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Generalized Kernels of Subsets through Ideals in Topological Spaces

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Abstract. In this research work, we introduce a generalization of the notion of kernel of a set in topological spaces endowed with an ideal, which is a fundamental tool to obtain new modifications of open sets and closed sets. Using this generalized kernel, we define and characterize new low separation axioms in other contexts obtained from a topological space endowed with an ideal. Also, we study the invariance of these low separation axioms under certain types of continuity defined in this novel theoretical framework.

Keywords: Topological kernel, Ideals, Co-local function, $(\tau^\star, \tau^\bullet)$-$g$-closed set, $\mathcal{I}$-λ-closed set.

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1. Introduction

The characterization of the points belong to topological closure of a set in terms of neighborhoods is well known in general topology. In similar way, other types of closures of a set have been given and characterized in terms of several collections of generalized open sets. On the other hand, the points belong to topological kernel can be characterized in the same style too. The notion of topological kernel begin to play relevance from 1986, when it was used by Maki [12] to introduce the concept of Λ-set in a topological space. The class of the Λ-sets and their complements, called V-sets, were appropriate to characterize the T_1 axiom (see [12]). After, in 1997, Arenas et al. [2], defined and studied the notions of λ-closed and λ-open sets, using Λ-sets and closed sets. In particular, these authors used the λ-closed sets to characterize the $T_{1/2}$ axiom. In 1990, Khalimsky et al. [10] studied the geometric and topological properties of digital images, and showed an important example of a space that is $T_{1/2}$ but is not T_1, called the digital line or Khalimsky's line. This fact motivated several studies that they involved variants of the topological kernel, which were made based on collections of generalized open sets, as we can see in [3, 4, 5, 6, 7, 8] and [13, 14, 15, 16, 17]. On the other hand, in 1933, Kuratowski [11] used the concept of ideal on a topological space in order to generalize the notion of the closure of a set, introducing the local function of a set with respect to an ideal and a topology. In 1990, Jankovic and Hamlett [9] studied local and global properties that involve the concept of ideal on a topological space. In particular, these authors defined a Kuratowski closure operator, $Cl^\star$, which induces a topology $\tau^\star$ that is finer than the topology originally given in the space. In this work, we introduced and studied a generalization of the notion of kernel of a set, in analogous with the generalizations of the notion of the closure of a set given in [9].

2. Preliminaries

In this section, we present the main definitions and results which will be useful in the sequel. Throughout this paper, if (X, τ) is a topological space, for $A \subseteq X$, we denote the closure of A and interior of A by $Cl(A)$ and $Int(A)$, respectively. For a subset A of X, the *θ-closure* of A [19], denoted by $Cl_\theta(A)$, is defined as the set of all points $x \in X$ such that $A \cap Cl(U) \neq \emptyset$ for each open set U containing x. Also, a subset A of a topological space (X, τ) is said to be *θ-closed* if $A = Cl_\theta(A)$. The complement of a θ-closed set is said to be a *θ-open* set. The collection of all θ-open sets in a topological space (X, τ), denoted by τ_θ, forms a topology on X. It is important to note that $\tau_\theta \subset \tau$ and that Cl_θ is not the closure of A with respect to τ_θ [19]. If A is a subset of X, the *kernel* of a set A [12], denoted by $\Lambda(A)$, is defined as the intersection of all open sets containing A. Observe that if A is an open set then $\Lambda(A) = A$. A subset A is said to be a Λ-*set* [12] if $A = \Lambda(A)$. The complements of the Λ-sets are called V-*sets*. The family of all V-sets are denoted by τ^V. It is well known that the pair (X, τ^V) is an Alexandroff space (i.e. is a topological space where any arbitrary intersection of open sets is open), see [12] for more details. A subset A of a topological space (X, τ) is said to be λ-*closed* [2], if $A = U \cap F$, where U is a Λ-set and F is a closed set. A subset A of a topological space (X, τ) is said to be $g\Lambda$-*set* [12] if $\Lambda(A) \subset F$ whenever $A \subset F$ and F is a closed set.

An *ideal* $\mathcal{I}$ on a set X is a non-empty collection of subsets of X, which satisfies the following properties: if $A \in \mathcal{I}$ and $B \subset A$, then $B \in \mathcal{I}$, also if $A, B \in \mathcal{I}$, then $A \cup B \in \mathcal{I}$. Throughout this work $(X, \tau, \mathcal{I})$ will denote a topological space (X, τ) with an ideal $\mathcal{I}$ on X and will be called a space. Given a space $(X, \tau, \mathcal{I})$ and a subset A of X, the *local function* [11] of A with respect to $\mathcal{I}$ and τ, denoted by $A^\star(\mathcal{I}, \tau)$, is defined as $A^\star(\mathcal{I}, \tau) = \{x \in X : U \cap A \notin \mathcal{I} \text{ for every } U \in \tau(x)\}$, where $\tau(x) = \{U \in \tau : x \in U\}$. We will denote $A^\star(\mathcal{I}, \tau)$ by $A^\star$ or $A^\star(\mathcal{I})$. Clearly if $(X, \tau, \mathcal{I})$ is a space, then $\emptyset^\star = \emptyset$, since for every $x \in X$ and every $U \in \tau(x)$, $\emptyset \cap U = \emptyset \in \mathcal{I}$. Obviously $X^\star \subset X$. In general $X^\star$ is a proper subset of X. For each subset A of X, $Cl^\star(A)$ [9] is defined as the union of A with $A^\star$; that is, $Cl^\star(A) = A \cup A^\star$. If $\mathcal{I} = \{\emptyset\}$, then for each subset A of X, $Cl^\star(A) = A \cup A^\star = A \cup Cl(A) = Cl(A)$. It is well known that $Cl^\star$ is a Kuratowski closure operator [9]. Using this fact, we denote by $\tau^\star$ (or $\tau^\star(\mathcal{I})$) to the topology generated by $Cl^\star$, that is, $\tau^\star = \{U \subset X : Cl^\star(X - U) = X - U\}$. The elements of $\tau^\star$ are called $\tau^\star$-open and the complement of a $\tau^\star$-open is called $\tau^\star$-closed. In the sequel we use the following two theorems.

Theorem 2.1. [9] *A subset A of a space $(X, \tau, \mathcal{I})$ is $\tau^\star$-closed if and only if $A^\star \subset A$.*

Theorem 2.2. [15] *Let $(X, \tau, \mathcal{I})$ be a space and A a subset of X. Then $A^\star - A$ does not contain any nonempty $\tau^\star$-open set.*

3. The co-local function

In this section, we introduce and study the concept of co-local function as a natural generalization of the kernel of a set in a topological space.

Definition 3.1. Let $(X, \tau, \mathcal{I})$ be a space. For each $A \subseteq X$, we define the *co-local function* of A with respect to $\mathcal{I}$ and τ as follows: $A^\bullet(\mathcal{I}, \tau) = \{x \in X : F \cap A \notin \mathcal{I}, \text{ for each } F \in \tau^c(x)\}$, where $\tau^c(x) = \{F : X - F \in \tau, \ x \in F\}$.

We will denote $A^\bullet(\mathcal{I}, \tau)$ by $A^\bullet$ or $A^\bullet(\mathcal{I})$. Observe that the co-local function can be seen as an operator from $\mathcal{P}(X)$ to $\mathcal{P}(X)$; that is, $(\ )^\bullet : \mathcal{P}(X) \to \mathcal{P}(X)$, defined by $A \mapsto A^\bullet$.

The co-local function is not a Kuratowski closure operator, since in general, it does not satisfy $A \subset A^{\bullet}$ for each $A \subseteq X$. In the case that $A \subset A^{\bullet}$, we say that A is a subset $\bullet$-*dense in itself*.

The following example shows that, in general, $X^{\bullet}$ is a proper subset of X; that is, X is not $\bullet$–dense in itself.

Example 3.2. Let $X = \mathbb{R}$ with the topology $\tau = \{\emptyset, \mathbb{R}, \mathbb{R} - A\}$ where A is any non-empty countable subset of $\mathbb{R}$ and $\mathcal{I} = \mathcal{C}$ the ideal of all countable subsets of $\mathbb{R}$. Observe that the only non-empty closed sets are $F_1 = \mathbb{R}$ and $F_2 = A$. Since $X \cap F_1 = F_1 \notin \mathcal{C}$ and $X \cap F_2 = A \in \mathcal{C}$, then is clear that $X^{\bullet} = \mathbb{R}^{\bullet} = \mathbb{R} - A \subsetneqq \mathbb{R} = X$.

Recall that the *co-kernel* of a subset A of X, denoted by $V(A)$, is defined as the union of all closed sets contained in A [12]. The following theorem gives some characterizations for the case that X is $\bullet$-dense in itself.

Theorem 3.3. *Let $(X, \tau, \mathcal{I})$ be a space. The following properties are equivalent:*

(1) $\tau^c \cap \mathcal{I} = \{\emptyset\}$, *where* $\tau^c = \{F : X - F \in \tau\}$.
(2) *If* $I \in \mathcal{I}$, *then* $V(I) = \emptyset$.
(3) $A \subset A^{\bullet}$ *for each* $A \in \tau^c$.
(4) $X = X^{\bullet}$.

Proof. (1) $\Rightarrow$ (2) Let $I \in \mathcal{I}$ and suppose that there exists a point $x \in X$ such that $x \in V(I)$. Then there exists $F \in \tau^c$ such that $x \in F \subset I$. By hereditary property of $\mathcal{I}$, it follows that $F \in \mathcal{I}$ and so, F is a nonempty set such that $F \in \tau^c \cap \mathcal{I}$, contradicting the fact that $\tau^c \cap \mathcal{I} = \{\emptyset\}$.

(2) $\Rightarrow$ (3) Let $x \in A$ and suppose that $x \notin A^{\bullet}$. Then there exists $F \in \tau^c(x)$ such that $A \cap F \in \mathcal{I}$. Since $A \in \tau^c$, it follows that $A \cap F \in \tau^c(x)$ and so $V(A \cap F) = A \cap F \in \tau^c(x)$, which implies that $V(A \cap F) \neq \emptyset$ and $A \cap F \in \mathcal{I}$. This contradicts the fact that $V(I) = \emptyset$ for each $I \in \mathcal{I}$.

(3) $\Rightarrow$ (4) It follows from the fact that X is closed.

(4) $\Rightarrow$ (1) If $X = X^{\bullet}$ then $X = \{x \in X : F \cap X = F \notin \mathcal{I}$ for each $F \in \tau^c(x)\}$, and this implies that $\tau^c \cap \mathcal{I} = \{\emptyset\}$. □

Proposition 3.4. *Let $(X, \tau, \mathcal{I})$ be a space. For each $A \subseteq X$, the following properties hold:*

(1) *If* $\mathcal{I} = \{\emptyset\}$, *then* $A^{\bullet} = \Lambda(A)$.
(2) *If* $\mathcal{I} = \mathcal{P}(X)$, *then* $A^{\bullet} = \emptyset$.

Now we give a characterization of the co-local function.

Theorem 3.5. *Let $(X, \tau, \mathcal{I})$ be a space. For each $A \subseteq X$, $A^{\bullet}(\mathcal{I}, \tau) = \{x \in X : Cl(\{x\}) \cap A \notin \mathcal{I}\}$.*

Proof. Let $S = \{x \in X : Cl(\{x\}) \cap A \notin \mathcal{I}\}$. Suppose that $y \notin S$, then $Cl(\{y\}) \cap A \in \mathcal{I}$. Since $Cl(\{y\})$ is a closed set containing y, then $y \notin A^{\bullet}$.

Conversely, if $y \notin A^\bullet$, then there exists a closed set F containing y such that $F \cap A \in \mathcal{I}$, so $Cl(\{y\}) \cap A \subset F \cap A$ and $Cl(\{y\}) \cap A \in \mathcal{I}$. Therefore, $y \notin S$. □

Corollary 3.6. *Let (X, τ) be a topological space. For each $A \subseteq X$, $\Lambda(A) = \{x \in X : Cl(\{x\}) \cap A \neq \emptyset\}$.*

The following lemma shows several properties that involve the co-local function

Lemma 3.7. *Let (X, τ) be a topological space with two ideals $\mathcal{I}$, $\mathcal{J}$ on X. If A, B are subsets of X, then the following properties hold:*

(1) *If $A \subset B$, then $A^\bullet \subset B^\bullet$.*
(2) *If $\mathcal{I} \subset \mathcal{J}$, then $A^\bullet(\mathcal{J}) \subset A^\bullet(\mathcal{I})$.*
(3) *$A^\bullet = \Lambda(A^\bullet) \subset \Lambda(A)$ ($A^\bullet$ is a Λ-set).*
(4) *$(A^\bullet)^\bullet \subset A^\bullet$.*
(5) *$\emptyset^\bullet = \emptyset$.*
(6) *$(A \cup B)^\bullet = A^\bullet \cup B^\bullet$.*
(7) *If F is a closed set, then $F \cap A^\bullet = F \cap (F \cap A)^\bullet \subset (F \cap A)^\bullet$.*
(8) *If $A \in \mathcal{I}$, then $A^\bullet = \emptyset$.*
(9) *$A^\bullet - B^\bullet = (A - B)^\bullet - B^\bullet$.*
(10) *If $B \in \mathcal{I}$, then $(A \cup B)^\bullet = A^\bullet = (A - B)^\bullet$.*
(11) *If $A \subset A^\bullet$, then $A^\bullet = \Lambda(A)$.*

Proof. The proof follows directly from the definition. □

Corollary 3.8. *Let $(X, \tau, \mathcal{I})$ be a space and $\{A_\omega : \omega \in \Omega\}$ a collection of subsets of X. The following properties hold:*

(1) $\left(\bigcap \{A_\omega : \omega \in \Omega\}\right)^\bullet \subset \bigcap \{A_\omega^\bullet : \omega \in \Omega\}$.
(2) $\left(\bigcup \{A_\omega : \omega \in \Omega\}\right)^\bullet = \bigcup \{A_\omega^\bullet : \omega \in \Omega\}$, *if Ω is finite.*

Since the co-local function is not a Kuratowski closure operator, it is necessary to introduce a new concept that allows us to obtain a new topology from it.

Definition 3.9. Let $(X, \tau, \mathcal{I})$ be a space. For each $A \subseteq X$, we define $Cl^\bullet(A) = A \cup A^\bullet$.

Remark 3.10. Let $(X, \tau, \mathcal{I})$ be a space.

(1) If $\mathcal{I} = \{\emptyset\}$ then $Cl^\bullet(A) = A \cup A^\bullet = A \cup \Lambda(A) = \Lambda(A)$.
(2) If $\mathcal{I} = \mathcal{P}(X)$, then $Cl^\bullet(A) = A \cup \emptyset = A$.

Theorem 3.11. *$Cl^\bullet$ is a Kuratowski closure operator.*

Proof. The proof is easy and is omitted. □

According with Theorem 3.11, if $(X, \tau, \mathcal{I})$ is a space, we denote by $\tau^\bullet(\mathcal{I})$ the topology generated by $Cl^\bullet$; that is $\tau^\bullet(\mathcal{I}) = \{U \subset X : Cl^\bullet(X - U) = X - U\}$.

When there is no chance for confusion, we will simply write $\tau^\bullet$ for $\tau^\bullet(\mathcal{I})$. The elements of $\tau^\bullet$ are called $\tau^\bullet$-*open* and the complement of a $\tau^\bullet$-open is called $\tau^\bullet$-*closed*. Note that if A is a subset of a space $(X, \tau, \mathcal{I})$, then: A is $\tau^\bullet$-closed if and only if $X - A \in \tau^\bullet$ if and only if $Cl^\bullet(X - (X - A)) = X - (X - A)$ if and only if $Cl^\bullet(X - (X - A)) = X - (X - A)$ if and only if $Cl^\bullet(A) = A$.

In general, $\tau^\bullet$ and τ are incomparable, as we can see in the following example.

Example 3.12. Consider $X = \{a, b, c\}$ with the topology $\tau = \{\emptyset, \{a, c\}, X\}$ and the ideal $\mathcal{I} = \{\emptyset, \{c\}\}$. Take $A = \{a, c\}$ and $B = \{c\}$. Since $(X - A)^\bullet = \{b\}^\bullet = \{a, b, c\} \not\subset \{b\} = X - A$, we have $X - A$ is not $\tau^\bullet$-closed and $A = \{a, c\}$ is not $\tau^\bullet$-open. On the other hand, $(X - B)^\bullet = \{a, b\}^\bullet = X - B$, we have $B = \{c\}$ is a $\tau^\bullet$-closed. Thus $\{a, b\}$ is a $\tau^\bullet$-open set, but $\{a, b\}$ is not a τ-open set.

Remark 3.13. Since $A^\bullet = \Lambda(A^\bullet) \subset \Lambda(A)$, then $Cl^\bullet(A) \subset \Lambda(A)$ for each subset A of X. Therefore, if A is a Λ-set, then A is $\tau^\bullet$-closed. It follows that each V-set is $\tau^\bullet$-open; that is $\tau^V \subset \tau^\bullet$.

Remark 3.14. According to [1], if A is a subset of X, we define the *local closure function* of A with respect to $\mathcal{I}$ and τ as follows: $\Gamma(A)(\mathcal{I}, \tau) = \{x \in X : Cl(U) \cap A \notin \mathcal{I},$ for each $U \in \tau(x)\}$, where $\tau(x) = \{U : U \in \tau,\ x \in U\}$. We denote $\Gamma(A)(\mathcal{I}, \tau)$ by $\Gamma(A)$. It is easy to see that $A^\bullet \subset \Gamma(A) \subset Cl_\theta(A)$, it follows that if A is a θ-closed set, then A is $\tau^\bullet$-closed. Hence $\tau_\theta \subset \tau^\bullet$.

Lemma 3.15. *If $\{A_\omega : \omega \in \Omega\}$ is a collection of $\tau^\bullet$-closed sets in a space $(X, \tau, \mathcal{I})$, then the following properties hold:*

(1) $\bigcap\{A_\omega : \omega \in \Omega'\}$ *is a $\tau^\bullet$-closed set for any subset Ω' of Ω.*
(2) $\bigcup\{A_\omega : \omega \in \Omega_0\}$ *is a $\tau^\bullet$-closed set for any finite subset Ω_0 of Ω.*

Proof. The proof is an immediate consequence of De Morgan's laws and the duality between the notions of $\tau^\bullet$-open and $\tau^\bullet$-closed sets. □

Proposition 3.16. *A subset A of a space $(X, \tau, \mathcal{I})$ is $\tau^\bullet$-closed if and only if $A^\bullet \subset A$.*

Proof. Suppose that A is $\tau^\bullet$-closed, then $Cl^\bullet(A) = A$. In consequence, $A \cup A^\bullet = A$ and hence, $A^\bullet \subset A$.

Conversely, suppose that $A^\bullet \subset A$. Since $Cl^\bullet(A) = A \cup A^\bullet$ and $A \cup A^\bullet \subset A$, then $Cl^\bullet(A) \subset A$. By Theorem 3.11, we have $A \subset Cl^\bullet(A)$ and so, we conclude that $Cl^\bullet(A) = A$. This shows that A is $\tau^\bullet$-closed. □

In a similar form as the base structure for the topology $\tau^\star$ given by Jankovic and Hamlett [9], we have the following result.

Proposition 3.17. *Let $(X, \tau, \mathcal{I})$ be a space. The collection $\kappa(\mathcal{I}, \tau) = \{F - J : X - F \in \tau$ and $J \in \mathcal{I}\}$ is a base for the topology $\tau^\bullet$.*

Proof. Consider F such that $X - F \in \tau$ and $J \in \mathcal{I}$. We assert that $A = X - (F - J) = X - [F \cap (X - J)] = (X - F) \cup J$ is a $\tau^\bullet$-closed set. Indeed, suppose that $x \notin A$, which is equivalent to saying that $x \in F - J$, it follows that $x \in F$ and $F \cap A = F \cap [X - (F - J)] = F \cap [(X - F) \cup J] = F \cap J \in \mathcal{I}$, hence $x \notin A^\bullet$ and so $A^\bullet \subset A$. This shows that $\kappa(\mathcal{I}, \tau) \subset \tau^\bullet$. On the other hand, it is easy to see that each element of $\tau^\bullet$ can be written as a union of sets belonging to $\kappa(\mathcal{I}, \tau)$. Therefore, $\kappa(\mathcal{I}, \tau)$ is a basis for the topology $\tau^\bullet$. □

Proposition 3.18. *Let $(X, \tau, \mathcal{I})$ be a space and $A \subseteq X$. Then $A^\bullet - A$ does not contain any nonempty $\tau^\bullet$-open set.*

Proof. Suppose that $A \subset X$ and U is a $\tau^\bullet$-open set such that $U \subset A^\bullet - A$, then $U \subset A^\bullet - A \subset X - A$, $A \subset X - U$ and $X - U$ is $\tau^\bullet$-closed. Using Lemma 3.7-(1) and Proposition 3.16, we obtain that $A^\bullet \subset (X - U)^\bullet \subset X - U$ and so, $U \subset X - A^\bullet$. Since $U \subset A^\bullet$, it follows that $U \subset (X - A^\bullet) \cap A^\bullet = \emptyset$, and hence $U = \emptyset$. Therefore, $A^\bullet - A$ does not contain any nonempty $\tau^\bullet$-open set. □

Corollary 3.19. *Let (X, τ) be a topological space and $A \subset X$. Then $\Lambda(A) - A$ does not contain any nonempty V-set.*

Theorem 3.20. *If $\mathcal{I}$ and $\mathcal{J}$ are ideals on a topological space (X, τ) such that $\mathcal{I} \subset \mathcal{J}$, then $\tau^\bullet(\mathcal{I}) \subset \tau^\bullet(\mathcal{J})$.*

Proof. Consider $A \in \tau^\bullet(\mathcal{I})$. We will prove that $B = X - A$ is a $\tau^\bullet(\mathcal{J})$-closed set, i.e. $B^\bullet(\mathcal{J}) \subset B$. Suppose that $x \notin B$, then $x \notin B^\bullet(\mathcal{I})$ because B is a $\tau^\bullet(\mathcal{I})$-closed set. Then, there exists a closed set F such that $x \in F$ and $B \cap F \in \mathcal{I}$. Since $\mathcal{I} \subset \mathcal{J}$, we have $B \cap F \in \mathcal{J}$ and so, $x \notin B^\bullet(\mathcal{J})$. This shows that $B^\bullet(\mathcal{J}) \subset B$ and B is a $\tau^\bullet(\mathcal{J})$-closed set. □

Theorem 3.21. *Let $\{\mathcal{I}_\omega : \omega \in \Omega\}$ be a collection of ideals on a topological space (X, τ). If $\mathcal{I} = \bigcap_{\omega \in \Omega} \mathcal{I}_\omega$ then $\tau^\bullet(\mathcal{I}) \subset \tau^\sharp$, where $\tau^\sharp = \bigcap_{\omega \in \Omega} \tau^\bullet(\mathcal{I}_\omega)$.*

Proof. The proof is clear and hence is omitted □

The following theorem gives a new characterization of the T_1-spaces, using the $\tau^\bullet$-closed sets.

Theorem 3.22. *If $(X, \tau, \mathcal{I})$ is a T_1-space, then for every $x \in X$, the singleton $\{x\}$ is $\tau^\bullet$-closed. The converse is true if each singleton is $\bullet$-dense in itself.*

Proof. Let x be any point of X. For every $y \in X$, $y \neq x$, there exists an open set U such that $x \in U$ and $y \notin U$. According to this, $y \in F = X - U$, F is closed and $\{x\} \cap F \subset U \cap F = \emptyset$. So, $\{x\} \cap F \in \mathcal{I}$ and hence, $y \notin \{x\}^\bullet$. This shows that $\{x\}^\bullet \subset \{x\}$ and $\{x\}$ is $\tau^\bullet$-closed. Conversely, suppose that each singleton is $\tau^\bullet$-closed and $\bullet$-dense in itself. Let x be any point of X and $y \in X - \{x\}$. Then, $\{y\}^\bullet = \{y\} \subset X - \{x\}$ and $x \notin \{y\}^\bullet$, so there exists a closed set F such that $x \in F$ and $F \cap \{y\} \in \mathcal{I}$. We assert that $F \cap \{y\} = \emptyset$.

Otherwise, we have that $F \cap \{y\} \neq \emptyset$, $\{y\} = F \cap \{y\} \in \mathcal{I}$. It follows that $\{y\}^\bullet = \emptyset$, contradicting the fact that $\{y\}^\bullet = \{y\}$. Thus, we have $F \cap \{y\} = \emptyset$, $y \in U = X - F \subset X - \{x\}$ and U is an open set. Therefore, $X - \{x\}$ is an open set and $\{x\}$ is a closed set. This shows that $(X, \tau, \mathcal{I})$ is a T_1-space. □

Corollary 3.23. [12] *A topological space (X, τ) is T_1 if and only if for each $x \in X$, the singleton $\{x\}$ is a Λ-set.*

4. $(\tau^\star, \tau^\bullet)$-$g$-closed sets

Using the topologies $\tau^\star$ and $\tau^\bullet$, we introduce a modification of the notion of g-closed set, called $(\tau^\star, \tau^\bullet)$-$g$-closed, in order to characterize a separation property called $(\tau^\star, \tau^\bullet)$-$T_{1/2}$.

Definition 4.1. A subset A of a space $(X, \tau, \mathcal{I})$ is said to be $(\tau^\star, \tau^\bullet)$-$g$-closed, if $Cl^\star(A) \subset U$ whenever $A \subset U$ and U is $\tau^\bullet$-open.

Proposition 4.2. *A subset A is $(\tau^\star, \tau^\bullet)$-$g$-closed if and only if $A^\star \subset U$ whenever $A \subset U$ and U is $\tau^\bullet$-open.*

Proof. The proof follows from the fact that $A^\star \subset Cl^\star(A)$. □

Proposition 4.3. *Let A and B be subsets of a space $(X, \tau, \mathcal{I})$. The following properties hold:*

(1) *If A is $\tau^\star$-closed, then A is $(\tau^\star, \tau^\bullet)$-$g$-closed.*
(2) *If A is $(\tau^\star, \tau^\bullet)$-$g$-closed and $\tau^\bullet$-open, then A is $\tau^\star$-closed.*
(3) *If A is $(\tau^\star, \tau^\bullet)$-$g$-closed and $A \subset B \subset A^\star$, then B is $(\tau^\star, \tau^\bullet)$-$g$-closed.*

Proof. The proof is clear. □

The following example shows that, in general, the converses of properties (1) and (2) of Proposition 4.3 are not true.

Example 4.4. Consider $X = \{a, b, c\}$ with the topology $\tau = \{\emptyset, \{a, c\}, X\}$ and the ideal $\mathcal{I} = \{\emptyset, \{c\}\}$. Take $A = \{a, c\}$ and $B = \{c\}$, then $A^\star = \{a, c\}^\star = X$ and $Cl^\star(A) = X$. Since $(X - A)^\bullet = \{b\}^\bullet = \{a, b, c\} \not\subset \{b\} = X - A$, we have $X - A$ is not $\tau^\bullet$-closed and $A = \{a, c\}$ is not $\tau^\bullet$-open, which implies that X is the only $\tau^\bullet$-open set containing A. Hence, $A = \{a, c\}$ is a $(\tau^\star, \tau^\bullet)$-$g$-closed set, but does not is $\tau^\star$-closed because $A^\star = \{a, c\}^\star = X \not\subset \{a, c\} = A$. On the other hand, as $B^\star = \{c\}^\star = \emptyset \subset \{c\} = B$, we have $B = \{c\}$ is a $\tau^\star$-closed set and hence is a $(\tau^\star, \tau^\bullet)$-$g$-closed set, but does not a $\tau^\bullet$-open set because $(X - B)^\bullet = \{a, b\}^\bullet = \{a, b, c\} \not\subset \{a, b\} = X - B$, that is $X - B$ does not is a $\tau^\bullet$-closed set.

Theorem 4.5. *A subset A of X is $(\tau^\star, \tau^\bullet)$-$g$-closed if and only if $A^\star \subset \Lambda^\bullet(A)$, where $\Lambda^\bullet(A)$ is the kernel of A in the topology $\tau^\bullet$.*

Proof. Suppose that A is $(\tau^\star, \tau^\bullet)$-$g$-closed and let U be any $\tau^\bullet$-open set such that $A \subset U$. Then, $Cl^\star(A) \subset U$ and hence, we get that $Cl^\star(A) \subset \Lambda^\bullet(A)$. Conversely, suppose that $Cl^\star(A) \subset \Lambda^\bullet(A)$ and let U be any $\tau^\bullet$-open set containing A. Then, $Cl^\star(A) \subset \Lambda^\bullet(A) \subset U$ and so, A is a $(\tau^\star, \tau^\bullet)$-$g$-closed set. □

Theorem 4.6. *If a subset A of a space $(X, \tau, \mathcal{I})$ is $(\tau^\star, \tau^\bullet)$-$g$-closed, then $A^\star - A$ does not contain any nonempty $\tau^\bullet$-closed set.*

Proof. The proof is clear. □

Theorem 4.7. *Let A be a $(\tau^\star, \tau^\bullet)$-$g$-closed subset of a space $(X, \tau, \mathcal{I})$. The following properties are equivalent:*

(1) *A is $\tau^\star$-closed.*
(2) *$A^\star - A = \emptyset$.*
(3) *$A^\star - A$ is a $\tau^\bullet$-closed set.*

Proof. (1) ⇔ (2) By Theorem 2.1, A is a $\tau^\star$-closed set if and only if $A^\star \subset A$, or equivalently $A^\star - A = \emptyset$.

(2) ⇒ (3) Follows from the fact that $\tau^\bullet$ is a topology.

(3) ⇒ (2) Suppose that $A^\star - A$ is a $\tau^\bullet$-closed set. Since A is a $(\tau^\star, \tau^\bullet)$-$g$-closed set, by Theorem 4.6 it follows that $A^\star - A = \emptyset$. □

Theorem 4.8. *Let $(X, \tau, \mathcal{I})$ be a space and Ω a finite index set. If $\{A_\omega : \omega \in \Omega\}$ is a collection of $(\tau^\star, \tau^\bullet)$-$g$-closed sets, then $\bigcup\{A_\omega : \omega \in \Omega\}$ is a $(\tau^\star, \tau^\bullet)$-$g$-closed set.*

Proof. The proof is clear and hence is omitted. □

Proposition 4.9. *For each $x \in X$, the singleton $\{x\}$ is $\tau^\bullet$-closed or $(\tau^\star, \tau^\bullet)$-$g$-open.*

Proof. Suppose that $\{x\}$ does not is $\tau^\bullet$-closed, then $X - \{x\}$ is not $\tau^\bullet$-open and the only $\tau^\bullet$-open set that contain $X - \{x\}$ is X. Since $Cl^\star(X - \{x\}) \subset X$, then $X - \{x\}$ is a $(\tau^\star, \tau^\bullet)$-$g$-closed set and hence, $\{x\}$ is a $(\tau^\star, \tau^\bullet)$-$g$-open set. □

Definition 4.10. The space $(X, \tau, \mathcal{I})$ is said to be $(\tau^\star, \tau^\bullet)$-$T_{1/2}$, if each $(\tau^\star, \tau^\bullet)$-$g$-closed set of X is $\tau^\star$-closed.

Now we characterize the $(\tau^\star, \tau^\bullet)$-$T_{1/2}$ spaces using the notions of $\tau^\star$-open and $\tau^\bullet$-closed sets.

Theorem 4.11. *The space $(X, \tau, \mathcal{I})$ is $(\tau^\star, \tau^\bullet)$-$T_{1/2}$ if and only if for each $x \in X$, the singleton $\{x\}$ is $\tau^\star$-open or $\tau^\bullet$-closed.*

Proof. Suppose that $(X, \tau, \mathcal{I})$ is a $(\tau^\star, \tau^\bullet)$-$T_{1/2}$ space and consider $x \in X$. If $\{x\}$ does not is $\tau^\bullet$-closed, then by Proposition 4.9, $\{x\}$ is a $(\tau^\star, \tau^\bullet)$-$g$-open and so, $X - \{x\}$ is a $(\tau^\star, \tau^\bullet)$-$g$-closed set. Since $(X, \tau, \mathcal{I})$ is $(\tau^\star, \tau^\bullet)$-$T_{1/2}$, then, $X - \{x\}$ is a $\tau^\star$-closed set. Therefore, $\{x\}$ is a $\tau^\star$-open. Conversely, suppose that A is a $(\tau^\star, \tau^\bullet)$-$g$-closed set and consider $x \in A^\star$. We have to analyze the following two cases:

Case 1: $\{x\}$ is a $\tau^\bullet$-closed set. By Theorem 4.7, $A^\star - A$ does not contain any nonempty $\tau^\bullet$-closed set and so $x \notin A^\star - A$. Since $x \in A^\star$, it follows that $x \in A$.

Case 2: $\{x\}$ is a $\tau^\star$-open set. By Theorem 2.2, $A^\star - A$ does not contain any nonempty $\tau^\star$-open set and hence, $x \notin A^\star - A$. Since $x \in A^\star$, we conclude that $x \in A$. Thus, in both cases $x \in A$ and therefore, $A^\star \subset A$. Thus, A is a $\tau^\star$-closed set and this shows that $(X, \tau, \mathcal{I})$ is a $(\tau^\star, \tau^\bullet)$-$T_{1/2}$ space. □

5. $\mathcal{I}$-λ-closed sets

In this section, we introduce a generalization of the notion of a λ-closed set in $(X, \tau, \mathcal{I})$, called $\mathcal{I}$-λ-closed set, in order to obtain two new separation properties, namely quasi $(\tau^\star, \tau^\bullet)$-$T_0$ and quasi $(\tau^\star, \tau^\bullet)$-$R_0$.

Definition 5.1. A subset A of a space $(X, \tau, \mathcal{I})$ is said to be $\mathcal{I}$-λ-closed, if $A = L \cap F$ where F is a $\tau^\bullet$-closed set and L is a $\tau^\star$-closed set.

Remark 5.2. Let $(X, \tau, \mathcal{I})$ be a space.

(1) If A is a λ-closed set, then A is a $\mathcal{I}$-λ-closed set, because each Λ-set is $\tau^\bullet$-closed and each closed set is $\tau^\star$-closed.

(2) If $\mathcal{I} = \{\emptyset\}$, then A is a $\mathcal{I}$-λ-closed set if and only if A is a λ-closed set.

Proposition 5.3. *Each $\tau^\bullet$-closed set is $\mathcal{I}$-λ-closed and each $\tau^\star$-closed set is $\mathcal{I}$-λ-closed.*

Proof. The proof is clear and hence is omitted. □

Theorem 5.4. *Let $(X, \tau, \mathcal{I})$ be a space and Ω be an index set. If $\{A_\omega : \omega \in \Omega\}$ is a collection of $\mathcal{I}$-λ-closed sets, then $\bigcap\{A_\omega : \omega \in \Omega\}$ is a $\mathcal{I}$-λ-closed set.*

Proof. The proof follows directly from the definition of $\mathcal{I}$-λ-closed set and the fact that $\tau^\bullet$ and $\tau^\star$ are topologies. □

Lemma 5.5. *Let A be a subset of a space $(X, \tau, \mathcal{I})$. The following properties are equivalent:*

(1) *A is a $\mathcal{I}$-λ-closed set.*

(2) *$A = L \cap Cl^\star(A)$, where L is a $\tau^\bullet$-closed set.*

(3) *$A = Cl^\bullet(A) \cap Cl^\star(A)$.*

Proof. The proof is clear and hence omitted. □

Definition 5.6. The space $(X, \tau, \mathcal{I})$ is said to be quasi $(\tau^\star, \tau^\bullet)$-$T_0$, if for each pair of distinct points $x, y \in X$ there exists a $\tau^\star$-open set U containing y but not x or there exists a $\tau^\bullet$-closed set F containing x but not y.

Remark 5.7. Each T_0 space is quasi $(\tau^\star, \tau^\bullet)$-$T_0$, because each open set is $\tau^\star$-open and $\tau^\bullet$-closed.

Theorem 5.8. *The space $(X, \tau, \mathcal{I})$ is quasi $(\tau^\star, \tau^\bullet)$-$T_0$ if and only if for each $x \in X$, the singleton $\{x\}$ is $\mathcal{I}$-λ-closed.*

Proof. For each $x \in X$, $\{x\} \subset Cl^\bullet(\{x\}) \cap Cl^\star(\{x\})$. If $y \neq x$, then we have to analyze the following two cases:
Case 1: There exists a $\tau^\star$-open set U containing y but not x or
Case 2: There exists a $\tau^\bullet$-closed set F containing x but not y.
In the Case 1, $y \notin Cl^\star(\{x\})$ and hence, $y \notin Cl^\star(\{x\}) \cap Cl^\bullet(\{x\})$. In Case 2, $y \notin Cl^\bullet(\{x\})$ and then, $y \notin Cl^\star(\{x\}) \cap Cl^\bullet(\{x\})$. This shows that $Cl^\star(\{x\}) \cap Cl^\bullet(\{x\}) \subset \{x\}$. Hence $\{x\} = Cl^\star(\{x\}) \cap Cl^\bullet(\{x\})$ and by Lemma 5.5, $\{x\}$ is a $\mathcal{I}$-λ-closed set.

Conversely, suppose that $(X, \tau, \mathcal{I})$ does not is a quasi $(\tau^\star, \tau^\bullet)$-$T_0$ space. Then there exist two distinct points $x, y \in X$ such that (i) $y \in F$ for each $\tau^\bullet$-closed set F containing x and (ii) $\{x\} \cap U \neq \emptyset$ for each $\tau^\star$-open set U containing y. From (i) and (ii) we obtain that $y \in Cl^\bullet(\{x\})$ and $y \in Cl^\star(\{x\})$, respectively. So $y \in Cl^\bullet(\{x\}) \cap Cl^\star(\{x\})$. Since $\{x\}$ is a $\mathcal{I}$-λ-closed set, by Lemma 5.5 it follows that $\{x\} = Cl^\bullet(\{x\}) \cap Cl^\star(\{x\})$ and from the above, we obtain that $y = x$, contradicting the fact that $x \neq y$. □

Remark 5.9. In [2, Theorem 2.5] was proved that a topological space (X, τ) is T_0 if and only if for each $x \in X$, the singleton $\{x\}$ is λ-closed. According to this result, if $\mathcal{I} = \{\emptyset\}$ we have: $(X, \tau, \mathcal{I})$ is a quasi $(\tau^\star, \tau^\bullet)$-$T_0$ space if and only if $(X, \tau, \mathcal{I})$ is a T_0 space.

Recall that a topological space (X, τ) is said to be a R_0 space [18], if for every $U \in \tau$ and every $x \in U$, $Cl(\{x\}) \subset U$. Now we introduce a notion that has a certain analogy with the notion of R_0 space.

Definition 5.10. The space $(X, \tau, \mathcal{I})$ is said to be quasi $(\tau^\star, \tau^\bullet)$-$R_0$, if for each $\tau^\bullet$-closed set F and each $x \in F$, $Cl^\star(\{x\}) \subset F$.

Theorem 5.11. *Let $(X, \tau, \mathcal{I})$ be a quasi $(\tau^\star, \tau^\bullet)$-$R_0$ space. A singleton $\{x\}$ is $\mathcal{I}$-λ-closed if and only if is $\tau^\star$-closed.*

Proof. Suppose that $\{x\}$ is a $\mathcal{I}$-λ-closed set. By Lemma 5.5, $\{x\} = Cl^\bullet(\{x\}) \cap Cl^\star(\{x\})$. For each $\tau^\bullet$-closed set F containing x, $Cl^\star(\{x\}) \subset F$; in particular, $Cl^\star(\{x\}) \subset Cl^\bullet(\{x\})$. Thus, we get that $\{x\} = Cl^\bullet(\{x\}) \cap Cl^\star(\{x\}) = Cl^\star(\{x\})$ and $\{x\}$ is $\tau^\star$-closed.

Conversely, suppose that $\{x\}$ is a $\tau^\star$-closed set. Then, $\{x\} = Cl^\star(\{x\})$ and follows that $\{x\} \subset Cl^\bullet(\{x\}) \cap Cl^\star(\{x\}) = Cl^\bullet(\{x\}) \cap \{x\} = \{x\}$. Therefore, $\{x\} = Cl^\bullet(\{x\}) \cap Cl^\star(\{x\})$ and again by Lemma 5.5, we conclude that $\{x\}$ is a $\mathcal{I}$-λ-closed set. □

Corollary 5.12. *Let $(X, \tau, \mathcal{I})$ be a quasi $(\tau^\star, \tau^\bullet)$-$R_0$ space. Then, $(X, \tau, \mathcal{I})$ is a quasi $(\tau^\star, \tau^\bullet)$-$T_0$ space if and only if $(X, \tau^\star)$ is a T_1 space.*

Proof. The proof is an immediate consequence of Theorem 5.11 and the characterization of a T_1 space. □

6. $\mathcal{I}$-$g\Lambda$-sets

Now we introduce the notion of $\mathcal{I}$-$g\Lambda$-set in order to extend the notion of $g\Lambda$-set and obtain some related results with the $(\tau^\star, \tau^\bullet)$-$T_{1/2}$ spaces.

Definition 6.1. A subset A of a space $(X, \tau, \mathcal{I})$ is said to be $\mathcal{I}$-$g\Lambda$-set, if $A^\bullet \subset F$ whenever $A \subset F$ and F is a $\tau^\star$-closed set.

Remark 6.2. If $\mathcal{I} = \{\emptyset\}$, then A an $\mathcal{I}$-$g\Lambda$-set if and only if A is a $g\Lambda$-set.

Proposition 6.3. *If A and B are subsets of a space $(X, \tau, \mathcal{I})$, then the following properties hold:*

(1) *If A is a $\tau^\bullet$-closed set, then A is a $\mathcal{I}$-$g\Lambda$-set.*
(2) *If A is a $\mathcal{I}$-$g\Lambda$-set and $\tau^\star$-closed, then A is $\tau^\bullet$-closed.*
(3) *If A is a $\mathcal{I}$-$g\Lambda$-set and $A \subset B \subset A^\bullet$, then B is a $\mathcal{I}$-$g\Lambda$-set.*

Proof. (1) Follows from the definition of $\tau^\bullet$-closed set,

(2) Follows from the definitions of $\mathcal{I}$-$g\Lambda$-set, $\tau^\star$-closed set and Proposition 3.16.

(3) Follows from (1) and (3) of Lemma 3.7. □

Theorem 6.4. *Let A be a subset of a space $(X, \tau, \mathcal{I})$. The following properties are equivalent:*

(1) *A is a $\mathcal{I}$-$g\Lambda$-set.*
(2) *$A^\bullet \cap U = \emptyset$ whenever $A \cap U = \emptyset$ and $U \in \tau^\star$.*
(3) *$A^\bullet \subset Cl^\star(A)$*

Proof. The equivalences follows directly and are omitted. □

Theorem 6.5. *Let A be a subset of a space $(X, \tau, \mathcal{I})$. If A is a $\mathcal{I}$-$g\Lambda$-set and F is a $\tau^\star$-closed set such that $(X - A^\bullet) \cup A \subset F$, then $F = X$.*

Proof. Suppose that A is a $\mathcal{I}$-$g\Lambda$-set and F is a $\tau^\star$-closed set such that $(X - A^\bullet) \cup A \subset F$. Then, $A \subset (X - A^\bullet) \cup A \subset F$ and so, $A^\bullet \subset F$. Therefore, $X - F \subset X - A^\bullet$. Since $(X - A^\bullet) \cup A \subset F$, it follows that $X - F \subset A^\bullet \cap (X - A) \subset A^\bullet$. In consequence, $X - F \subset (X - A^\bullet) \cap A^\bullet = \emptyset$ and so $X = F$. □

Theorem 6.6. *Let A be a $\mathcal{I}$-$g\Lambda$-set of a space $(X, \tau, \mathcal{I})$. Then, $(X - A^\bullet) \cup A$ is a $\tau^\star$-closed set if and only if A is a $\tau^\bullet$-closed set.*

Proof. The proof follows from Theorem 6.5. □

Proposition 6.7. *For each $x \in X$, the singleton $\{x\}$ is $\tau^\star$-open or $X - \{x\}$ is a $\mathcal{I}$-$g\Lambda$-set.*

Proof. Suppose that $\{x\}$ does not is a $\tau^\star$-open set, then $X - \{x\}$ is not a $\tau^\star$-closed set and the only $\tau^\star$-closed set that contain $X - \{x\}$ is X. Thus, $(X - \{x\})^\bullet \subset X^\bullet \subset X$ and hence, $X - \{x\}$ is a $\mathcal{I}$-$g\Lambda$-set. □

Theorem 6.8. *Let A be a subset of a space $(X, \tau, \mathcal{I})$. If A is a $\mathcal{I}$-$g\Lambda$-set, then $A^\bullet - A$ does not contain any nonempty $\tau^\star$-open set.*

Proof. The proof is clear and is omitted. □

Theorem 6.9. *Let A be a $\mathcal{I}$-$g\Lambda$-set in a space $(X, \tau, \mathcal{I})$. Then, A is a $\tau^\bullet$-closed set if and only if $A^\bullet - A$ is a $\tau^\star$-open set.*

Proof. If A is a $\tau^\bullet$-closed set, then $A^\bullet \subset A$ and $A^\bullet - A = \emptyset$. Since $\tau^\star$ is a topology, then $A^\bullet - A$ is a $\tau^\star$-open set.

Conversely, si $A^\bullet - A$ is a $\tau^\star$-open set. Since A is a $\mathcal{I}$-$g\Lambda$-set, by Theorem 6.8 it follows that $A^\bullet - A = \emptyset$ and hence, $A^\bullet \subset A$. This shows that A is a $\tau^\bullet$-closed set. □

Theorem 6.10. *If each $\mathcal{I}$-$g\Lambda$-set in a space $(X, \tau, \mathcal{I})$ is a $\tau^\bullet$-closed set, then $(X, \tau, \mathcal{I})$ is a $(\tau^\star, \tau^\bullet)$-$T_{1/2}$ space.*

Proof. The proof follows from Proposition 6.7 and Theorem 4.11. □

7. Invariance under certain forms of continuity

In this section, we introduce certain types of continuity in order to analyze the invariance of some separation properties studied in the previous sections under the action of these new classes of functions. Next we consider a function f defined from a space (X, τ, I) to a space (Y, σ, J).

Definition 7.1. A function $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ is called:

(1) $\bullet$-irresolute if $f^{-1}(B)$ is a $\tau^\bullet$-closed set whenever B is a $\sigma^\bullet$-closed set.
(2) $(\star, \bullet)$-g-irresolute if $f^{-1}(B)$ is a $(\tau^\star, \tau^\bullet)$-$g$-closed set whenever B is a $(\sigma^\star, \sigma^\bullet)$-$g$-closed set.
(3) $(\mathcal{I}, \mathcal{J})$-$\lambda$-irresolute if $f^{-1}(B)$ is an $\mathcal{I}$-λ-closed set whenever B is an $\mathcal{J}$-λ-closed set.

Theorem 7.2. *Let $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ be a $(\star, \bullet)$-g-irresolute surjection such that $(f(A))^\star \subset f(A^\star)$ for each $\tau^\star$-closed set A. If $(X, \tau, \mathcal{I})$ is a $(\tau^\star, \tau^\bullet)$-$T_{1/2}$ space, then $(Y, \sigma, \mathcal{J})$ is a $(\sigma^\star, \sigma^\bullet)$-$T_{1/2}$ space.*

Proof. Assume that $(X, \tau, \mathcal{I})$ is a $(\tau^\star, \tau^\bullet)$-$T_{1/2}$ space and let B be any $(\sigma^\star, \sigma^\bullet)$-$g$-closed set. Since f is $(\star, \bullet)$-g-irresolute, $f^{-1}(B)$ is a $(\tau^\star, \tau^\bullet)$-$g$-closed set and, hence, it is $\tau^\star$-closed. Thus, we obtain $B^\star = (f(f^{-1}(B)))^\star \subset f((f^{-1}(B))^\star) \subset f(f^{-1}(B)) = B$ and, so B is a $\sigma^\star$-closed set. This shows that $(Y, \sigma, \mathcal{J})$ is a $(\sigma^\star, \sigma^\bullet)$-$T_{1/2}$ space. □

Theorem 7.3. *Let $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ be a $(\mathcal{I}, \mathcal{J})$-$\lambda$-irresolute bijection. If $(X, \tau, \mathcal{I})$ is a quasi $(\tau^\star, \tau^\bullet)$-$T_0$ space, then $(Y, \sigma, \mathcal{J})$ is a quasi $(\sigma^\star, \sigma^\bullet)$-$T_0$ space.*

Proof. This follows from Theorem 5.8. □

Theorem 7.4. *Let $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ be a $\bullet$-irresolute surjection such that $(f(\{x\}))^\star \subset f(\{x\}^\star)$ for each $x \in X$. If $(X, \tau, \mathcal{I})$ is a quasi $(\tau^\star, \tau^\bullet)$-$R_0$ space, then $(Y, \sigma, \mathcal{J})$ is a quasi $(\sigma^\star, \sigma^\bullet)$-$R_0$ space.*

Proof. Straightforward. □

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