R}\setminus\{0\}$. A function $f : I \to \mathbb{R}$ is said to be a p-convex function, if

$$f([\alpha x^p + (1-\alpha)y^p]^{\frac{1}{p}}) \le \alpha f(x) + (1-\alpha)f(y),$$

for all $x,y \in I$ and $\alpha \in [0,1]$.

In [8, Theorem 5], if we take $I \subset (0,\infty)$, $p \in \mathbb{R}\setminus\{0\}$ and $h(t) = t$, then we have the following theorem.

Theorem 1.12. [16] *Let $f : I \subset (0,\infty) \to \mathbb{R}$ be a p-convex function, $p \in \mathbb{R}\setminus\{0\}$ and $a,b \in I$ with $a < b$. If $f \in L^1[a,b]$, then the following inequalities hold:*

$$f\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right) \le \frac{p}{b^p-a^p}\int_a^b \frac{f(x)}{x^{1-p}}\,dx \le \frac{f(a)+f(b)}{2}. \tag{1.12}$$

Theorem 1.13. [22] *Let $f : I \subset (0,\infty) \to \mathbb{R}$ be a p-convex function, $p \in \mathbb{R}\setminus\{0\}$, $\alpha > 0$ and $a,b \in I$ with $a < b$. If $f \in L^1[a,b]$, then the following inequalities hold:*
(i) if $p > 0$

$$\begin{aligned} f\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right) &\le \frac{\Gamma(\alpha+1)}{2^{1-\alpha}(b^p-a^p)^{\alpha}}\left[J^{\alpha}_{\frac{a^p+b^p}{2}+}(f\circ g)(b^p) + J^{\alpha}_{\frac{a^p+b^p}{2}-}(f\circ g)(a^p)\right] \\ &\le \frac{f(a)+f(b)}{2}, \end{aligned} \tag{1.13}$$

with :

$$g(x) = x^{\frac{1}{p}}, x \in [a^p, b^p],$$

(ii) if $p < 0$

$$\begin{aligned} f\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right) &\leq \frac{\Gamma(\alpha+1)}{2^{1-\alpha}(a^p-b^p)^\alpha}\left[J^\alpha_{\frac{a^p+b^p}{2}+}(f\circ g)(a^p)+J^\alpha_{\frac{a^p+b^p}{2}-}(f\circ g)(b^p)\right] \\ &\leq \frac{f(a)+f(b)}{2}, \end{aligned} \tag{1.14}$$

with:

$$g(x) = x^{\frac{1}{p}}, x \in [b^p, a^p].$$

Definition 1.14. Let $p \in \mathbb{R} \setminus \{0\}$. A function $w : [a, b] \subset (0, \infty) \to \mathbb{R}$ is said to be p-symmetric with respect to $\left[\frac{a^p+b^p}{2}\right]^{1/p}$ if $w(x) = w\left(\left[a^p + b^p - x^p\right]^{1/p}\right)$ holds for all $x \in [a, b]$.

In [25], Kunt and İşcan presented a Hermite–Hadamard–Fejer type inequality for p-convex functions as follows:

Theorem 1.15. *Let* $f : I \subset (0, \infty) \to \mathbb{R}$ *be a* p*-convex function,* $p \in \mathbb{R} \setminus \{0\}$ *and* $a, b \in I$ *with* $a < b$*. If* $f \in L^1([a, b])$ *and* $w : [a, b] \subset \mathbb{R} \setminus \{0\} \to \mathbb{R}$ *is nonnegative, integrable and* p*-symmetric with respect to* $\left(\frac{a^p+b^p}{2}\right)^{1/p}$*, then the following inequalities hold:*

$$f\left(\left[\frac{a^p+b^p}{2}\right]^{1/p}\right)\int_a^b \frac{w(x)}{x^{1-p}}\mathrm{d}x \leqslant \int_a^b \frac{f(x)w(x)}{x^{1-p}}\mathrm{d}x \leqslant \frac{f(a)+f(b)}{2}\int_a^b \frac{w(x)}{x^{1-p}}\mathrm{d}x. \tag{1.15}$$

The aim of this paper is to give some refinements for the inequalities (1.9), (1.10), (1.11), (1.12), (1.13), (1.14) and (1.15).

2. Refinements of some inequalities for harmonically convex functions

The main aim of this section is to prove estimations better than those of (1.9), (1.10) and (1.11).

Theorem 2.1. *Let* $f : I \subset \mathbb{R} \setminus \{0\} \to \mathbb{R}$ *be a harmonically convex function and* $a, b \in I$ *with* $a < b$ *, if* $f \in L^1([a, b])$ *and* $w : [a, b] \subset \mathbb{R} \setminus \{0\} \to \mathbb{R}$ *is nonnegative, integrable and harmonically symmetric with respect to* $\frac{2ab}{a+b}$*, then*

$$\begin{aligned} f\left(\frac{2ab}{a+b}\right)\int_a^b \frac{w(x)}{x^2}\mathrm{d}x &\leqslant \ell\int_a^b \frac{w(x)}{x^2}\mathrm{d}x \leqslant \int_a^b \frac{f(x)w(x)}{x^2}\,\mathrm{d}x \\ &\leqslant L\int_a^b \frac{w(x)}{x^2}\mathrm{d}x \leqslant \frac{f(a)+f(b)}{2}\int_a^b \frac{w(x)}{x^2}\mathrm{d}x, \end{aligned} \tag{2.1}$$

where

$$\ell = \frac{1}{2}f\left(\frac{4ab}{b+3a}\right)+\frac{1}{2}f\left(\frac{4ab}{3b+a}\right), \tag{2.2}$$

$$L = \frac{1}{2}\left[\frac{f(b)+f(b)}{2}+f\left(\frac{2ab}{a+b}\right)\right]. \tag{2.3}$$

Proof. Applying (1.11) on each of the subintervals $[a,H]$ and $[H,b]$ with $H = \frac{2ab}{a+b}$

Firstly, applying (1.11) on $[a,H]$, we get:

$$f\left(\frac{2aH}{a+H}\right)\int_a^H \frac{w(x)}{x^2}\mathrm{d}x \leqslant \int_a^H \frac{f(x)w(x)}{x^2}\,\mathrm{d}x \leqslant \frac{f(a)+f(H)}{2}\int_a^H \frac{w(x)}{x^2}\mathrm{d}x,$$

which implies

$$f\left(\frac{4ab}{3b+a}\right)\int_a^H \frac{w(x)}{x^2}\mathrm{d}x \leqslant \int_a^H \frac{f(x)w(x)}{x^2}\,\mathrm{d}x \leqslant \frac{f(a)+f(H)}{2}\int_a^H \frac{w(x)}{x^2}\mathrm{d}x. \tag{2.4}$$

Secondly, applying (1.9) on $[H,b]$, we obtain

$$f\left(\frac{2bH}{b+H}\right)\int_H^b \frac{w(x)}{x^2}\mathrm{d}x \leqslant \int_H^b \frac{f(x)w(x)}{x^2}\,\mathrm{d}x \leqslant \frac{f(b)+f(H)}{2}\int_H^b \frac{w(x)}{x^2}\mathrm{d}x,$$

thus

$$f\left(\frac{4ab}{b+3a}\right)\int_H^b \frac{w(x)}{x^2}\mathrm{d}x \leqslant \int_H^b \frac{f(x)w(x)}{x^2}\,\mathrm{d}x \leqslant \frac{f(b)+f(H)}{2}\int_H^b \frac{w(x)}{x^2}\mathrm{d}x. \tag{2.5}$$

Since g is harmonically symmetric with respect to $\frac{2ab}{a+b}$, we have

$$\int_a^H \frac{w(x)}{x^2}\mathrm{d}x = \int_H^b \frac{w(x)}{x^2}\mathrm{d}x = \frac{1}{2}\int_a^b \frac{w(x)}{x^2}\mathrm{d}x.$$

So, summing up (2.5) and (2.4) side by side we get:

$$\begin{aligned}\frac{1}{2}\left[f\left(\frac{4ab}{b+3a}\right)+f\left(\frac{4ab}{3b+a}\right)\right]\int_a^b \frac{w(x)}{x^2}\mathrm{d}x &\leqslant \int_a^b \frac{f(x)w(x)}{x^2}\,\mathrm{d}x \\ &\leqslant \frac{1}{2}\left[\frac{f(b)+f(H)}{2}+\frac{f(a)+f(H)}{2}\right]\int_a^b \frac{w(x)}{x^2}\mathrm{d}x \\ &= \frac{1}{2}\left[\frac{f(b)+f(b)}{2}+f(H)\right]\int_a^b \frac{w(x)}{x^2}\mathrm{d}x.\end{aligned}$$

As f is a harmonically convex function, so

$$\begin{aligned} f\left(\frac{2ab}{a+b}\right) &= f\left[\frac{2\frac{4ab}{3a+b}\frac{4ab}{3b+a}}{\frac{4ab}{3a+b}+\frac{4ab}{3b+a}}\right] \\ &\leqslant \frac{1}{2}f\left(\frac{4ab}{b+3a}\right)+\frac{1}{2}f\left(\frac{4ab}{3b+a}\right) \\ &= \ell, \end{aligned} \tag{2.6}$$

and

$$\begin{aligned} \frac{f(a)+f(b)}{2} &= \frac{1}{2}[f(a)+f(b)] \\ &\geqslant \frac{1}{2}\left[\frac{f(b)+f\,(b)}{2}+f\,(H)\right] \\ &= L, \end{aligned} \tag{2.7}$$

which ends the proof. ☐

Corollary 2.2. *If we choose* $w(x)=1$ *in (2.1), then we get*

$$f\left(\frac{2ab}{a+b}\right) \leqslant \ell \leqslant \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}\,\mathrm{d}x \ \leqslant L \leqslant \frac{f(a)+f(b)}{2}.$$

Corollary 2.3. *If we choose* $w(x)=\frac{1}{\Gamma(\alpha)}\left[\left(\frac{1}{a}-\frac{1}{x}\right)^{\alpha-1}+\left(\frac{1}{x}-\frac{1}{b}\right)^{\alpha-1}\right]$ *in (2.1), then by using the equalities*

$$\begin{aligned} & J^{\alpha}_{\frac{1}{b}+}(f\circ g)(\frac{1}{a})+J^{\alpha}_{\frac{1}{a}-}(f\circ g)(\frac{1}{b}) \\ = \ & \frac{1}{\Gamma(\alpha)}\int_a^b\left[\left(\frac{1}{a}-\frac{1}{x}\right)^{\alpha-1}+\left(\frac{1}{x}-\frac{1}{b}\right)^{\alpha-1}\right]\frac{f(x)}{x^2}dx,\ g(x)=1/x, \end{aligned}$$

and

$$\frac{1}{\Gamma(\alpha)}\int_a^b\left[\left(\frac{1}{a}-\frac{1}{x}\right)^{\alpha-1}+\left(\frac{1}{x}-\frac{1}{b}\right)^{\alpha-1}\right]\frac{1}{x^2}dx=\frac{2}{\Gamma(\alpha+1)}\left(\frac{b-a}{ab}\right)^{\alpha},$$

we get

$$\begin{aligned} f\left(\frac{2ab}{a+b}\right) \ &\leqslant \ \ell \leqslant \frac{\Gamma(\alpha+1)}{2}\left(\frac{ab}{b-a}\right)^{\alpha}\left[J^{\alpha}_{\frac{1}{b}+}(f\circ g)(\frac{1}{a})+J^{\alpha}_{\frac{1}{a}-}(f\circ g)(\frac{1}{b})\right] \\ &\leqslant \ L \leqslant \frac{f(a)+f(b)}{2}. \end{aligned}$$

Corollary 2.4. *If we choose*

$$w(x)=\begin{cases} \frac{1}{\Gamma(\alpha)}\left(\frac{1}{a}-\frac{1}{x}\right)^{\alpha-1}, & x\in[a,H] \\ \frac{1}{\Gamma(\alpha)}\left(\frac{1}{x}-\frac{1}{b}\right)^{\alpha-1}, & x\in[H,b] \end{cases}$$

in (2.1), then by using the equalities

$$\begin{aligned} & J^{\alpha}_{\frac{a+b}{2ab}+}(f\circ g)(\frac{1}{a})+J^{\alpha}_{\frac{a+b}{2ab}-}(f\circ g)(\frac{1}{b}) \\ = \ & \frac{1}{\Gamma(\alpha)}\left\{\int_a^H\left(\frac{1}{a}-\frac{1}{x}\right)^{\alpha-1}\frac{f(x)}{x^2}dx+\int_H^b\left(\frac{1}{x}-\frac{1}{b}\right)^{\alpha-1}\frac{f(x)}{x^2}dx\right\}, \end{aligned}$$

and

$$\begin{aligned}&\frac{1}{\Gamma(\alpha)}\left\{\int_a^H\left(\frac{1}{a}-\frac{1}{x}\right)^{\alpha-1}\frac{1}{x^2}dx+\int_H^b\left(\frac{1}{x}-\frac{1}{b}\right)^{\alpha-1}\frac{1}{x^2}dx\right\}\\ =\ &\frac{2^{1-\alpha}}{\Gamma(\alpha+1)}\left(\frac{b-a}{ab}\right)^{\alpha},\end{aligned}$$

we get

$$\begin{aligned}f\left(\frac{2ab}{a+b}\right) &\leqslant \ell\leqslant\frac{\Gamma(\alpha+1)}{2^{1-\alpha}}\left(\frac{ab}{b-a}\right)^{\alpha}\left[J^{\alpha}_{\frac{a+b}{2ab}+}(f\circ g)(\frac{1}{a})+J^{\alpha}_{\frac{a+b}{2ab}-}(f\circ g)(\frac{1}{b})\right]\\ &\leqslant L\leqslant\frac{f(a)+f(b)}{2}. \qquad (2.8)\end{aligned}$$

So, we obtain a refinement of the inequality (1.10).

Theorem 2.5. *Let $f : I \subset \mathbb{R}\setminus\{0\}\to\mathbb{R}$ be a harmonically convex function and $a, b \in I$ with $a < b$, if $f \in L^1([a, b])$. Then there exist two real functions l, L such that* (1.9) *takes the following form:*

$$f\left(\frac{2ab}{a+b}\right)\leqslant\ell(t)\leqslant\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}\mathrm{d}x\ \leqslant L(t)\leqslant\frac{f(a)+f(b)}{2}, \qquad (2.9)$$

where

$$\begin{aligned}\ell(t)&=(1-t)f\left(\frac{2ab}{(1-t)b+(1+t)a}\right)+tf\left(\frac{2ab}{(2-t)b+ta}\right),\\ L(t)&=\frac{1}{2}\left[f\left(\frac{ab}{tb+(1-t)a}\right)+tf(a)+(1-t)f(b)\right],\ t\in[0,1].\end{aligned}$$

Proof. Applying (1.9) on the subinterval $[a, H_t]$ with $H_t = \frac{ab}{(1-t)b+ta}$ and $t \neq 0$ we get

$$f\left[\frac{2aH_t}{a+H_t}\right]\leqslant\frac{aH_t}{H_t-a}\int_a^{H_t}\frac{f(x)}{x^2}\mathrm{d}x\leqslant\frac{f(a)+f(H_t)}{2},$$

so

$$f\left(\frac{2ab}{(2-t)b+ta}\right)\leqslant\frac{ab}{t[b-a]}\int_a^{H_t}\frac{f(x)}{x^2}\mathrm{d}x\leqslant\frac{f(a)+f(H_t)}{2}. \qquad (2.10)$$

Applying (1.9) on the subinterval $[H_t, b]$ with $t \neq 1$
we obtain 6

$$f\left[\frac{2H_tb}{H_t+b}\right]\leqslant\frac{H_tb}{b-H_t}\int_{H_t}^b\frac{f(x)}{x^2}\mathrm{d}x\leqslant\frac{f(b)+f(H_t)}{2},$$

thus

$$f\left(\frac{2ab}{(1-t)b+(1+t)a}\right)\leqslant\frac{ab}{(1-t)[b-a]}\int_{H_t}^b\frac{f(x)}{x^2}\mathrm{d}x\leqslant\frac{f(b)+f(H_t)}{2}. \qquad (2.11)$$

Multiplying (2.10) by t and (2.11) by $(1-t)$ then by summing up the obtained inequalities side to side, we can deduce that

$$\underbrace{(1-t)f\left(\frac{2ab}{(1-t)b+(1+t)a}\right)+tf\left(\frac{2ab}{(2-t)b+ta}\right)}_{\ell(t)} \leqslant \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}\mathrm{d}x$$
$$\leqslant \underbrace{\left[f\left(H_t\right)+(1-t)f(b)+tf(a)\right]}_{L(t)}.$$

As f is a harmonically convex function on $[a,b]$, we get:

$$\begin{aligned} f\left(\frac{2ab}{a+b}\right) &= f\left[\frac{\frac{2ab}{(1-t)b+(1+t)a}\,\frac{2ab}{(2-t)b+ta}}{t\left(\frac{2ab}{(1-t)b+(1+t)a}\right)+(1-t)\left(\frac{2ab}{(2-t)b+ta}\right)}\right] \\ &\leqslant (1-t)f\left(\frac{2ab}{(1-t)b+(1+t)a}\right)+tf\left(\frac{2ab}{(2-t)b+ta}\right) \\ &\leqslant \ell(t), \end{aligned}$$

and

$$\begin{aligned} \frac{f(a)+f(b)}{2} &= \frac{1}{2}[f(a)+f(b)] \\ &\geqslant \frac{1}{2}\left[f\left(H_t\right)+(1-t)f(b)+tf(a)\right], \end{aligned}$$

this achieves the proof. □

Lemma 2.6. *Let f and g be two integrable functions on I. Then one has*

$$\left(\frac{ab}{b-a}\right)^{\alpha}\Gamma(\alpha)\left[J^{\alpha}_{\frac{a+b}{2ab}+}(f\circ g)(\frac{1}{a})+J^{\alpha}_{\frac{a+b}{2ab}-}(f\circ g)(\frac{1}{b})\right]$$

$$= \left(\frac{ab}{b-a}\right)^{\alpha}\left\{\int_{\frac{a+b}{2ab}}^{\frac{1}{a}}(\frac{1}{a}-x)^{\alpha-1}f\left(\frac{1}{x}\right)\,\mathrm{d}x+\int_{\frac{1}{b}}^{\frac{a+b}{2ab}}(x-\frac{1}{b})^{\alpha-1}f\left(\frac{1}{x}\right)\mathrm{d}x\right\} \tag{2.12}$$

$$= \left(\frac{ab}{b-a}\right)^{\alpha}\left\{\int_{\frac{1}{b}}^{\frac{a+b}{2ab}}(x-\frac{1}{b})^{\alpha-1}f\left(\frac{1}{\frac{1}{b}+\frac{1}{a}-x}\right)\,\mathrm{d}x+\int_{\frac{1}{b}}^{\frac{a+b}{2ab}}(x-\frac{1}{b})^{\alpha-1}f\left(\frac{1}{x}\right)\mathrm{d}x\right\}$$

$$= \int_0^{\frac{1}{2}} t^{\alpha-1}\left[\frac{f\left(\frac{ab}{ta+(1-t)b}\right)+\left(\frac{ab}{tb+(1-t)a}\right)}{2}\right]\mathrm{d}t, \tag{2.13}$$

with $\alpha>0$ *and* $g(x)=\frac{1}{x}, x\in[\frac{1}{a},\frac{1}{b}], \quad a,b\in I \quad a<b.$

Proof. We use the change of variables
$y=\frac{1}{a}+\frac{1}{b}-x$ to prove (2.12) and $x=\dfrac{ta+(1-t)b}{ab}$ to prove (2.13).
For $x=\frac{1}{b}$, $y=\frac{1}{b}+\frac{1}{a}-\frac{1}{b} \implies y=\frac{1}{a}$ for $x=\frac{a+b}{2ab}$, $y=\frac{1}{b}+\frac{1}{a}-\frac{a+b}{2ab} \implies$ $y=\frac{a+b}{2ab}$ which gives (2.12), for $x=\frac{1}{a}$, $a=\dfrac{ab}{ta+(1-t)b} \implies t=0$ for $x=\frac{1}{b}$, $b=\dfrac{ab}{ta+(1-t)b} \implies t=1$

for $x = \frac{a+b}{2ab}$, $\frac{a+b}{2ab} = \frac{ta+(1-t)b}{ab} \implies t = \frac{1}{2}$
for $x = \dfrac{ta+(1-t)b}{ab} \implies \mathrm{d}t = -\frac{ab}{b-a}\mathrm{d}x$. Using (2.12) we get (2.13). □

Theorem 2.7. *Let $f : I \subset (0,\infty) \to \mathbb{R}$ be a harmonically convex function and $a, b \in I$ with $a < b$, if $f \in L^1([a,b])$. Then there exists a real function which depends on α such that the inequalities* (1.10) *take the following form:*

$$f\left(\frac{2ab}{a+b}\right) \leqslant K(\alpha) \leqslant \left(\frac{ab}{b-a}\right)^{\alpha} \Gamma(\alpha)\left[J^{\alpha}_{\frac{a+b}{ab}+}(f\circ g)(\frac{1}{a}) + J^{\alpha}_{\frac{a+b}{ab}-}(f\circ g)(\frac{1}{b})\right] \leqslant \frac{f(a)+f(b)}{2}, \tag{2.14}$$

with $\alpha > 0$ and $g(x) = \frac{1}{x}, x \in [\frac{1}{a}, \frac{1}{b}]$,
with $K(\alpha) = \frac{\alpha}{2^{1-\alpha}} \int_0^{\frac{1}{2}} t^{\alpha-1}\left[f\left(\frac{4ab}{(1+2t)b+(3-2t)a}\right) + f\left(\frac{4ab}{(1+2t)a+(3-2t)b}\right)\right] \mathrm{d}t$.

Proof. Let $H_t = \frac{ab}{(1-t)a+tb}$, $t \in [0,1]$ .As f is a harmonically convex function on $I = [a,b]$, we have for all $x, y \in I$ (by taking $t = \frac{1}{2}$ in (1.8)) we get

$$f\left(\frac{2xy}{x+y}\right) = f\left[\frac{2\frac{4xy}{3x+y}\frac{4xy}{3y+x}}{\frac{4xy}{3x+y}+\frac{4xy}{3y+x}}\right] \leqslant \frac{1}{2}f\left(\frac{4xy}{y+3x}\right) + \frac{1}{2}f\left(\frac{4xy}{3y+x}\right) \leqslant \frac{f(x)+f(y)}{2}, \tag{2.15}$$

for $x = H_{1-t}$, $y = H_t$ and for all $t \in [0,1]$, we obtain from the inequality (2.15), the following result

$$\begin{aligned} f\left(\frac{2ab}{a+b}\right) &\leqslant \frac{1}{2}f\left(\frac{4H_{1-t}H_t}{H_t+3H_{1-t}}\right) + \frac{1}{2}f\left(\frac{4H_{1-t}H_t}{H_t+H_{1-t}}\right) \\ &\leqslant \frac{f(H_{1-t})+f(H_t)}{2}, \end{aligned}$$

which allows us to write

$$\begin{aligned} f\left(\frac{2ab}{a+b}\right) &\leqslant \frac{f\left(\frac{4ab}{(1+2t)b+(3-2t)a}\right) + f\left(\frac{4ab}{(1+2t)a+(3-2t)b}\right)}{2} \\ &\leqslant \frac{f(H_{1-t})+f(H_t)}{2} \leqslant \frac{f(a)+f(b)}{2}. \end{aligned}$$

By multiplying the above inequality by $2t^{\alpha-1}$, then integrating the obtained inequality with respect to t on $[0, \frac{1}{2}]$, we get :

$$\begin{aligned} f\left(\frac{2ab}{a+b}\right) &\leqslant \frac{\alpha}{2^{1-\alpha}} \int_0^{\frac{1}{2}} t^{\alpha-1}\left[f\left(\frac{4ab}{(1+2t)b+(3-2t)a}\right) + f\left(\frac{4ab}{(1+2t)a+(3-2t)b}\right)\right] \mathrm{d}t \\ &\leqslant \frac{\alpha}{2^{1-\alpha}} \int_0^{\frac{1}{2}} t^{\alpha-1}\left[f(H_{1-t}) + f(H_t)\right] \mathrm{d}t \leqslant \frac{[f(a)+f(b)]}{2}. \end{aligned}$$

Applying Lemma (2.6) we obtain:

$$\begin{aligned} f\left(\frac{2ab}{a+b}\right) &\leqslant \underbrace{\frac{\alpha}{2^{1-\alpha}} \int_0^{\frac{1}{2}} t^{\alpha-1}\left[f\left(\frac{4ab}{(1+2t)b+(3-2t)a}\right) + f\left(\frac{4ab}{(1+2t)a+(3-2t)b}\right)\right] \mathrm{d}t}_{K(\alpha)} \\ &\leqslant \frac{1}{2^{1-\alpha}}\left(\frac{ab}{b-a}\right)^{\alpha} \Gamma(\alpha+1)\left[J^{\alpha}_{\frac{a+b}{ab}+}(f\circ g)(\frac{1}{a}) + J^{\alpha}_{\frac{a+b}{ab}-}(f\circ g)(\frac{1}{b})\right] \leqslant \frac{[f(a)+f(b)]}{2}. \end{aligned}$$

□

3. Refinements of some inequalities for p-convex functions

The main aim of this section is to prove estimations better than those of (1.12), (1.13), (1.14) and (1.15). Also, some of the results obtained in this section are reduced to the results obtained in the previous section in special cases, as well as the improvements of the inequalities (1.1), (1.2) and (1.3) in special cases.

Theorem 3.1. *Let $f : I \subset (0,\infty) \to \mathbb{R}$ be a p-convex function $p \in \mathbb{R}\setminus\{0\}$. and $a, b \in I$ with $a < b$. If $f \in L^1[a,b]$, then there exist two real numbers $\mathcal{M}$ and $\mathcal{F}$ such that* (1.12) *takes the following*

$$f\left(\frac{[a^p+b^p]^{\frac{1}{p}}}{2}\right)\int_a^b \frac{w(x)}{x^{1-p}}\mathrm{d}x \leqslant \mathcal{M}\int_a^b \frac{w(x)}{x^{1-p}}\mathrm{d}x \leqslant \int_a^b \frac{f(x)w(x)}{x^{1-p}}\mathrm{d}x \leqslant \mathcal{F}\int_a^b \frac{w(x)}{x^{1-p}}\mathrm{d}x \leqslant \frac{f(a)+f(b)}{2}\int_a^b \frac{w(x)}{x^{1-p}}\mathrm{d}x, \quad (3.1)$$

where

$$\mathcal{M} = \frac{1}{2}\left[f\left(\left[\frac{3a^p+b^p}{4}\right]^{1/p}\right) + f\left(\left[\frac{a^p+3b^p}{4}\right]^{1/p}\right)\right],$$

$$\mathcal{F} = \frac{1}{2}\left[\frac{f(a)+f(b)}{2} + f\left(\left[\frac{a^p+b^p}{2}\right]^{1/p}\right)\right].$$

Proof. Applying (1.15) on each of the subintervals $[a, A_p]$ and $[A_p, b]$ with $A_p = \left[\frac{a^p+b^p}{2}\right]^{1/p}$.

Firstly, on the subinterval $[a, A_p]$ inequality (1.15) transform to :

$$f\left(\left[\frac{a^p+A_p^p}{2}\right]^{\frac{1}{p}}\right)\int_a^{A_p} \frac{w(x)}{x^{1-p}}\mathrm{d}x \leqslant \int_a^{A_p} \frac{f(x)w(x)}{x^{1-p}}\mathrm{d}x \leqslant \frac{f(a)+f(A_p)}{2}\int_a^{A_p} \frac{w(x)}{x^{1-p}}\mathrm{d}x,$$

which can be written

$$f\left(\left[\frac{3a^p+b^p}{4}\right]^{1/p}\right)\int_a^{A_p} \frac{w(x)}{x^{1-p}}\mathrm{d}x \leqslant \int_a^{A_p} \frac{f(x)w(x)}{x^{1-p}}\mathrm{d}x \leqslant \frac{f(a)+f(A_p)}{2}\int_a^{A_p} \frac{w(x)}{x^{1-p}}\mathrm{d}x. \quad (3.2)$$

Secondly on the subinterval $[A_p, b]$ the inequality (1.15) leads to :

$$f\left(\left[\frac{A_p^p+b^p}{2}\right]^{\frac{1}{p}}\right)\int_{A_p}^{b} \frac{w(x)}{x^{1-p}}\mathrm{d}x \leqslant \int_{A_p}^{b} \frac{f(x)w(x)}{x^{1-p}}\mathrm{d}x \leqslant \frac{f(A_p)+f(b)}{2}\int_{A_p}^{b} \frac{w(x)}{x^{1-p}}\mathrm{d}x,$$

which takes the form:

$$f\left(\left[\frac{a^p+3b^p}{4}\right]^{1/p}\right)\int_{A_p}^{b} \frac{w(x)}{x^{1-p}}\mathrm{d}x \leqslant \int_{A_p}^{b} \frac{f(x)w(x)}{x^{1-p}}\mathrm{d}x \leqslant \frac{f(A_p)+f(b)}{2}\int_{A_p}^{b} \frac{w(x)}{x^{1-p}}\mathrm{d}x. \quad (3.3)$$

Since g is p-symmetric with respect to A_p, we have

$$\int_a^{A_p} \frac{w(x)}{x^{1-p}}\mathrm{d}x = \int_{A_p}^{b} \frac{w(x)}{x^{1-p}}\mathrm{d}x = \frac{1}{2}\int_a^b \frac{w(x)}{x^{1-p}}\mathrm{d}x. \quad (3.4)$$

By summing (3.2) and (3.3), also via (3.4), we get :

$$\mathcal{M}\int_a^b \frac{w(x)}{x^{1-p}}\mathrm{d}x \leqslant \int_{A_p}^b \frac{f(x)w(x)}{x^{1-p}}\mathrm{d}x \leqslant \mathcal{F}\int_a^b \frac{w(x)}{x^{1-p}}\mathrm{d}x.$$

As f is p-convex function on $[a,b]$, so :

$$\begin{aligned} f(A_p) &= f\left(\left[\frac{\frac{3a^p}{4}+\frac{b^p}{4}+\frac{a^p}{4}+\frac{3b^p}{4}}{2}\right]^{1/p}\right) \\ &\leqslant \frac{1}{2}\left[f\left(\left[\frac{a^p+3b^p}{4}\right]^{1/p}\right)+f\left(\left[\frac{3a^p+b^p}{4}\right]^{1/p}\right)\right] = \mathcal{M}, \end{aligned}$$

and

$$\frac{f(a)+f(b)}{2} \geqslant \frac{1}{2}\left[\frac{f(a)+f(b)}{2}+2f\left(\left[\frac{a^p+b^p}{2}\right]^{1/p}\right)\right] = \mathcal{F}.$$

This ends the proof. □

Corollary 3.2. *If we choose $w(x)=1$ in (3.1), then we get*

$$f\left(\frac{[a^p+b^p]^{\frac{1}{p}}}{2}\right) \leqslant \mathcal{M} \leqslant \frac{p}{b^p-a^p}\int_a^b \frac{f(x)w(x)}{x^{1-p}}\mathrm{d}x \leqslant \mathcal{F} \leqslant \frac{f(a)+f(b)}{2}. \quad (3.5)$$

Remark 3.3. (i) For $p=1$, the inequality (3.5) reduces to the inequality (1.4).
(ii) For $p=-1$, the inequality (3.5) reduces to the inequality (2.1).

Corollary 3.4. *If we choose*

$$w(x) = \begin{cases} \frac{1}{\Gamma(\alpha)}\left[(x^p-a^p)^{\alpha-1}+(b^p-x^p)^{\alpha-1}\right] & p>0, \\ \frac{1}{\Gamma(\alpha)}\left[(a^p-x^p)^{\alpha-1}+(x^p-b^p)^{\alpha-1}\right] & p<0, \end{cases}$$

in (3.1), then by using the equalities

$$\begin{aligned} & J^{\alpha}_{a^p+}(f\circ g)(b^p)+J^{\alpha}_{b^p-}(f\circ g)(a^p) \\ = \; & \frac{p}{\Gamma(\alpha)}\int_a^b \left[(x^p-a^p)^{\alpha-1}+(b^p-x^p)^{\alpha-1}\right]\frac{f(x)}{x^{1-p}}dx, \; g(x)=x^{1/p}, p>0, \end{aligned}$$

$$\begin{aligned} & J^{\alpha}_{b^p+}(f\circ g)(a^p)+J^{\alpha}_{a^p-}(f\circ g)(b^p) \\ = \; & \frac{p}{\Gamma(\alpha+1)}\int_a^b \left[(a^p-x^p)^{\alpha-1}+(x^p-b^p)^{\alpha-1}\right]\frac{f(x)}{x^{1-p}}dx, \; g(x)=x^{1/p}, p<0, \end{aligned}$$

and

$$\begin{aligned} \frac{p}{\Gamma(\alpha)}\int_a^b \left[(x^p-a^p)^{\alpha-1}+(b^p-x^p)^{\alpha-1}\right]\frac{1}{x^{1-p}}dx &= \frac{2}{\Gamma(\alpha+1)}(b^p-a^p)^{\alpha}, p>0, \\ \frac{p}{\Gamma(\alpha)}\int_a^b \left[(a^p-x^p)^{\alpha-1}+(x^p-b^p)^{\alpha-1}\right]\frac{1}{x^{1-p}}dx &= \frac{2}{\Gamma(\alpha+1)}(b^p-a^p)^{\alpha}, p<0, \end{aligned}$$

we get

$$f\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right) \leqslant \mathcal{M} \leqslant \frac{\Gamma(\alpha+1)}{2\left(b^p-a^p\right)^{\alpha}}\left[J^{\alpha}_{a^p+}(f\circ g)(b^p)+J^{\alpha}_{b^p-}(f\circ g)(a^p)\right] \leqslant \mathcal{F} \leqslant \frac{f(a)+f(b)}{2}, p>0, \tag{3.6}$$

and

$$f\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right) \leqslant \mathcal{M} \leqslant \frac{\Gamma(\alpha+1)}{2\left(a^p-b^p\right)^{\alpha}}\left[J^{\alpha}_{a^p-}(f\circ g)(b^p)+J^{\alpha}_{b^p+}(f\circ g)(a^p)\right] \leqslant \mathcal{F} \leqslant \frac{f(a)+f(b)}{2}, p<0. \tag{3.7}$$

Remark 3.5. (i) For $p=1$, the inequality (3.6) reduces to a refinement of the inequality (1.3).

(ii) For $p=-1$, the inequality (3.7) reduces to the inequality

$$f\left(\frac{2ab}{a+b}\right) \leqslant \ell \leqslant \left(\frac{ab}{b-a}\right)^{\alpha}\frac{\Gamma(\alpha+1)}{2}\left[J^{\alpha}_{a^{-1}-}(f\circ g)(b^{-1})+J^{\alpha}_{b^{-1}+}(f\circ g)(a^{-1})\right] \leqslant L \leqslant \frac{f(a)+f(b)}{2},$$

where ℓ and L are as in (2.2) and (2.3).

Corollary 3.6. *If we choose*

$$w(x)=\begin{cases}\frac{1}{\Gamma(\alpha)}\left(x^p-a^p\right)^{\alpha-1}, & x\in[a,A_p]\\ \frac{1}{\Gamma(\alpha)}\left(b^p-x^p\right)^{\alpha-1}, & x\in[A_p,b]\end{cases}, A_p=\frac{a^p+b^p}{2}, p>0,$$

in (3.1), then for $p>0$, by using the equalities

$$\begin{aligned}&J^{\alpha}_{A_p^p+}(f\circ g)(b^p)+J^{\alpha}_{A_p^p-}(f\circ g)(a^p)\\ =\;&\frac{1}{\Gamma(\alpha)}\left\{\int_{A_p}^{b}\left(b^p-x^p\right)^{\alpha-1}\frac{f(x)}{x^{1-p}}dx+\int_a^{A_p}\left(x^p-a^p\right)^{\alpha-1}\frac{f(x)}{x^{1-p}}dx\right\}\end{aligned}$$

and

$$\begin{aligned}&\frac{1}{\Gamma(\alpha)}\left\{\int_{A_p}^{b}\left(b^p-x^p\right)^{\alpha-1}\frac{1}{x^{1-p}}dx+\int_a^{A_p}\left(x^p-a^p\right)^{\alpha-1}\frac{1}{x^{1-p}}dx\right\}\\ =\;&\frac{2^{1-\alpha}}{\Gamma(\alpha+1)}\left(b^p-a^p\right)^{\alpha},\end{aligned}$$

we get

$$f\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right) \leqslant \mathcal{M} \leqslant \frac{\Gamma(\alpha+1)}{2^{1-\alpha}(b^p-a^p)^{\alpha}}\left[J^{\alpha}_{\frac{a^p+b^p}{2}+}(f\circ g)(b^p)+J^{\alpha}_{\frac{a^p+b^p}{2}-}(f\circ g)(a^p)\right] \leqslant \mathcal{F} \leqslant \frac{f(a)+f(b)}{2}. \tag{3.8}$$

So, we obtain a refinement of the inequality (1.13).

Similarly, for $p<0$, if we choose

$$w(x)=\begin{cases}\frac{1}{\Gamma(\alpha)}\left(a^p-x^p\right)^{\alpha-1}, & x\in[a,A_p]\\ \frac{1}{\Gamma(\alpha)}\left(x^p-b^p\right)^{\alpha-1}, & x\in[A_p,b]\end{cases},\ A_p=\frac{a^p+b^p}{2},$$

in (3.1), then we get

$$\begin{aligned} f\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right) &\leqslant \mathcal{M}\leqslant \frac{\Gamma(\alpha+1)}{2^{1-\alpha}(b^p-a^p)^{\alpha}}\left[J^{\alpha}_{\frac{a^p+b^p}{2}+}(f\circ g)(a^p)+J^{\alpha}_{\frac{a^p+b^p}{2}-}(f\circ g)(b^p)\right]\\ &\leqslant \mathcal{F}\leqslant\frac{f(a)+f(b)}{2}. \end{aligned} \tag{3.9}$$

So, we obtain a refinement of the inequality (1.14).

Remark 3.7. (i) For $p=1$, the inequality (3.8) reduces to the inequality

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leqslant s\leqslant\frac{\Gamma(\alpha+1)}{2^{1-\alpha}(b-a)}\left[J^{\alpha}_{\frac{a+b}{2}+}f(b)+J^{\alpha}_{\frac{a+b}{2}-}f(a)\right]\\ &\leqslant S\leqslant\frac{f(a)+f(b)}{2}, \end{aligned}$$

where s and S are given in (1.5) and (1.6).

(ii) For $p=-1$, the inequality (3.9) reduces to the inequality (2.8).

Theorem 3.8. *Let $f:I\subset(0,\infty)\to\mathbb{R}$ be a p-convex function $p\in\mathbb{R}\setminus\{0\}$ and $a,b\in I$ with $a<b$. If $f\in L^1[a,b]$, then there exist two real functions $\mathcal{M}(t)$ and $\mathcal{F}(t)$ such that* (1.12) *takes the following*

$$f\left(\frac{[a^p+b^p]^{\frac{1}{p}}}{2}\right)\leqslant\mathcal{M}(t)\leqslant\frac{p}{b^p-a^p}\int_a^b\frac{f(x)}{x^{1-p}}\mathrm{d}x\leqslant\mathcal{F}(t)\leqslant\frac{f(a)+f(b)}{2}, \tag{3.10}$$

where

$$\mathcal{M}(t)=(1-t)f\left(\left[\frac{(1-t)a^p+(1+t)b^p}{2}\right]^{1/p}\right)+tf\left(\left[\frac{(2-t)a^p+tb^p}{2}\right]^{1/p}\right),$$

$$\mathcal{F}(t)=\frac{1}{2}\left[tf(a)+(1-t)f(b)+f([(1-t)a^p+tb^p]^{1/p})\right],\ t\in[0,1].$$

Proof. Let $A_p(t)=[(1-t)a^p+tb^p]^{1/p},\ t\in[0,1]$. Applying inequality (1.12) on each of the subintervals $[a,A_p(t)]$; $[A_p(t),b]$ for all $t\in[0;1]$.
Firstly on the subinterval $[a,A_p(t)]$ with $t\neq1$ inequality (1.12) becomes :

$$\begin{aligned} f\left(\frac{[a^p+A_p(t)]^{\frac{1}{p}}}{2}\right)&\leqslant\frac{p}{A_p^p(t)-a^p}\int_a^{A_p(t)}\frac{f(x)}{x^{1-p}}\mathrm{d}x\\ &\leqslant\frac{f(a)+f(A_p(t))}{2}, \end{aligned}$$

which gives

$$f\left(\left[\frac{(2-t)a^p+tb^p}{2}\right]^{1/p}\right)\leqslant\frac{p}{t(b^p-a^p)}\int_a^{A_p(t)}\frac{f(x)}{x^{1-p}}\mathrm{d}x\leqslant\frac{f(a)+f(A_p(t))}{2}. \tag{3.11}$$

Secondly on the subinterval $[A_p(t), b]$ inequality (1.12) becomes :

$$f\left(\frac{[b^p+[(1-t)a^p+tb^p]]^{\frac{1}{p}}}{2}\right) \leqslant \frac{p}{[b^p-A_p^p(t)]}\int_{A_p(t)}^{b}\frac{f(x)}{x^{1-p}}\mathrm{d}x \leqslant \frac{f(b)+f(A_p(t))}{2},$$

which allows us to write

$$f\left(\left[\frac{(1-t)a^p+(1+t)b^p}{2}\right]^{1/p}\right) \leqslant \frac{p}{(1-t)b^p-a^p}\int_{A_p(t)}^{b}\frac{f(x)}{x^{1-p}}\mathrm{d}x \leqslant \frac{f(b)+f(A_p(t))}{2}. \tag{3.12}$$

Multiplying (3.12) by $(1-t)$ and (3.11) by t and summing up side to side, we get :

$$\underbrace{(1-t)f\left(\left[\frac{(1-t)a^p+(1+t)b^p}{2}\right]^{1/p}\right)+tf\left(\left[\frac{(2-t)a^p+tb^p}{2}\right]^{1/p}\right)} \leqslant \frac{p}{b^p-a^p}\int_a^b\frac{f(x)}{x^{1-p}}\mathrm{d}x$$

$$\leqslant \underbrace{\frac{1}{2}\left[tf(a)+(1-t)f(b)+f(A_p(t))\right]}_{\mathcal{F}(t)}.$$

As f is a p-convex function on $[a, b]$, so we can easily write :

$$f\left(\left[\frac{a^p+b^p}{2}\right]^{1/p}\right) \leqslant \mathcal{M}(t),$$

and

$$\frac{f(a)+f(b)}{2} \geqslant \mathcal{F}(t),$$

the proof is ended. □

Remark 3.9. (i) For $p = 1$, the inequality (3.10) reduces to the inequality (1.7).
(ii) For $p = -1$, the inequality (3.10) reduces to the inequality (2.9).

Theorem 3.10. *Let $f : I \subset (0,\infty) \to \mathbb{R}$ be a p-convex function $p \in \mathbb{R}\setminus\{0\}$ and $a, b \in I$ with $a < b$. If $f \in L^1[a, b]$, then there exists a real function $S(\alpha)$ such that Theorem (1.13) takes the following form :*
(i) if $p > 0$

$$f\left(\frac{[a^p+b^p]^{\frac{1}{p}}}{2}\right) \leq S(\alpha) \leq \frac{\Gamma(\alpha+1)}{2^{1-\alpha}(b^p-a^p)^{\alpha}}\left[J^{\alpha}_{\frac{a^p+b^p}{2}+}(f\circ g)(b^p)+J^{\alpha}_{\frac{a^p+b^p}{2}-}(f\circ g)(a^p)\right] \leq \frac{f(a)+f(b)}{2}, \tag{3.13}$$

with :

$$g(x) = x^{\frac{1}{p}}, x \in [a^p, b^p]$$

where

$$S(\alpha) = \frac{\alpha}{2^{1-\alpha}} \int_0^{1/2} t^{\alpha-1} \left[f\left(\left[\frac{(1+2t)a^p + (1-2t)b^p}{4} \right]^{1/p} \right) + f\left(\left[\frac{(1+2t)b^p + (1-2t)a^p}{4} \right]^{1/p} \right) \right] \mathrm{d}t,$$

(*ii*) *if* $p < 0$

$$\begin{aligned} f\left(\frac{[a^p + b^p]^{\frac{1}{p}}}{2} \right) &\leq S(\alpha) \leq \frac{\Gamma(\alpha+1)}{2^{1-\alpha}(b^p - a^p)^{\alpha}} \left[J^{\alpha}_{\frac{a^p+b^p}{2}+} (f \circ g)(a^p) + J^{\alpha}_{\frac{a^p+b^p}{2}-} (f \circ g)(b^p) \right] \\ &\leq \frac{f(a) + f(b)}{2}, \end{aligned} \tag{3.14}$$

with

$$g(x) = x^{\frac{1}{p}}, x \in [b^p, a^p].$$

Proof. Let $p > 0$, and f be p-convex function, so we have :

$$\begin{aligned} f\left(\left[\frac{x^p + y^p}{2} \right]^{1/p} \right) &= f\left(\left[\frac{\frac{3x^p}{4} + \frac{y^p}{4} + \frac{x^p}{4} + \frac{3y^p}{4}}{2} \right]^{1/p} \right) \\ &\leqslant \frac{f\left(\left[\frac{x^p+3y^p}{4} \right]^{1/p} \right) + f\left(\left[\frac{3x^p+y^p}{4} \right]^{1/p} \right)}{2} \end{aligned} \tag{3.15}$$

and

$$\frac{f(x) + f(y)}{2} \geq \frac{f\left(\left[\frac{x^p+3y^p}{4} \right]^{1/p} \right) + f\left(\left[\frac{3x^p+y^p}{4} \right]^{1/p} \right)}{2}. \tag{3.16}$$

If we choose $x = A_p(1-t) = (ta^p + (1-t)b^p)^{\frac{1}{p}}$ and $y = A_p(t) = (tb^p + (1-t)a^p)^{\frac{1}{p}}$ in (3.15) and (3.16), then we get

$$f\left(\left[\frac{a^p + b^p}{2} \right]^{1/p} \right) \leq \frac{f\left(\left[\frac{(1+2t)a^p+(1-2t)b^p}{4} \right]^{1/p} \right) + f\left(\left[\frac{(1+2t)b^p+(1-2t)a^p}{4} \right]^{1/p} \right)}{2}$$

and

$$\frac{f(A_p(1-t)) + f(A_p(t))}{2} \geq \frac{f\left(\left[\frac{(1+2t)a^p+(1-2t)b^p}{4} \right]^{1/p} \right) + f\left(\left[\frac{(1+2t)b^p+(1-2t)a^p}{4} \right]^{1/p} \right)}{2}$$

but, we have $\frac{f(a)+f(b)}{2} \geq \frac{f(A_p(1-t))+f(A_p(t))}{2}$ i.e

$$\begin{aligned} f\left(\left[\frac{a^p + b^p}{2} \right]^{1/p} \right) &\leq \frac{f\left(\left[\frac{(1+2t)a^p+(1-2t)b^p}{4} \right]^{1/p} \right) + f\left(\left[\frac{(1+2t)b^p+(1-2t)a^p}{4} \right]^{1/p} \right)}{2} \\ &\leq \frac{f(A_p(1-t)) + f(A_p(t))}{2} \leq \frac{f(a) + f(b)}{2}. \end{aligned}$$

Multiplying the above inequalities by $2t^{\alpha-1}$, then integrating the result inequalities with respect to t on $[0, 1/2)$, we obtain :

$$f\left(\left[\frac{a^p+b^p}{2}\right]^{1/p}\right) \leq \frac{\alpha}{2^{1-\alpha}}\int_0^{1/2} t^{\alpha-1}\left[f\left(\left[\frac{(1+2t)a^p+(1-2t)b^p}{4}\right]^{1/p}\right)\right.$$
$$\left.+f\left(\left[\frac{(1+2t)b^p+(1-2t)a^p}{4}\right]^{1/p}\right)\right]\mathrm{d}t$$
$$\leq \frac{\alpha}{2^{1-\alpha}}\int_0^{1/2} t^{\alpha-1}\left[f(A_p(1-t)))+f(A_p(t))\right]\mathrm{d}t \leq \frac{f(a)+f(b)}{2}.$$

As:

$$\begin{aligned}&\frac{\alpha}{2^{1-\alpha}}\int_0^{1/2} t^{\alpha-1}\left[f(A_p(1-t)))+f(A_p(t))\right]\mathrm{d}t\\ =\;&\frac{\Gamma(\alpha+1)}{2^{1-\alpha}(b^p-a^p)^\alpha}\left[J^\alpha_{\frac{a^p+b^p}{2}+}(f\circ g)(b^p)+J^\alpha_{\frac{a^p+b^p}{2}-}(f\circ g)(a^p)\right]\end{aligned}$$

and

$$\frac{\alpha}{2^{1-\alpha}}\int_0^{1/2} t^{\alpha-1}\left[f\left(\left[\frac{(1+2t)a^p+(1-2t)b^p}{4}\right]^{1/p}\right)+f\left(\left[\frac{(1+2t)b^p+(1-2t)a^p}{4}\right]^{1/p}\right)\right]\mathrm{d}t = S(\alpha).$$

So, we get

$$f\left(\left[\frac{a^p+b^p}{2}\right]^{1/p}\right) \leq S(\alpha) \leq \frac{\Gamma(\alpha+1)}{2^{1-\alpha}(b^p-a^p)^\alpha}\left[J^\alpha_{\frac{a^p+b^p}{2}+}(f\circ g)(b^p)+J^\alpha_{\frac{a^p+b^p}{2}-}(f\circ g)(a^p)\right]$$
$$\leq \frac{f(a)+f(b)}{2}.$$

We prove (ii) as (i) . ☐

Remark 3.11. (i) For $p = 1$, the inequality (3.13) reduces to the inequality

$$\begin{aligned}f\left(\frac{[a^p+b^p]^{\frac{1}{p}}}{2}\right) &\leq S(\alpha) \leq \frac{\Gamma(\alpha+1)}{2^{1-\alpha}(b^p-a^p)^\alpha}\left[J^\alpha_{\frac{a^p+b^p}{2}+}(f\circ g)(b^p)+J^\alpha_{\frac{a^p+b^p}{2}-}(f\circ g)(a^p)\right]\\ &\leq \frac{f(a)+f(b)}{2},\end{aligned}$$

where $S(\alpha) = \frac{\alpha}{2^{1-\alpha}}\int_0^{1/2} t^{\alpha-1}\left[f\left(\frac{(1+2t)a+(1-2t)b}{4}\right)+f\left(\frac{(1+2t)b+(1-2t)a}{4}\right)\right]\mathrm{d}t$

(ii) For $p = -1$, the inequality (3.14) reduces to the inequality

$$\begin{aligned}f\left(\frac{[a^p+b^p]^{\frac{1}{p}}}{2}\right) &\leq S(\alpha) \leq \frac{\Gamma(\alpha+1)}{2^{1-\alpha}(b^p-a^p)^\alpha}\left[J^\alpha_{\frac{a^p+b^p}{2}+}(f\circ g)(a^p)+J^\alpha_{\frac{a^p+b^p}{2}-}(f\circ g)(b^p)\right]\\ &\leq \frac{f(a)+f(b)}{2},\end{aligned}$$

where

$$S(\alpha) = \frac{\alpha}{2^{1-\alpha}}\int_0^{1/2} t^{\alpha-1}\left[f\left(\left[\frac{(1+2t)a^{-1}+(1-2t)b^{-1}}{4}\right]^{-1}\right)+f\left(\left[\frac{(1+2t)b^{-1}+(1-2t)a^{-1}}{4}\right]^{-1}\right)\right]\mathrm{d}t.$$

4. Conclusion

In the article, we provide refinements of the Hermite-Hadamard inequality, harmonically convex and p-convex functions. Our proved results can be presented as improvements and extensions of some exiting results. We can obtain more results with respect to the type of convexity.

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